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SELF-SIMILARITY AND BRANCHING IN GROUP THEORY

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Introduction

The idea of self-similarity is one of the most basic and fruitful ideas in mathematics of all times and populations. In the last few decades it established itself as the central notion in areas such as fractal geometry, dynamical systems, and statistical physics. Recently, self-similarity started playing a role in algebra as well, first of all in group theory.

Regular rooted trees are well known self-similar objects (the subtree of the regular rooted tree hanging below any vertex looks exactly like the whole tree). The self-similarity of the tree induces the self-similarity of its group of automorphisms and this is the context in which we talk about self-similar groups. Of particular interest are the finitely generated examples, which can be constructed by using finite automata. Groups of this type are extremely interesting and usually difficult to study as there are no general means to handle all situations. The difficulty of study is more than fairly compensated by the beauty of these examples and the wealth of areas and problems where they can be applied.

Branching is another idea that plays a major role in many areas, first of all in Probability Theory, where the study of branching processes are one of the main directions.

The idea of branching entered Algebra via the so called branch groups that were introduced by the first author at St. Andrews Group Theory Conference in Bath 1997.

Branch groups are groups that have actions "of branch type" on spherically homogeneous rooted trees. The phrase "of branch type" means that the dynamics of the action (related to the subnormal subgroup structure) mimics the structure of the tree. Spherically homogenous trees appear naturally in this context, both because they are the universal models for homogeneous ultra-metric spaces and

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because a group is residually finite if and only if it has a faithful action on a spherically homogeneous tree.

The importance of the choice of the "branch type" action is reflected in the fact that it is the naturally opposite to the so called diagonal type. While any residually finite group can act faithfully on a rooted homogeneous tree in diagonal way, the actions of branch type are more restrictive and come with some structural implications.

Actions of branch type give rise to many examples of just-infinite groups (thus answering a question implicitly raised in [**21**] on existence of exotic examples of just infinite groups) and to a number of examples of "small groups" (or atomic groups) in the sense of S. Pride [**78**].

The ideas of self-similarity and branching interact extremely well in group theory. There is a large intersection between these two classes of groups and this article demonstrates some important features and examples of this interaction.

This survey article is based on the course of four talks that were given by the first author at St. Andrews Group Theory Conference 2005 (although we do indicate here some new examples and links). We hope that it will serve as an accessible and quick introduction into the subject.

The article is organized as follows.

After a quick overview of several self-similar objects and basics notions related to actions on rooted trees in Section 1 and Section 2, we define the notion of a self-similar group in Section 3 and explain how such groups are related to finite automata. Among the examples we consider are the Basilica group, the 3-generated 2-group of intermediate growth known as "the first group of intermediate growth" and the Hanoi Towers groups H^k, which model the popular (in life and in mathematics) Hanoi Towers Problem on k pegs, $k \geq 3$.

Section 4 contains a quick introduction to the theory of iterated monodromy groups developed by Nekrashevych [**70**, **71**]. This theory is a wonderful example of application of group theory in dynamical systems and, in particular, in holomorphic dynamics. We mention here that the well known Hubbard Twisted Rabbit Problem in holomorphic dynamics was recently solved by Bartholdi and Nekrashevych [**12**] by using self-similar groups arising as iterated monodromy groups.

Section 5 deals with branch groups. We give two versions of the definition (algebraic and geometric) and mention some of basic properties of this groups. We show that the Hanoi Group $\mathcal{H}^{(3)}$ is branch and hence the other Hanoi groups are at least weakly branch.

Section 6 and Section 7 deal with important asymptotic characteristics of groups such as growth and amenability. Basically, all currently known results on groups of intermediate growth and on amenable but not elementary amenable groups are based on self-similar and/or branch groups. Among various topics related to amenability we discus (following the article [**19**]) the question on the range of Tarski numbers and amenability of groups generated by bounded automata and their generalizations, introduced by Said Sidki.

In the last sections we give an account of the use of Schreier graphs in the circle of questions described above, related to self-similarity, amenability and geometry of Julia sets and other fractal type sets, substitutional systems and the spectral problem. We finish with an example of a computation of the spectrum in a problem related to Sierpiński gasket.

Some of the sections end with a short list of open problems.

The subject of self-similarity and branching in group theory is quite young and the number of different directions, open questions, and applications is growing rather quickly. We hope that this article will serve as an invitation to this beautiful, exciting, and extremely promising subject.

1. *Self-similar objects*

It is not our intention in this section to be very precise and define the notion of self-similarity in any generality. Rather, we provide some examples of objects that, for one reason or another, may be considered self-similar and that will be relevant in the later sections.

