

CHAPTER 14 – APPORTIONMENT

An *apportionment problem* is to round a set of fractions so their sum is maintained at its original value.

The rounding procedure used in an apportionment problem is called an *apportionment method*.

Notation

- Round q to the nearest integer is $[q]$. Round half integers up.
- Round q down is $\lfloor q \rfloor$
- Round q up is $\lceil q \rceil$

The total population, p , divided by the house size, h , is called the *standard divisor, s* .

$$s = \frac{p}{h}$$

A group's quota q_i is the group's population, p_i , divided by the standard divisor, s .

$$q_i = \frac{p_i}{s}$$

Hamilton Method

1. Round each quota down.
2. Calculate the number of seats left to be assigned.
3. Assign the seats to those with the largest fractional parts.

Use the Hamilton method to apportion 36 silver coins to Doris, Mildred, and Henrietta if Doris paid \$5900, Mildred paid \$7600, and Henrietta paid \$1400.

$$S = \frac{14,900}{36} = 413.8889$$

		q	$Lq \downarrow$	N
Doris	5900	$\frac{5900}{413.8889} = 14.2550$	14	14
Mildred	7600	$\frac{7600}{413.8889} = 18.3624$	18	18
Henrietta	1400	$\frac{1400}{413.8889} = 3.3825$	3 + 1	4 ← b/c largest fractional part
	\$14,900		35 1 coin left	36

When the bag of coins is opened, there are 37 coins. How should they be apportioned?

$$S = \frac{14,900}{37} = 402.7027$$

		q	$Lq \downarrow$	N
Doris	5900	$\frac{5900}{402.7027} = 14.6510$	14 + 1	15
Mildred	7600	$\frac{7600}{402.7027} = 18.8725$	18 + 1	19
Henrietta	1400	$\frac{1400}{402.7027} = 3.4765$	3	3
			35 2 coins left	

Alabama Paradox occurred

A **paradox** is a statement that is seemingly contradictory or opposed to common sense and yet is perhaps true.

The **Alabama paradox** occurs when a state loses a seat as the result of an increase in the house size. house size was only change

The **new states paradox** occurs in a reapportionment in which an increase in the total number of ~~seats~~ ^{States} causes a shift in the apportionment of existing states. New state + change prev apportionment

The **population paradox** occurs when there are a fixed number of seats and a reapportionment causes a state to lose a seat to another state even though the percent increase in the population of the state that loses the seat is larger than the percent increase of the state that wins the seat.

Consider two numbers, A and B , where $A > B$. pop changed + ~~change doesn't match~~
change doesn't match
% inc rankings

The **absolute difference** between the two numbers is $A - B$

The **relative difference** between the two numbers is $\frac{A - B}{B} \times 100\%$

Find the absolute and relative difference between the following numbers

- Absolute difference*
- (a) 1 and 2 $2 - 1 = 1$ $\frac{2-1}{1} \cdot 100\% = 100\%$
- (b) 10 and 11 $11 - 10 = 1$ $\frac{11-10}{10} \cdot 100\% = 10\%$
- (c) 2000 and 2001 $2001 - 2000 = 1$ $\frac{2001-2000}{2000} \cdot 100\% = 0.05\%$

A county has 4 districts, North, South, East, and West. They will apportion for a 100 member advisory council using the method of Hamilton. Determine the number of council members from each district.

$$S = \frac{64920}{100} = 649.2$$

		$\frac{27460}{649.2} = 42.2982$	$\lfloor 42 \rfloor$	
North	27,460	$\frac{27460}{649.2} = 42.2982$	42	42
South	17,250	$\frac{17250}{649.2} = 26.5712$	26 + 1	27
East	19,210	$\frac{19210}{649.2} = 29.5903$	29 + 1	30
West	1000	$\frac{1000}{649.2} = 1.5404$	1	1
	64,920		98	100

2 left

Ten years later the county will reapportion for the 100 seats using new population data. Determine the number of council members from each district. Did a paradox occur?

$$S = \frac{65910}{100} = 659.1$$

		$\frac{28140}{659.1} = 42.69$	$\lfloor 42 \rfloor$	new		
old	North	28,140	$\frac{28140}{659.1} = 42.69$	42 + 1	43 ^{gain}	$\frac{28140 - 27460}{27460} = 2.476\%$
27	South	17,450	$\frac{17450}{659.1} = 26.48$	26	26 ^{loss}	$\frac{17450 - 17250}{17250} = 1.159\%$
30	East	19,330	$\frac{19330}{659.1} = 29.33$	29	29 ^{loss}	$\frac{19330 - 19210}{19210} = 0.625\%$
1	West	990	$\frac{990}{659.1} = 1.50$	1 + 1	2 ^{gain}	$\frac{990 - 1000}{1000} = -1.01\%$
	65,910		98	100		

2 left

So population paradox occurred b/c west had smallest % gain (was even negative) but gained a seat.

