The rank gradient and the Lamplighter Group

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Abstract

We introduce the notion of the gradient rank function of a descending chain of subgroups of finite index and show that the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ has uncountably many 2-chains (that is chains in which each next group has index 2 in previous one) with pairwise different gradient rank functions. Meanwhile we get some information on subgroups of finite index in the lamplighter group.

1 Introduction

The Lamplighter group, by which we mean the wreath product of the group of order 2 with the infinite cyclic group, denoted $\mathcal{L}$ (thus $\mathcal{L} = \mathbb{Z}_2 \wr \mathbb{Z}$) is a popular object in group theory and its applications. Just one illustration of this is Chapter 6 in [1] and some sections in [7]. It is a 2-step solvable group (i.e. metableian group), of exponential growth, infinitely presented and scale invariant [2, 3], which is the cornerstone in all known results about the range of L^2-Betti numbers of groups on compact manifolds. In particular, the Atiyah's problem about the existence of closed manifold with non-integer and even irrational L^2-Betti numbers was completely solved on the base of considerations related to $\mathcal{L}$, [2, 9, 10].

In [4], Lackenby introduced an interesting group-theoretical notion, the rank gradient, which happens to be useful in topology, the theory of countable equivalence relations, the study of amenable groups and other areas. Given a group G and a descending sequence $\{H_n\}_{n=1}^{\infty}$ of subgroups of finite index one can define

$$RG(G, \{H_n\}) = \lim_{n\to\infty} \frac{d(H_n) - 1}{[G : H_n]}$$

to be the rank gradient of the sequence $\{H_n\}$ with respect to G where $d(H)$ denotes the minimal number of generators of a group H.

Amenable groups were introduced by J. von Neumann in 1929 and play an important role in many areas of mathematics [3]. There are a number of results due to Lackenby, M. Abért, A. Jaikin-Zapirain and N. Nikolov showing that amenability of G or of certain normal subgroup of G usually implies vanishing of the rank gradient. For instance, finitely generated infinite amenable groups have $RG = 0$ with respect to any normal chain with trivial intersection (see Theorem 5 in [5]). It is reasonable to study the rank gradient for sequences $\{H_n\}$ with trivial core (i.e. no nontrivial normal subgroups in the intersection $\cap_n H_n$). Indeed $RG(G, \{H_n\}) = RG(G/N, \{H_n/N\})$ if $N \triangleleft G, N < \cap_n H_n$. The most attention is given to the case when $\cap_n H_n = \{1\}$. One of the remaining open questions is

Question 1.1. *[5] Let G be a finitely generated infinite amenable group. Is it true that $RG(G, \{H_n\}) = 0$ for any chain with trivial intersection?*

If $\cap_{n=1}^{\infty} H_n = H$ then H is a closed subgroup with respect to the profinite topology and $RG(G, \{H_n\})$ somehow is a characteristic of the pair (G, H) which in some situations may characterize the pair (G, H) up to isomorphism. We say two pairs $(G, H), (P, Q)$ are isomorphic is there is an isomorphism $\phi : G \to P$ such that $\phi(H) = Q$.

If $RG(G, \{H_n\}) = 0$ then one may be interested in the decay of the function of natural argument $n \in \mathbb{N}$ given by

$$rg(n) = rg_{(G,\{H_n\})}(n) = \frac{d(H_n) - 1}{[G : H_n]}$$

which we call the *rank gradient function*. We may omit the index $(G, \{H_n\})$ if the group and chain in consideration is understood. Again, the rate of decay of $rg(n)$ may be an invariant of the pair (G, H) and may characterize the way H lies in G as a subgroup. Note that the same subgroup can be obtained as the intersection of distinct chains: one can delete certain elements in H_n thereby allowing $rg(n)$ to decay as fast as one would like (and this is not the only way to get different chains with the same intersection). Thus, we restrict our definition to the case when for some prime p, we have $[H_{n+1} : H_n] = p$ and in this case we say the chain is a *p-chain*. Our main result shows that $rg(n)$ may be used to show that the lamplighter contains 2-chains with different decay of the rank gradient function.