The unit interval $I = [0, 1]$ is one of the simplest mathematical objects that may be considered self-similar. Indeed, I is equal to the union of two intervals, namely $[0, 1/2]$ and $[1/2, 1]$, both of which are similar to I. The similarities between I and the intervals $I_0 = [0, 1/2]$ and $I_1 = [1/2, 1]$ are the affine maps ϕ_0 and ϕ_1 given by

$$\phi_0(x) = \frac{x}{2}, \qquad \phi_1(x) = \frac{x}{2} + \frac{1}{2}.$$

The free monoid $X^* = X_k^*$ over the alphabet $X = X_k = \{0, \dots, k-1\}$ is another example of an object that may be considered self-similar (we usually omit the index in X_k). Namely

$$X^* = \{\emptyset\} \cup 0X^* \cup \dots \cup (k-1)X^*, \tag{1.1}$$

where $\emptyset$ denotes the empty word and, for each letter x in X, the map ϕ_x defined by

$$\phi_x(w) = xw,$$

for all words w over X, is a "similarity" from the set X^* of all words over X to the set xX^* of all words over X that start in x.

Both examples so far may be considered a little bit imperfect. Namely, the intersection of I_0 and I_1 is a singleton (so the decomposition $I = I_0 \cup I_1$ is not disjoint), while the union decomposing X^* involves an extra singleton (that is not similar to X^* in any reasonable sense). However, in both cases the apparent imperfection can be easily removed.

For example, denote by J the set of points on the unit interval that is the complement of the set of diadic rational points in I. Thus $J = I \setminus D$ where

$$D = \left\{ \frac{p}{q} \mid 0 \le p \le q,\ p \in \mathbb{Z},\ q = 2^m, \text{ for some non-negative integer } m \right\}$$

Then $\phi_0(J) = J_0$ and $\phi_1(J) = J_1$ are sets similar to J, whose disjoint union is J.

Before we discuss how to remove the apparent imperfection from the decomposition (1.1) we define more precisely the structure that is preserved under the similarities ϕ_x, $x \in X$.

The free monoid X^* has the structure of a rooted k-ary tree $\mathcal{T} = \mathcal{T}^{(k)}$. The empty word $\emptyset$ is the root, the set X^n of words of length n over X is the level n, denoted $\mathcal{L}_n$, and every vertex u in $\mathcal{T}$ has k children, namely ux, $x \in X$. Figure 1 depicts the ternary rooted tree. Let $u = x_1 \dots x_n$ be a word over X. The set uX^* of words over X that start in u is a subtree of $\mathcal{T}$, denoted $\mathcal{T}_u$, which is canonically isomorphic to the whole tree through the isomorphism ϕ_u defined as the composition $\phi_u = \phi_{x_1} \dots \phi_{x_n}$ (see Figure 2). In particular, each ϕ_x, $x \in X$, is a canonical tree

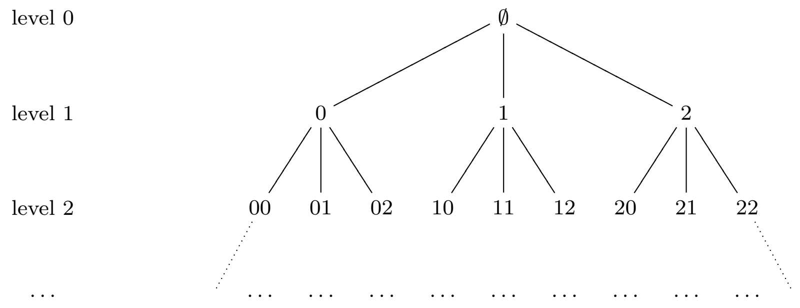


FIGURE 1. *The ternary rooted tree*

isomorphism between the tree $\mathcal{T}$ and the tree $\mathcal{T}_x$ hanging below the vertex x on the first level of $\mathcal{T}$.