A country has two states, Solid and Liquid. Use Hamilton's method to apportion 12 seats for their congress.

$$S = \frac{203,995}{12} = 16999.5833$$

Solid	144,899	$\frac{144899}{16999.5833} = 8.5237$	8 + 1	9
Liquid	59,096	$\frac{59096}{16999.5833} = 3.4763$	3	3
	203,995		11	12

1 left

Another state, Plasma, wants to join. If there are 38,240 people in that state, how many representatives should they receive?

$$\frac{38,240}{16999.5833} = 2.2495 \approx 2$$

Apportion using Hamilton's method. 12 original + 2 new representatives added b/c of Plasma

$$S = \frac{242,235}{14} = 17302.5$$

Solid	144,899	$\frac{144899}{17302.5} = 8.374$	8	8
Liquid	59,096	$\frac{59096}{17302.5} = 3.415$	3 + 1	4
Gas Plasma	38,240	$\frac{38,240}{17302.5} = 2.210$	2	2
	242,235		13	14

1 left

Did any paradoxes occur?

Yes New State Paradox

Divisor Methods

The standard divisor, s , represents the average district population. Apportionment can be done by adjusting the average district population to be a specific value called the *adjusted divisor*, d .

A *divisor method* of apportionment determines each state's apportionment by dividing its population by a common divisor d and rounding the resulting quota. Different divisor methods use different rounding rules.

A *critical divisor* is a divisor that will produce a quota for each population that gives a correct total number of seats.

The Jefferson Method

1. Find the standard divisor, s . Then find $q_i = \left\lfloor \frac{p_i}{s} \right\rfloor$
2. If the total number of seats is not correct, find the new divisors that correspond to giving each state one more seat. $d_i = \frac{p_i}{q_i + 1}$
3. Assign a seat to the state with the largest d_i . If the total number of seats is correct, stop. Otherwise repeat step 2.
4. The adjusted divisor d will be the exact value of the last divisor found in step 3.

Use the Jefferson method to apportion 36 silver coins to Doris, Mildred, and Henrietta if Doris paid \$5900, Mildred paid \$7600, and Henrietta paid \$1400. Comment on your results.

$$S = \frac{14900}{36} = 413.8889$$

		q	Lq	$d = \frac{p}{q_i + 1}$	
Doris	5900	$\frac{5900}{413.8889} = 14.2550$	14	$\frac{5900}{14+1} = 393.33$	14
Mildred	7600	$\frac{7600}{413.8889} = 18.3624$	18	$\frac{7600}{18+1} = 400$ * largest divisor +1	19
Henrietta	1400	$\frac{1400}{413.8889} = 3.3825$	3	$\frac{1400}{3+1} = 350$	3
	14900		35 1 left	critical divisor = 400	36

When the bag of coins is opened, there are 37 coins. How should they be apportioned? Comment on your results

$$S = \frac{14900}{37} = 402.7027$$

		q	Lq	$d = \frac{p}{q_i + 1}$	
Doris	5900	$\frac{5900}{402.7027} = 14.65$	14	$\frac{5900}{14+1} = 393.33$	14
Mildred	7600	$\frac{7600}{402.7027} = 18.87$	18	$\frac{7600}{18+1} = 400 + 1$	19
Henrietta	1400	$\frac{1400}{402.7027} = 3.48$	3	$\frac{1400}{3+1} = 350$	3
	14900		35 2 left		36 1 left

	$d = \frac{p}{q_i + 1}$		
Doris	$\frac{5900}{14+1} = 393.33$ * +1	15	Critical divisor = 393.33
Mildred	$\frac{7600}{19+1} = 380$	19	
Henrietta	$\frac{1400}{3+1} = 350$	3	
		37 coins	

A school offers four different art classes with the enrollments shown below. Ten new teachers will be hired using Jefferson's method. Determine who gets the new teachers and comment on the results.