Theorem 1.2. *The group $\mathcal{L}$ has uncountably many 2-chains with pairwise different rank gradient functions.*

This result is obtained by explicitly describing subgroups of index 2 in the "higher rank" lamplighter groups $\mathcal{L}_n = \mathbb{Z}_2^n \wr \mathbb{Z}$.

We also prove

Theorem 1.3. *For any 2-chain $\{H_n\}$ in $\mathcal{L}$ each member H_n is isomorphic to $\mathcal{L}_i = \mathbb{Z}_2^i \wr \mathbb{Z}$ for some $i \leq n$.*

2 Subgroups of index 2 in $\mathcal{L}_n$

Let $\mathcal{L}_n = \mathbb{Z}_2^n \wr \mathbb{Z} = \oplus_{\mathbb{Z}} \mathbb{Z}_2^n \rtimes \mathbb{Z}$ (by $\mathbb{Z}_2$ we mean the group of order 2, the generator of active group $\mathbb{Z}$ acts as a shift in the direct sum) and $\mathcal{A}_n = \oplus_{\mathbb{Z}} \mathbb{Z}_2^n$ be the base group of $\mathcal{L}_n$. Observe that $\mathcal{L}_n$ is generated by elements $a_i, i = 1, 2, \ldots, n$ and t where t is a generator of infinite multiplicative cyclic group which we nevertheless denote in the additive way $\mathbb{Z}$, and $a_i \in \mathcal{A}_n, i = 1, 2, \ldots, n$ are elements given by an $n \times \infty$ matrix with all entries zero except one located in $i - th$ row and column at place 0 (we assume that the columns are enumerated by the elements of $\mathbb{Z}$). So $\mathcal{L}_n = \langle a_1, \ldots, a_n, t \rangle$ and we will use similar notation for generation in the future. Observe that if we identify elements of the base group $\mathcal{A}_n$ with two side infinite sequences of columns of dimension n over $\mathbb{Z}_2$ then the conjugation by t acts on them as the shift τ in the set of sequences. We will later use this fact.

Theorem 2.1. *Let $H < \mathcal{L}_n$ be a subgroup of index 2. Then either $H \simeq \mathcal{L}_n$ or $H \simeq \mathcal{L}_{2n}$. There are $2^{n+1} - 2$ subgroups of the first type and 1 subgroup of the second type.*

In the proof, we use the following well known result.

Lemma 2.2. *Let $M = \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p \oplus \cdots$ be a finite or infinite direct sum of cyclic groups $\mathbb{Z}_p$ with p a prime. Then every subgroup $P < M$ is a direct summand: $M = P \oplus Q$ for some Q (see Chapter 10 in [6]).*

Note that we will often interpret $\mathbb{Z}_p^n$ as a vector space of dimension n over the prime field $\mathbb{F}_p \backsimeq \mathbb{Z}_p$. Before we present a proof of Theorem 2.1, we will need the following lemma.

Lemma 2.3. *Let $M = \mathbb{Z}_p^n$. Every subgroup $P < M$ of index p has a unique "orthogonal" complement $Q < M$ such that $M = P \oplus Q$.*

The group Q is generated by the element $\bar{a} = (a_1, \ldots, a_n)$ which is determined by P. Then P consists of elements $\bar{x} = (x_1, \ldots, x_n)$ whose coordinates satisfy the "orthogonality" condition

$$a_1x_1 + \cdots a_nx_n \equiv 0 \pmod{p}.$$

Proof. Let $[M : P] = p$. Consider subgroup P as a vector subspace of the vector space $M = \mathbb{Z}_p^n$. Choose a basis $\bar{b}_1, \ldots, \bar{b}_{n-1}$ of P:

$$\begin{aligned} \bar{b}_1 &= (b_{1,1}, \ldots, b_{1,n}) \\ &\vdots \\ \bar{b}_{n-1} &= (b_{n-1,1}, \ldots, b_{n-1,n}), \end{aligned}$$

$b_{i,j} \in \mathbb{Z}_p$. Now define the $(n-1) \times n$ matrix $B = (b_{ij})$ which has rank $n-1$ and consider the system of equations over $\mathbb{Z}_p$.