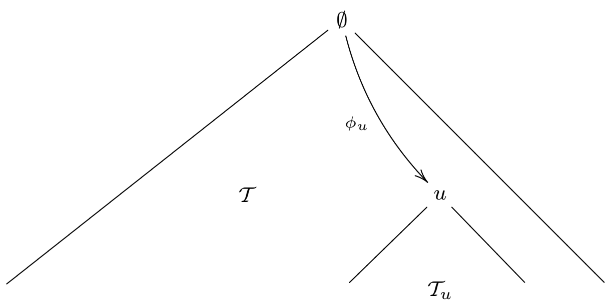


FIGURE 2. *Canonical isomorphism between $\mathcal{T}$ and $\mathcal{T}_u$*

The boundary of the tree $\mathcal{T}$, denoted $\partial\mathcal{T}$, is an ultrametric space whose points are the infinite geodesic rays in $\mathcal{T}$ starting at the root. In more detail, each infinite geodesic ray in $\mathcal{T}$ starting at the root is represented by an infinite (to the right) word ξ over X, which is the limit $\xi = \lim_{n\to\infty} \xi_n$ of the sequence of words $\{\xi_n\}_{n=0}^{\infty}$ such that ξ_n is the word of length n representing the unique vertex on level n on the ray. Denote the set of all infinite words over X by X^ω and, for u in X^*, the set of infinite words starting in u by uX^ω. The set X^ω of infinite words over X decomposes as disjoint union as

$$X^\omega = 0X^\omega \cup \cdots \cup (k-1)X^\omega.$$

Defining the distance between rays ξ and ζ by

$$d(\xi,\zeta) = \frac{1}{k^{|\xi\wedge\zeta|}}$$

where $|\xi \wedge \zeta|$ is the length of the longest common prefix $\xi \wedge \zeta$ of the rays ξ and ζ turns X^ω into a metric space, denoted $\partial\mathcal{T}$. Moreover, for each x in X, the map ϕ_x given by

$$\phi_x(w) = xw,$$

for all infinite words w in X^ω, is a contraction (by a factor of k) from $\partial\mathcal{T}$ to the subspace $\partial\mathcal{T}_x$ consisting of those rays in $\partial\mathcal{T}$ that pass through the vertex x. The space $\partial\mathcal{T}$ is homeomorphic to the Cantor set. Its topology is just the Tychonov product topology on $X^{\mathbb{N}}$, where X has the discrete topology. A measure on $\partial\mathcal{T}$ is defined as the Bernoulli product measure on $X^{\mathbb{N}}$, where X has the uniform measure.

The Cantor middle thirds set C is a well known self-similar set obtained from I by removing all points whose base 3 representation necessarily includes the digit 1. In other words, the open middle third interval is removed from I, then the open middle thirds are removed from the two obtained intervals, etc (the first three steps in this procedure are illustrated in Figure 3) In this case C is the disjoint union of

Figure 3. *The first three steps in the construction of the middle thirds Cantor set*

the two similar subsets $C_0 = \phi_0(C)$ and $C_1 = \phi_1(C)$ where the similarities ϕ_0 and ϕ_1 are given by

$$\phi_0(x) = \frac{x}{3}, \qquad \phi_1(x) = \frac{x}{3} + \frac{2}{3}.$$

A homeomorphism between the boundary of the binary tree $\partial\mathcal{T}^{(2)}$ and the Cantor middle thirds set is given by

$$x_1x_2x_3\cdots \leftrightarrow 0.(2x_1)(2x_2)(2x_3)\dots\ .$$

The number of the right is the ternary representation of a point in C (consisting solely of digits 0 and 2).

Another well known self-similar set is the Sierpiński gasket. It is the set of points in the plane obtained from the set of points bounded by an equilateral triangle by successive removal of the middle triangles. A set of points homeomorphic to the Sierpiński gasket is given in Figure 15.

The set in Figure 15 is the Julia set of a rational post-critically finite map on the Riemann Sphere. Julia sets of such maps provide an unending supply of self-similar subsets of the complex plane. For example, the Julia set of the quadratic map $z \mapsto z^2 - 1$ is given in Figure 4.

In fact, some of the other examples we already mentioned are (up to homeomorphism) also Julia sets of quadratic maps. Namely, the interval $[-2, 2]$ is the Julia set of the quadratic map $z \mapsto z^2 - 2$, while the Julia set of the quadratic map $z \mapsto z^2 + c$, for $|c| > 2$, is a Cantor set.

2. *Actions on rooted trees*

The self-similarity decomposition

$$X^* = \{\emptyset\} \cup 0X^* \cup \dots (k-1)X^*$$

of the k-ary rooted tree $\mathcal{T} = \mathcal{T}^{(k)}$, defined over the alphabet $X = \{0, \dots, k-1\}$, induces the self-similarity of the group of automorphisms $\mathsf{Aut}(\mathcal{T})$ of the tree $\mathcal{T}$.