$$S = \frac{1192}{10} = 119.2$$



	q	$d = \frac{P}{q_i + 1}$	q_i	$d = \frac{P}{q_i + 1}$	q_i	$d = \frac{P}{q_i + 1}$	
Ceramics	785	$\frac{785}{119.2} = 6.59$	6	$\frac{785}{6+1} = 112.14$ <small>largest +1</small>	7	$\frac{785}{7+1} = 98.125$ <small>largest +1</small>	8
Painting	152	$\frac{152}{119.2} = 1.28$	1	$\frac{152}{1+1} = 76$	1	76	1
Dance	160	$\frac{160}{119.2} = 1.34$	1	$\frac{160}{1+1} = 80$	1	80	1
Theatre	95	$\frac{95}{119.2} = 0.80$	0	$\frac{95}{0+1} = 95$	0	95	0
	1192		8		9		10
			2 left		1 left		

Violates quota rule
 b/c ceramics does not have
 $\lceil 6.59 \rceil = 7$ or $\lfloor 6.59 \rfloor = 6$ representatives

Quota Rule says that the number assigned to each represented unit must be the standard quota, q_i , rounded up or rounded down.

Balinski and Young found that no apportionment method that satisfies the quota rule is free of paradoxes.

The Adams Method

1. Find the standard divisor, s . Then find $N_i = \lceil q_i \rceil = \left\lceil \frac{p_i}{s} \right\rceil$ 
2. If the total number of seats is not correct, find the new divisors that correspond to giving each state one fewer seat. $d_i = \frac{p_i}{q_i - 1}$ 
3. Remove a seat from the state with the smallest d_i . If the total number of seats is correct, stop. Otherwise repeat step 2.
4. The adjusted divisor d will be the exact value of the last divisor found in step 3.

The Webster Method

1. Find the standard divisor, s . Then find $N_i = \left[\frac{p_i}{s} \right]$
2. If the total number of seats is correct, the process is done.
3. If the total number of seats is too few, use a critical divisor of d^+ and the state with the largest critical divisor gets a next seat

$$d_i^+ = \frac{p_i}{N_i + \frac{1}{2}}$$

4. If the total number of seats is too many, use a critical divisor of d^- and the state with the smallest critical divisor loses a seat

$$d_i^- = \frac{p_i}{N_i - \frac{1}{2}}$$

Apportion the regions below using both the Adams method and the Webster method for a house size of 16. Comment on your results.

$$S = \frac{45136}{16} = 2821$$

Adams

		q	$\lceil q \rceil$	$d = \frac{P}{q_i - 1}$	smallest	
Beach	28,204	$\frac{28204}{2821} = 9.998$	10	$\frac{28204}{10-1} = 3134$	-1	9
Forest	11,267	$\frac{11267}{2821} = 3.99$	4	$\frac{11267}{4-1} = 3756$		4
Plains	4203	$\frac{4203}{2821} = 1.49$	2	$\frac{4203}{2-1} = 4203$		2
Swamp	1462	$\frac{1462}{2821} = 0.52$	1	$\frac{1462}{1-1} = \text{Big \#}$		1
	45136		17			16

1 too many

Webster $S = \frac{45136}{16} = 2821$

		q	$\lceil q \rceil$
Beach	28,204	$\frac{28204}{2821} = 9.998$	10
Forest	11,267	$\frac{11267}{2821} = 3.99$	4
Plains	4203	$\frac{4203}{2821} = 1.49$	1
Swamp	1462	$\frac{1462}{2821} = 0.52$	1
	45136		16

Geometric Mean

The arithmetic mean of two numbers a and b is given by

$$\bar{x} = \frac{a+b}{2}$$

The *geometric mean* of two numbers a and b is given by

$$G(a,b) = \sqrt{ab}$$

Example

Find the arithmetic and geometric means for the following numbers.

(a) 0 and 1.