$$\begin{aligned} b_{1,1}x_1 + \cdots + b_{1,n}x_n &= 0 \\ &\vdots \\ b_{n-1,1}x_1 + \cdots + b_{n-1,n}x_n &= 0. \end{aligned}$$

Note that this system has a nontrivial solution $\bar{a} = (a_1, \ldots, a_n)$ and every other solution is some constant multiple of $\bar{a}$. It is then easy to see that $M = P \oplus \langle \bar{a} \rangle$. It is also clear that given some $\bar{a} \in M, \bar{a} = (a_1, \ldots, a_n)$ with $a \neq 0$, the set of solutions of $a_1x_1 + \cdots a_nx_n \equiv 0 \pmod{p}$ yields a subgroup P of index p in M. □

Observe that using the linear algebra tools the notion of orthogonal complement can be defined in a similar way as we did for a subgroup of index p for arbitrary subgroup in elementary p-group of finite rank. We will use the notation $H^\perp$ to denote the orthogonal complement of a subgroup $H < M$ in M.

Corollary 2.4. *There is a bijection between subgroups of index p in $M = \mathbb{Z}_p^n$ and subgroups of order p given by:*

$$H \to H^\perp.$$

We now restrict our attention to the case when $p = 2$. The following is a proof of Theorem 2.1.

Proof. First, observe that the abelianization $A := (\mathcal{L}_n)_{ab}$ is isomorphic to $\mathbb{Z}_2^n \times \mathbb{Z}$. Define $A^2 < A$ to be the subgroup generated by the squares of elements in A. Then, $A/A^2 \backsimeq \mathbb{Z}_2^{n+1} = \langle \bar{a}_1, \ldots, \bar{a}_n, \bar{t} \rangle$ where as before $\mathbb{Z} = \langle t \rangle$ is the notation for the multiplicative infinite cyclic group generated by t, and $\bar{a}_i$ or $\bar{t}$ denotes the corresponding to a_i or t element of the quotient group $\mathcal{L}_n/[\mathcal{L}_n, \mathcal{L}_n]\mathcal{L}_n^2 \simeq \mathbb{Z}_2^{n+1}$. If we consider a_i as an $n \times \infty$ matrix, then it is of the form

$$\begin{pmatrix} \ldots & 0 & 0 & 0 & \ldots \\ & \vdots & \vdots & \vdots & \\ \ldots & 0 & 1 & 0 & \ldots \\ & \vdots & \vdots & \vdots & \\ \ldots & 0 & 0 & 0 & \ldots \end{pmatrix},$$

where the 1 is in the i-th row and the 0-th column. Recall that each a_i is the i-th generator of $\mathcal{A}_n^0$ where we define

$$\mathcal{A}_n = \bigoplus_{\mathbb{Z}} \mathbb{Z}_2^n = \bigoplus_{j\in\mathbb{Z}} \mathcal{A}_n^j.$$

The number of subgroups of index 2 in $\mathcal{L}_n$ is equal to the number of subgroups of index 2 in $\mathbb{Z}_2^{n+1}$, which is equal to the number of epimorphisms $\mathcal{L}_n \to \mathbb{Z}_2$, which is equal to $2^{n+1}-1$ (as a kernel of any such epimorphism is an orthogonal complement to a subgroup of order 2 generated by a nonidentity element). We have a short exact sequence

$$1 \to \mathcal{A}_n \to \mathcal{L}_n \xrightarrow{\phi} \langle t\rangle \to 1,$$

where ϕ is the natural projection onto $\mathbb{Z} = \langle t\rangle$. Let $H < \mathcal{L}_n$ be of index 2. H is normal in $\mathcal{L}_n$ and therefore shift invariant.

There are two cases: either $\phi[H] = \langle t^2\rangle$ or $\phi[H] = \langle t\rangle$.