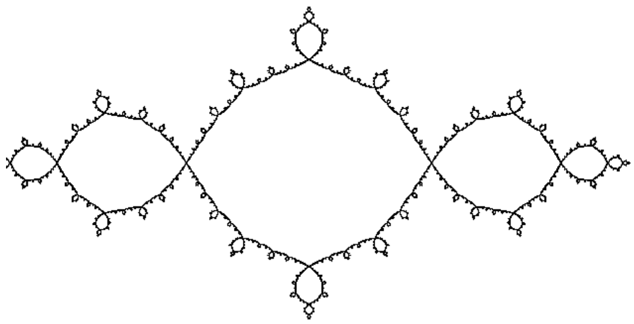


FIGURE 4. *Julia set of the map* $z \mapsto z^2 - 1$

Namely, each automorphism of $\mathcal{T}$ can be decomposed as

$$g = \pi_g \; (g_0, \dots, g_{k-1}), \tag{2.1}$$

where π_g is a permutation in $\mathsf{S}_k = \mathsf{Sym}(X_k)$, called the *root permutation* of g and, for x in X, g_x is an automorphism of $\mathcal{T}$, called *section* of g at x. The root permutation and the sections of g are uniquely determined by the relation

$$g(xw) = \pi_g(x) g_x(w),$$

for x a letter in X and w a word over X. The automorphisms g_x, x in X, represent the action of g on the subtrees $\mathcal{T}_x$ hanging below the vertices on level 1, which are then permuted according to the root permutation π_g (see Figure 5).

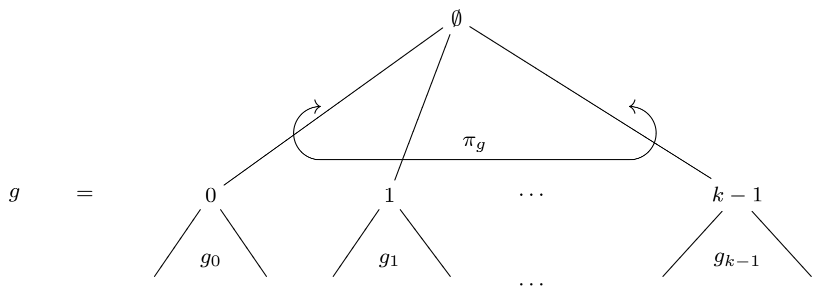


FIGURE 5. *Decomposition* $g = \pi_g \; (g_0, \dots, g_{k-1})$ *of an automorphism of* $\mathcal{T}$

Algebraically, $\mathsf{Aut}(\mathcal{T})$ decomposes as

$$\mathsf{Aut}(\mathcal{T}) = \mathsf{S}_k \ltimes (\mathsf{Aut}(\mathcal{T}) \times \cdots \times \mathsf{Aut}(\mathcal{T})) = \mathsf{S}_k \ltimes \mathsf{Aut}(\mathcal{T})^X = \mathsf{S}_k \wr_X \mathsf{Aut}(\mathcal{T}). \tag{2.2}$$

The product $\wr_X$ is the permutational wreath product defined by the permutation action of S_k on X, i.e., S_k acts on $\mathsf{Aut}(\mathcal{T})^X$ by permuting the coordinates (we usually omit the subscript in $\wr_X$). For $f, g \in \mathsf{Aut}(\mathcal{T})$, we have

$$\begin{aligned} gh &= \pi_g\ (g_0, \dots, g_{k-1})\ \pi_h\ (h_0, \dots, h_{k-1}) = \\ &= \pi_g \pi_h\ (g_0, \dots, g_{k-1})^{\pi_h} (h_0, \dots, h_{k-1}) = \\ &= \pi_g \pi_h\ (g_{\pi_h(0)} h_0, \dots, g_{\pi_h(k-1)} h_{k-1}). \end{aligned}$$

Thus

$$\pi_{gh} = \pi_g \pi_h \qquad \text{and} \qquad (gh)_x = g_{h(x)} h_x,$$

for x in X. The decomposition (2.2) can be iterated to get

$$\mathsf{Aut}(\mathcal{T}) = \mathsf{S}_k \wr \mathsf{Aut}(\mathcal{T}) = \mathsf{S}_k \wr (\mathsf{S}_k \wr \mathsf{Aut}(\mathcal{T})) = \dots = \mathsf{S}_k \wr (\mathsf{S}_k \wr (\mathsf{S}_k \wr \dots)).$$

Thus $\mathsf{Aut}(\mathcal{T})$ has the structure of iterated permutational wreath product of copies of the symmetric group S_k.

The sections of an automorphism g of $\mathcal{T}$ are also automorphisms of $\mathcal{T}$ (describing the action on the first level subtrees). The sections of these sections are also automorphisms of $\mathcal{T}$ (describing the action on the second level subtrees) and so on. Thus, we may recursively define the sections of g at the vertices of $\mathcal{T}$ by

$$g_{ux} = (g_u)_x,$$

for x a letter and u a word over X. By definition, the section of g at the root is g itself. Then we have, for any words u and v over X,

$$g(uv) = g(u) g_u(v).$$

Definition 1. Let G be a group acting by automorphisms on a k-ary rooted tree $\mathcal{T}$.