$$\bar{x} = \frac{0+1}{2} = 0.5$$

Geometric

$$\sqrt{0 \cdot 1} = 0$$

(b) 4 and 5

$$\frac{4+5}{2} = 4.5$$

$$\sqrt{4 \cdot 5} \approx 4.4721$$

(c) 25 and 26

$$\frac{25+26}{2} = 25.5$$

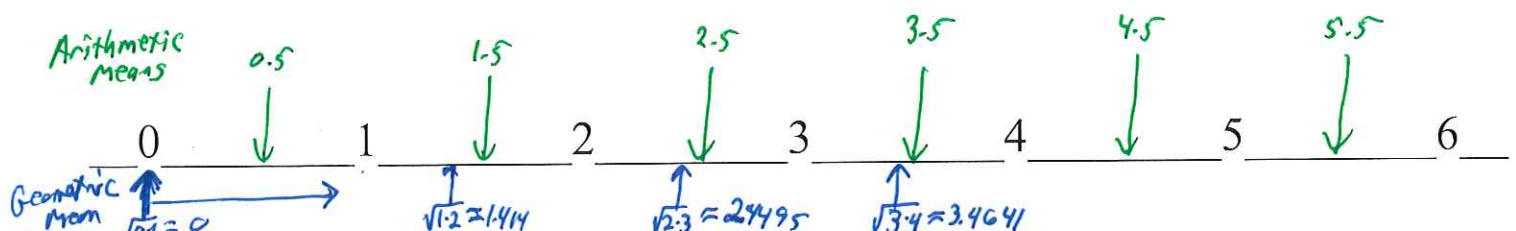
$$\sqrt{25 \cdot 26} \approx 25.4951$$

(d) Round 0.2 and 3.47 using their geometric means.

rounds to 1 *rounds to 4*

$$\sqrt{0.1} = 0$$

$$\sqrt{3.4} \approx 3.4641$$



The Hill-Huntington Method

1. Find the standard divisor, s . Then find $q_i = \frac{P_i}{s}$
2. Round each quota q_i up or down by comparing it to $q_i^* = \sqrt{\lfloor q_i \rfloor \times \lceil q_i \rceil}$
3. If the total number of seats is correct, the process is done.
4. If the total number of seats is too few, implement Jefferson's method

with a critical divisor of $d_i^+ = \frac{P_i}{\sqrt{N \times (N+1)}}$

5. If the total number of seats is too many, implement Adams' method

with a critical divisor of $d_i^- = \frac{P_i}{\sqrt{N \times (N-1)}}$

Apportion the regions below using the Hill-Huntington method for a house size of 16.

$S = \frac{45136}{16} = 2821$

		q	q^*	rounded quota	$d^- = \frac{P}{\sqrt{N(N-1)}}$	N
Beach	28,204	9.998	$\sqrt{9 \cdot 10} \approx 9.48$	10	$\frac{28204}{\sqrt{10(10-1)}} = 2973$	10
Forest	11,267	3.99	$\sqrt{3 \cdot 4} \approx 3.46$	4	$\frac{11267}{\sqrt{4(4-1)}} = 3253$	4
Plains	4203	1.49	$\sqrt{1 \cdot 2} \approx 1.41$	2	$\frac{4203}{\sqrt{2(2-1)}} = 2972$	1
Swamp	1462	0.52	$\sqrt{0 \cdot 1} \approx 0$	1	$\frac{1462}{\sqrt{1(1-1)}} = \text{Big \#}$	1
	45136			17		16

Handwritten notes:
 - "rounded quota compare to q^* " (above the rounded quota column)
 - "Smallest loses a seat" (with an arrow pointing to the Plains row)
 - "1 too many" (below the total seat count 17)

A country has 5 states and a house size of 40. Apportion for their congress using the following methods: Hamilton, Jefferson, Adams, Webster and Hill-Huntington.

Largest Free Part

Hamilton - round down largest free parts

		$\lfloor q \rfloor$	$\lfloor q \rfloor$	Final
Northwest	130,451	10.3219	10	10
Southwest	157,633	12.4767	12 + 1	13
Central	25,879	2.0477	2	2
Southeast	88,956	7.0386	7	7
Northeast	102,611	8.1191	8	8
			39	40

Largest $d_i = \frac{P_i}{q_i + 1}$

Jefferson q

		$\lfloor q \rfloor$	d_i	N
Northwest	130,451	10	11859.18	10
Southwest	157,633	12	12125.62 *	13
Central	25,879	2	8626.33	2
Southeast	88,956	7	11119.50	7
Northeast	102,611	8	11401.22	8
				40

Adjusted divisor = 12125.62

adjusted divisor = 13045.10

Smallest
 $d_i = \frac{p_i}{q_i - 1}$

	Adams	q	$\lceil q \rceil$	d_i	N	d_i	N	d_i	N	d_i	N
Northwest	130,451	10.3219	11	13045.10	11		11		11		11
Southwest	157,633	12.4767	13	13136.05	13		13		13		13
Central	25,879	2.0477	3	12939.50	3		3		2	25879	2
Southeast	88,956	7.0386	8	12708.00*	7	14826	7		7		7
Northeast	102,611	8.1191	9	12826.38	9	*	8	14658.71	8		8
			44		43		42		41		40