Case 1. Assume $\phi[H] = \langle t^2\rangle$. In this case $H \cap \mathcal{A}_n = \mathcal{A}_n$, otherwise $[\mathcal{L}_n : H] \geq 4$, and there is only one subgroup H of index 2 in $\mathcal{L}_n$ with this property. Furthermore, $t^2 \in H$ and $H = \mathcal{A}_n \rtimes \langle t^2\rangle$.

Let $D_0 < \mathcal{A}_n, D_0 \simeq \mathbb{Z}_2^{2n}$ be a subgroup of $n\times\infty$ matrices where the only nonzero entries belong to columns placed at 0 and 1. Define $D_j = t^{-2j}D_0t^{2j}$. Then notice $D_i \cap D_j = 0$ for $i \neq j$ and $\mathcal{A}_n = \oplus_{j\in\mathbb{Z}} D_j$. The element t^2 acts by conjugation on $\oplus_{j\in\mathbb{Z}} D_j$ as the one-step shift τ. This implies $H \backsimeq \mathcal{L}_{2n}$.

Case 2. Now we assume $\phi[H] = \langle t\rangle$. We have $2^{n+1}-2$ such subgroups H. In this case, $H\cap\mathcal{A}_n = P$ is a shift invariant subgroup of index 2 in $\mathcal{A}_n$. Because P is shift invariant, there must be some $x \in \mathcal{A}_n$ whose matrix representation has only one nonzero column, namely the column with place 0, such that $x \notin P$. Let $q \in \mathbb{Z}_2^n$ be the vector with coordinates the same as x. That is, we consider x as an $n\times 1$ vector and relabel it q for clarity. Then let Q^0 be the orthogonal complement to $\langle q\rangle$:

$$\mathcal{A}_n^0 = \langle q\rangle \oplus Q^0$$

where as before we have $\mathcal{A}_n = \oplus_{i\in\mathbb{Z}}\mathcal{A}_n^i$. Note that we are considering Q^0 and $\langle q\rangle$ as subgroups of $\mathcal{A}_n^0$ and so Q^0 is a subgroup of H since otherwise we would have $[\mathcal{L} : H] \geq 4$. Define $Q = \oplus_{i\in\mathbb{Z}} Q^i$ where $Q^i = t^{-i}Q^0t^i$.

Let $\mathcal{R} = \mathbb{Z}_2[t, t^{-1}]$ be the ring of Laurent polynomials in $\mathbb{Z}_2$. It is isomorphic to the group ring $\mathbb{Z}_2[\mathbb{Z}]$ where as before $\mathbb{Z}$ is and additive notation for the multiplicative infinite cyclic group generated by t. The group $\mathcal{A}_n$ can be converted into an $\mathcal{R}$-module M_n by the agreement that the generator t of cyclic group acts on $\mathcal{A}_n$ as the right shift τ (remember that elements of $\mathcal{A}_n$ are viewed to be two sided infinite sequences of columns representing the elements of $\mathbb{Z}_2^n$). Moreover $\mathcal{A}_n$ is the additive group of this module, M_n is a free $\mathcal{R}$-module of rank n and is isomorphic to $\mathcal{R}^n$.

Observe that Q is a shift invariant subgroup of H. Because of lemma 2.2 there is a subgroup $S < P$ such that the decomposition $P = Q \oplus S$ holds. S also is a shift invariant subgroup of P and therefore can be interpreted as $\mathcal{R}$-module. Therefore P, Q and S can be considered as submodules of M_n and the decomposition of modules $P = Q \oplus S$ holds (we will not change the notation for P, Q, S when converting them in modules or vise versa, each time it will be clear if we consider these objects as abelian groups or as $\mathcal{R}$-modules).

We will need the following lemma. Any graduate level textbook in Algebra will contain the fact that a ring of polynomials with coefficients in some field is a principal ideal domain. The ring $\mathcal{R}$ is the localization of the polynomial ring in the multiplicative set consisting of the non-negative powers of t [8]. Many properties of the Laurent polynomial ring follow from the general properties of localization as well as the next one which is well known fact, but being unable to find a reference we add a proof of it.