The *vertex stabilizer* of a vertex u in $\mathcal{T}$ is

$$\mathsf{St}_G(u) = \{\ g \in G \mid g(u) = u\ \}.$$

The *level stabilizer* of level n in $\mathcal{T}$ is

$$\mathsf{St}_G(\mathcal{L}_n) = \{\ g \in G \mid g(u) = u,\ \text{for all } u \in \mathcal{L}_n\ \} = \bigcap_{u \in \mathcal{L}_n} \mathsf{St}_G(u).$$

Proposition 2.1. *The n-th level stabilizer of a group G acting on a k-ary tree is a normal subgroup of G. The group $G/\mathsf{St}_G(\mathcal{L}_n)$ is isomorphic to a subgroup of the group*

$$\underbrace{\mathsf{S}_k \wr (\mathsf{S}_k \wr (\mathsf{S}_k \wr \dots \wr (\mathsf{S}_k \wr \mathsf{S}_k) \dots))}_{n \text{ copies}}$$

and the index $[G : \mathsf{St}_G(\mathcal{L}_n)]$ is finite and bounded above by $(k!)^{1+k+\dots+k^{n-1}}$.

Since the intersection of all level stabilizers in a group of tree automorphisms is trivial (an automorphism fixing all the levels fixes the whole tree) we have the following proposition.

Proposition 2.2. *Every group acting faithfully on a regular rooted tree is residually finite.*

The group $\mathsf{S}_k \wr (\mathsf{S}_k \wr (\mathsf{S}_k \wr \cdots \wr (\mathsf{S}_k \wr \mathsf{S}_k) \dots))$ that appears in Proposition 2.1 is the automorphism group, denoted $\mathsf{Aut}(\mathcal{T}_{[n]})$ of the finite rooted k-ary tree $\mathcal{T}_{[n]}$ consisting of levels 0 through n in $\mathcal{T}$.

The group $\mathsf{Aut}(\mathcal{T})$ is a pro-finite group. Indeed, it is the inverse limit of the sequence

$$1 \leftarrow \mathsf{Aut}(\mathcal{T}_{[1]}) \leftarrow \mathsf{Aut}(\mathcal{T}_{[2]}) \leftarrow \dots$$

of automorphism groups of the finite k-ary rooted trees, where the surjective homomorphism $\mathsf{Aut}(\mathcal{T}_{[n]}) \leftarrow \mathsf{Aut}(\mathcal{T}_{[n+1]})$ is given by restriction of the action of $\mathsf{Aut}(\mathcal{T}_{[n+1]})$ on $\mathcal{T}_{[n]}$.

Since tree automorphisms fix the levels of the tree the highest degree of transitivity that they can achieve is to act transitively on all levels.

Definition 2. A group acts *spherically transitively* on a rooted tree if it acts transitively on every level of the tree.

Let G act on $\mathcal{T}$ and u be a vertex in $\mathcal{T}$. Then the map

$$\varphi_u : \mathsf{St}_G(u) \to \mathsf{Aut}(\mathcal{T})$$

given by

$$\varphi_u(g) = g_u$$

is a homomorphism.

Definition 3. The homomorphism φ_u is called the *projection* of G at u. The image of φ_u is denoted by G_u and called the *upper companion group* of G at u.

The map

$$\psi_n : \mathsf{St}_G(\mathcal{L}_n) \to \prod_{u \in \mathcal{L}_n} \mathsf{Aut}(\mathcal{T})$$

given by

$$\psi_n(g) = (\varphi_u(g))_{u \in \mathcal{L}_n} = (g_u)_{u \in \mathcal{L}_n}$$

is a homomorphism. We usually omit the index in ψ_n when $n = 1$.

In the case of $G = \mathsf{Aut}(\mathcal{T})$ the maps $\varphi_u : \mathsf{St}_{\mathsf{Aut}(\mathcal{T})}(u) \to \mathsf{Aut}(\mathcal{T})$ and $\psi_n : \mathsf{St}_{\mathsf{Aut}(\mathcal{T})}(\mathcal{L}_n) \to \prod_{u \in \mathcal{L}_n} \mathsf{Aut}(\mathcal{T})$ are isomorphisms, for any word u over X and any $n \geq 0$.