Largest
 $d_i^+ = \frac{p_i}{N_i + \frac{1}{2}}$
 Smallest
 $d_i^- = \frac{p_i}{N_i - \frac{1}{2}}$

	Webster	q	$\lceil q \rceil$	d_i^+	N
Northwest	130,451	10.3219	10	12423.90	10
Southwest	157,633	12.4767	12	12610.64*	13
Central	25,879	2.0477	2	10351.60	2
Southeast	88,956	7.0386	7	11860.80	7
Northeast	102,611	8.1191	8	12071.88	8

adjusted divisor = 12610.64

Hill-Huntington
 $d_i^+ = \frac{p_i}{\sqrt{N_i(N_i+1)}}$
 $d_i^- = \frac{p_i}{\sqrt{N_i(N_i-1)}}$

	Hill-Huntington	q	$q^* = \sqrt{\lceil q \rceil \lfloor q \rfloor}$	d_i^+	N
Northwest	130,451	10.3219	10.48809	10	12438.01
Southwest	157,633	12.4767	12.49000	12	12620.74*
Central	25,879	2.0477	2.44949	2	10565.06
Southeast	88,956	7.0386	7.48331	7	11887.25
Northeast	102,611	8.1191	8.48528	8	12092.82

39

adjusted divisor = 12620.74

SAMPLE EXAM QUESTIONS FROM CHAPTER 14

1. Match the definition to the term

- (A) Quota Rule is III
- (B) Alabama Paradox is II
- (C) Population Paradox is IV

- I. An increase in the total number of ^{states} seats causes a shift in the apportionment of existing states. ~~Alabama Paradox~~
- II. A state loses a seat as the result of an increase in the house size.
- III. The number assigned to each represented unit must be the standard quota, q_i , rounded up or rounded down.
- IV. A reapportionment causes a state to lose a seat to another state even though the percent increase in the population of the state that loses the seat is larger than the percent increase of the state that wins the seat.

2. Round the following numbers using the geometric mean.

$q^* = \sqrt{0.1} = 0$ $q^* = \sqrt{3.4} \approx 3.4641$
 0.39 rounds to 1 3.47 rounds to 4

3. Use the Webster method of apportionment to distribute 12 seats on a city council to three districts with the populations shown below.

$S = \frac{75,000}{12} = 6250$

District	Population	quota	[2]	$d = \frac{P}{N - \frac{1}{2}}$	smallest loses seat
North	30,000	$\frac{30000}{6250} = 4.8$	5	$\frac{30000}{5 - \frac{1}{2}} = 6667$	} (North, Central, South)
Central	10,100	$\frac{10100}{6250} = 1.616$	2	$\frac{10100}{2 - \frac{1}{2}} = 6733$	
South	34,900	$\frac{34900}{6250} = 5.584$	6	$\frac{34900}{6 - \frac{1}{2}} = 6345$	
		75,000	13		12

(seat too many)

4. A school is offering Gothic literature, Hispanic literature and Women writers. There are enough teachers to teach 10 sections of literature. We know that 58 people signed up for G, 20 for H and 42 for W.

(a) Use Hamilton's method to determine the number of sections of each class that will be taught.

(b) Use Jefferson's method to determine the number of sections of each class that will be taught. $S = \frac{120}{10} = 12$

Hamilton

	q	Lq	N
G	58 $\frac{58}{12} = 4.83$	4 + 1	5
H	20 $\frac{20}{12} = 1.67$	1 + 1	2
W	42 $\frac{42}{12} = 3.5$	3	3
120		8	10

Need 2 more

Jefferson

	q	Lq	$d = \frac{P}{N+1}$	N	$d = \frac{P}{N+1}$	N
G	58 $\frac{58}{12} = 4.83$	4	$\frac{58}{4+1} = 11.6$	5	$\frac{58}{5+1} = 9.67$	5
H	20 $\frac{20}{12} = 1.67$	1	$\frac{20}{1+1} = 10$	1	$\frac{20}{1+1} = 10$	1
W	42 $\frac{42}{12} = 3.5$	3	$\frac{42}{3+1} = 10.5$	3	$\frac{42}{3+1} = 10.5$	4
120		8		9		10

Need 2 more

Need 1 more

Critical divisor = 10.5

largest gets seat

largest gets seat