Lemma 2.5. *The ring $\mathcal{R}$ is a principal ideal domain.*

Proof. Let I be an ideal in $\mathcal{R}$. Then $I \cap \mathbb{Z}_2[t]$ is an ideal in $\mathbb{Z}_2[t]$, and since the ring of polynomials over the field is a principal ideal domain, $I \cap \mathbb{Z}_2[t] = (f)$ for some $f \in \mathbb{Z}_2[t]$. Then $\mathcal{R}f \subset I$. Each $h \in I$ has a form $h = t^{-k}g$ for some $k \in \mathbb{N}$ and $g \in \mathbb{Z}_2[t]$. Thus $g \in I \cap \mathbb{Z}_2[t] = (f)$, and so $h = t^{-k}fa \in \mathcal{R}f$ for some a in $\mathcal{R}$. Therefore $\mathcal{R}f = I$. □

Being a submodules of a free finitely generated module $M_n \simeq \mathcal{R}^n$ over a principal ideal domain $\mathcal{R}$ the modules P, Q and S are also free. As P is a subgroup of index 2 in $\mathcal{A}_n$, the module P is free of rank n, Q is free of rank $n-1$ and S is free of rank 1. Thus the $\mathcal{R}^n$-module P being converted into a group generated by the additive group P and the element t acting by conjugation on P as shift τ becomes isomorphic to $\mathcal{R}^n \rtimes \mathbb{Z} \simeq \mathcal{L}_n$.

We have $2^{n+1}-2$ subgroups H which can be obtained in the second case. Indeed, there are 2^n-1 choices for vector q and therefore the subgroup Q. And to each choice of Q we have two choices to construct H: either to assume that $t \in H$ or that t is not in H. Thus we get $2(2^n-1) = 2^{n+1}-2$ subgroups in the case (2). This finishes the proof of the first theorem.

□

The theorem 1.3 is a corollary of theorem 2.1.

Since $\mathbb{Z}_2[x, x^{-1}]$ is a principal ideal domain by Lemma 2.5, a shift invariant subgroup T of $\mathcal{A}_1 = \oplus_{\mathbb{Z}}\mathbb{Z}_2$ correspond to a principal ideal $\mathcal{J}$ such that

$$\mathbb{Z}_2[x, x^{-1}]/\mathcal{J} \backsimeq \mathbb{Z}_2$$

which is a field. This implies that $\mathcal{J}$ is a maximal ideal generated by some irreducible polynomial of degree 1. Thus, $\mathcal{J} = \langle f \rangle$ with $\deg(f) = 1$ so $f = x + 1$. The corresponding element of T is then $\xi = (\ldots, 0, 1, 1, 0, \ldots)$ where the 1's are in the 0 and 1 places respectively. Additionally, ξ is a generator of T as an $\mathcal{R}$- module. One then concludes that T consists of sequences

$$(\ldots, a_{-1}, a_0, a_1, \ldots)$$

where

$$\sum_n a_n \equiv 0 \pmod 2. \qquad (1)$$

This observation gives an effective way to construct a subgroup H of index 2 in $\mathcal{L}_n$ with $H \simeq \mathcal{L}_n$. Choose a basis E in $\mathbb{Z}_2^n$ and write elements of $\mathcal{A}_n$ as $n \times \mathbb{Z}$ matrices with respect to this basis at each place $i \in \mathbb{Z}$. Then take a subgroup of $\mathcal{A}_n$ consisting of elements which satisfy the relation (1) in the first row. And after make a choice: $t \in H$ or not.