The group $\mathsf{Aut}(\mathcal{T})$ is isomorphic to the group $\mathsf{Isom}(\partial\mathcal{T})$ of isometries of the boundary $\partial\mathcal{T}$. The isometry corresponding to an automorphism g of $\mathcal{T}$ is naturally defined by

$$g(\xi) = \lim_{n \to \infty} g(\xi_n),$$

where ξ_n is the prefix of length n of the infinite word ξ over X.

3. *Self-similar groups*

3.1. *General definition*

We observed that the self-similarity of the k-ary rooted tree $\mathcal{T}$ induces the self-similarity of its automorphism group. The decomposition of X^* in (1.1) is reflected in the decomposition of tree automorphisms in (2.1) and in the decomposition of $\mathsf{Aut}(\mathcal{T})$ in (2.2). The most obvious manifestation of the self-similarity of $\mathsf{Aut}(\mathcal{T})$ is that all sections of all tree automorphisms are again tree automorphisms. We use this property as the basic feature defining a self-similar group.

Definition 4. A group G of k-ary tree automorphisms is *self-similar* if all sections of all elements in G are elements in G.

Proposition 3.1. *A group G of k-ary tree automorphisms is self-similar if and only if, for every element g in G and every letter x in X, there exists a letter y in X and an element h in G such that, for all words w over X,*

$$g(xw) = yh(w).$$

Definition 5. A self-similar group G of k-ary tree automorphisms is *recurrent* if, for every vertex u in $\mathcal{T}$, the upper companion group of G at u is G, i.e.,

$$\varphi_u(\mathsf{St}_G(u)) = G.$$

Proposition 3.2. *A self-similar group G of k-ary tree automorphisms is recurrent if for every letter x in X the upper companion group of G at x is G, i.e., for every element g in G and every letter x, there exists an element h in the stabilizer of x whose section at u is g.*

Example 1. (Odometer) For $k \geq 2$, define a k-ary tree automorphism a by

$$a = \rho\ (1, \dots, 1, a),$$

where $\rho = (0\ 1\ \dots\ k-1)$ is the standard cycle that cyclically permutes (rotates) the symbols in X. The automorphism a is called the k-ary *odometer* because of the way in which it acts on the set of finite words over X. Namely if we interpret the word $w = x_1 \dots x_n$ over X as the number $\sum x_i k^i$ then

$$a(w) = w + 1, \text{ for } w \neq (k-1)\dots(k-1), \qquad a((k-1)\dots(k-1)) = 0\dots0.$$

Thus the automorphism a acts transitively on each level of $\mathcal{T}$, $\mathbb{Z} \cong \langle a \rangle$ and since

$$a^k = (a, a, \dots, a)$$

the group generated by the odometer is a recurrent group. Therefore $\mathbb{Z}$ has a self-similar, recurrent action on any k-ary tree.

Proposition 3.3. *A k-ary tree automorphism acts spherically transitively on $\mathcal{T}$ if and only if it is conjugate in $\mathsf{Aut}(\mathcal{T})$ to the k-ary odometer automorphism.*

3.2. *Automaton groups*

We present a simple way to construct finitely generated self-similar groups. Let $\pi_1, \dots, \pi_m$ be permutations in S_k and let $S = \{s^{(1)}, \dots, s^{(m)}\}$ be a set of m distinct symbols. Consider the system

$$\begin{cases} s^{(1)} = \pi_1(s_0^{(1)}, \dots, s_{k-1}^{(1)}) \\ \dots \\ s^{(m)} = \pi_m(s_0^{(m)}, \dots, s_{k-1}^{(m)}) \end{cases} \tag{3.1}$$

where each $s_x^{(i)}$, $i = 1, \dots, m$, x in X, is a symbol in S. Such a system defines a unique set of k-ary tree automorphisms, denoted by the symbols in S, whose first level decompositions are given by the equations in (3.1). Moreover, the group $G = \langle S \rangle$ is a self-similar group of tree automorphisms, since all the sections of all of its generators are in G.

The language of finite automata (in fact finite transducers, or sequential machines, or Mealy automata) is well suited to describe the groups of tree automorphisms that arise in this way.

Definition 6. A finite automaton is a quadruple $\mathcal{A} = (S, X, \tau, \pi)$ where S is a finite set, called set of *states*, X is a finite alphabet, and $\tau : S \times X \to S$ and $\pi : S \times X \to X$ are maps, called the *transition map* and the *output map* of $\mathcal{A}$.

The automaton $\mathcal{A}$ is *invertible* if, for all s in S, the restriction $\pi_s : X \to X$ given by $\pi_s(x) = \pi(s, x)$ is a permutation of X.