3 Construction of chains

We know that $\mathcal{L}$ contains 3 subgroups of index 2, 2 of which are isomorphic to $\mathcal{L}$ and the other is isomorphic to $\mathcal{L}_2$. Furthermore, $\mathcal{L}_2$ has 7 subgroups of index 2, 1 of which is isomorphic to $\mathcal{L}_4$ and 6 are isomorphic to $\mathcal{L}_2$, etc. If we take a subgroup $H < \mathcal{L}$ of index 2^k obtained from $\mathcal{L}$ by taking a descending chain of subgroups of index 2 in the previous member of the chain then we have $H \simeq \mathcal{L}_{2^i}$ for some $i \leq k$. We can then take a subgroup of index 2 isomorphic to $\mathcal{L}_{2^i}$ (call this choice type (0)) or to $\mathcal{L}_{2^{i+1}}$ (call this choice type (1)). It is clear that $d(\mathcal{L}_n) = n+1$ (obviously $\mathcal{L}_n$ is generated by $n+1$ elements and the abealization of $\mathcal{L}_n$ is $\mathbb{Z}_2^n \times \mathbb{Z}$ and is $n+1$ generated). Now let $\omega \in \{0,1\}^{\mathbb{N}}$ be a sequence. Then these two types of choices for subgroups of index 2 allow us to construct a chain $\{H_n^\omega\}$ such that the term H_n^ω is obtained from the previous term H_{n-1}^ω by looking at the n^{th} term ω_n in the sequence ω. That is, a 0 dictates to make a choice of type (0) and a 1 dictates to make a choice of type (1). It is clear that in such a way we obtain uncountably many different chains $\{H_n^\omega\}$ such that each of the functions $rg^\omega(n)$ are distinct. This provides the proof of the Theorem 1.2.

Remark: if $r^\omega = \lim_{n\to\infty} rg^\omega(n) > 0$ then $r^\omega = \frac{1}{2^k}$ for some k, the rank gradient of the chain $\{H_n^\omega\}$ is positive and the number of 0's in the sequence ω is k. In this case, $H^\omega = \cap_n H_n^\omega$ contains a nontrivial normal subgroup. In all other cases the rank gradient of the 2-chain is 0.

4 Conclusion

It is clear that the same method used to construct uncountably many rank gradient functions of 2-chains in $\mathcal{L}$ allows to construct uncountably many 2-chains with pairwise different types of decay of the rank gradient function. For instance, for this purpose one can consider a family of functions $\delta_\alpha(n) = 2^{-n^\alpha}$ with $0 < \alpha < 1$ and to each such function correspond a sequence ω with the property that the rank gradient function $rg^\omega(n)$ is the "best approximation" of the function 2^{-n^α}. By the " best approximation" here, for instance, we mean the following. Starting with any subgroup $H_1 \simeq \mathcal{L}_2$ of index 2 in $\mathcal{L}$ (which corresponds to the value $\omega_1 = 1$ of the first member of the sequence ω, and the value $rg^\omega(1) = 1 > \frac{1}{2} = \delta_\alpha(1)$) one can apply operation (0) till the first time when the value of the rank gradient function will become less than the value of the function $\delta_\alpha(n)$ (for corresponding value of argument n). Then apply operation (1) till the rank gradient function will become greater or equal to the value of $\delta_\alpha(n)$ for corresponding value of n. Then again apply the operation (0) etc. Continue in such way we will construct a 2-chain which approximates $\delta_\alpha(n)$ in what we have called the best possible way. As the rates of decay of functions $\delta_\alpha(n)$ for different values of α clearly are different the same hold for the corresponding gradient rank functions.

Our study is the first step in understanding which decays of the rank gradient function may arise in the case of finitely generated residually finite amenable groups.

If $\{H_n\}_{n=1}^\infty$ is a descending chain of subgroups of finite index in residually finite group G then the intersection $H_* = \bigcap_{n=1}^\infty H_n$ is a subgroup of G closed in profinite topology (that is topology generated by subgroups of finite index), and any closed subgroup can be obtained in that way. The rank gradient function of the chain $\{H_n\}_{n=1}^\infty$ introduced by us may serve as certain characteristic of a subgroup H_*. Right now it is unclear how $rg(n)$ depends on the chain $\{H_n\}_{n=1}^\infty$ with the fixed intersection H_*. Even in the case $H_* = \{1\}$ it may happen that different p-chains (p prime) with trivial intersection have different decay of $rg(n)$ but we do not have examples of this sort. Also it is reasonable to consider chains with the property that H_{n+1} is a maximal subgroup in $\{H_n\}$.

We have considered the case of the lamplighter group. It would be interesting to study the decay of gradient rank function in other amenable groups, for instance in 3-generated infinite torsion 2-group of intermediate growth constructed by the first author in [11, 12].

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