All the automata in the rest of the text are invertible and we will not emphasize this fact. Each state of an automaton $\mathcal{A}$ acts on words over X as follows. When the automaton is in state s and the current input letter is x, the automaton produces the output letter $y = \pi_s(x) = \pi(s, x)$ and changes its state to $s_x = \tau(s, x)$. The state s_x then handles the rest of the input (a schematic description is given in Figure 6)

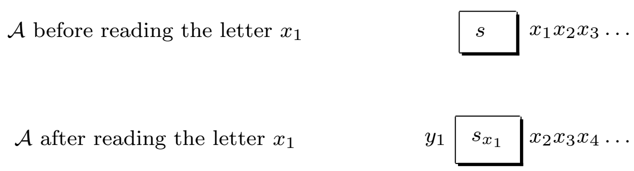


Figure 6. *An automaton $\mathcal{A}$ processing an input word $x_1x_2x_3\dots$ starting at state s*

Thus we have

$$s(xw) = \pi_s(x)s_x(w),$$

for x a letter and w a word over X, and we see that the state s acts on $\mathcal{T}$ as the automorphism with root permutation π_s and sections given by the states s_x, x in X.

Definition 7. Let $\mathcal{A}$ be a finite invertible automaton. The *automaton group* of $\mathcal{A}$, denoted $G(\mathcal{A})$, is the finitely generated self-similar group of tree automorphisms $G(\mathcal{A}) = \langle S \rangle$ generated by the states of $\mathcal{A}$.

Example 2. (Basilica group $\mathcal{B}$) Automata are often encoded by directed graphs such as the one in Figure 7. The vertices are the states, each state s is labeled by its corresponding root permutation π_s and, for each pair of a state s and a letter x, there is an edge from s to s_x labeled by x. We use () to label the states with trivial root permutations (sometimes we just leave such states unlabeled). The state 1 represents the identity automorphism of $\mathcal{T}$. Consider the corresponding automaton

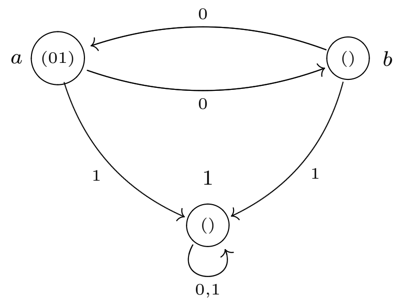


Figure 7. *The binary automaton generating the Basilica group*

group $\mathcal{B} = \langle a, b, 1\rangle = \langle a, b\rangle$. The decompositions of the binary tree automorphisms a and b are given by

$$\begin{aligned} a &= (01)\ (b,\ 1), \\ b &= \qquad\ \ (a,\ 1). \end{aligned}$$

Note that we omit writing trivial root permutations in the decomposition. The group $\mathcal{B}$ is recurrent. Indeed we have $\mathsf{St}_{\mathcal{B}}(\mathcal{L}_1) = \langle b, a^2, a^{-1}ba\rangle$ and

$$\begin{aligned} b &= (a,\ 1), \\ a^2 &= (b,\ b), \\ a^{-1}ba &= (1,\ a) \end{aligned}$$

which shows that $\varphi_0(\mathsf{St}_{\mathcal{B}}(\mathcal{L}_1)) = \varphi_1(\mathsf{St}_{\mathcal{B}}(\mathcal{L}_1)) = \langle a, b\rangle = \mathcal{B}$.

Example 3. (The group $\mathcal{G}$) Consider the automaton group $\mathcal{G} = \langle a, b, c, d\rangle$ generated by the automaton in Figure 8. The decompositions of the binary tree automorphisms a, b, c and d are given by

$$\begin{aligned} a &= (01)\ (1,\ 1), \\ b &= \qquad\ \ (a,\ c), \\ c &= \qquad\ \ (a,\ d), \\ d &= \qquad\ \ (1,\ b), \end{aligned}$$

The group $\mathcal{G}$ is recurrent. Indeed we have $\mathsf{St}_{\mathcal{G}}(\mathcal{L}_1) = \langle b, c, d, aba, aca, ada\rangle$ and

$$\begin{aligned} b &= (a,\ c), & aba &= (c,\ a), \\ c &= (a,\ d), & aca &= (d,\ a), \\ d &= (1,\ b), & ada &= (b,\ 1), \end{aligned}$$

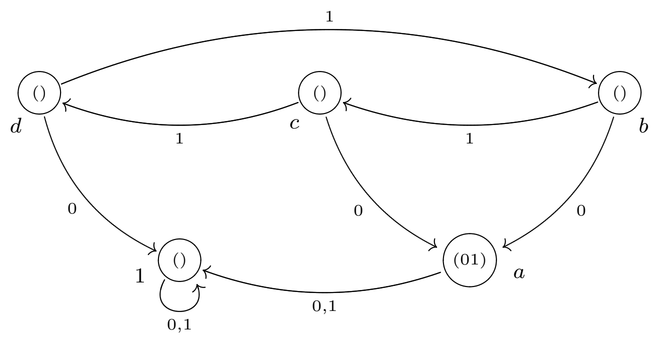


FIGURE 8. *The binary automaton generating the group $\mathcal{G}$*

which shows that $\varphi_0(\mathsf{St}_{\mathcal{G}}(\mathcal{L}_1)) = \varphi_1(\mathsf{St}_{\mathcal{G}}(\mathcal{L}_1)) = \langle a, b, c, d\rangle = \mathcal{G}$.

EXAMPLE 4. (Hanoi Towers groups) Let $k \geq 3$. For a permutation α in S_k define a k-ary tree automorphism a_α by

$$a_\alpha = \alpha\ (a_0, \dots, a_{k-1})$$

where $a_i = 1$ if $i \in \mathsf{Supp}(\alpha)$ and $a_i = a_\alpha$ if $i \notin \mathsf{Supp}(\alpha)$. Define a group

$$\mathcal{H}^{(k)} = \langle\ a_{(ij)} \mid 0 \leq i < j \leq k-1\ \rangle$$

of k-ary tree automorphisms, generated by a_α corresponding to the transpositions in S_k. For example, the automaton generating $\mathcal{H}^{(4)}$ is given in Figure 9. The state in the middle represents the identity and its loops are not drawn. We note that

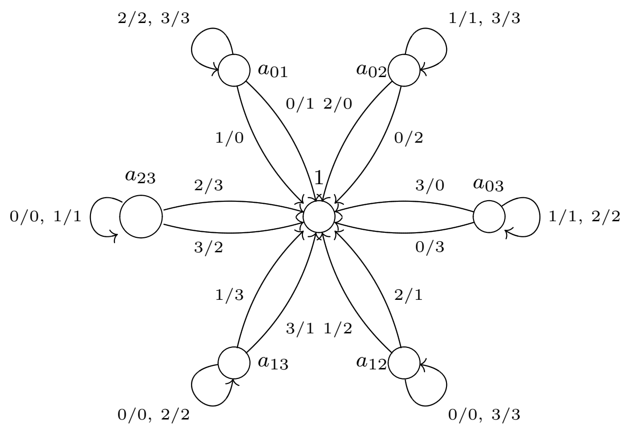


FIGURE 9. *Automaton generating the Hanoi Towers group $\mathcal{H}^{(4)}$ on 4 pegs*

the directed graph in Figure 9 follows another common convention to encode the transition and output function of an automaton. Namely, the vertices are the states and for each pair of a state s and a letter x there is an edge connecting s to s_x labeled by $x|s(x)$.

The effect of the automorphism $a_{(ij)}$ on a k-ary word is that it changes the first occurrence of the symbol i or j to the other symbol (if such an occurrence exists) while leaving all the other symbols intact. A recursive formula for $a_{(ij)}$ is given by

$$a_{(ij)}(iw) = jw, \qquad a_{(ij)}(jw) = iw, \qquad a_{(ij)}(xw) = xa_{(ij)}(w), \text{ for } x \notin \{i, j\}.$$

The group $\mathcal{H}^{(k)}$ is called the *Hanoi Towers group* on k pegs since it models the Hanoi Towers Problem on k pegs (see, for example, [**57**]). We quickly recall this classical problem. Given n disks of distinct size, labeled $1, \dots, n$ by their size, and k pegs, $k \geq 3$, labeled $0, \dots, k-1$, a configuration is any placement of the disks on the pegs such that no disk is placed on top of a smaller disk. Figure 10 depicts a configuration of 5 disks on 3 pegs. In a single step, the top disk from one peg

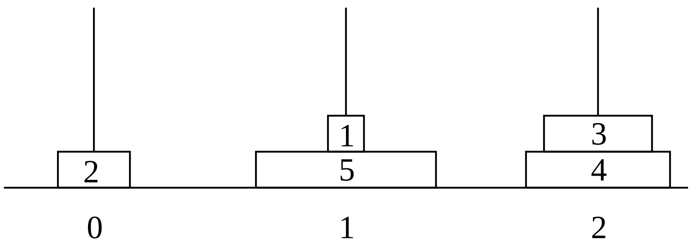


FIGURE 10. *A configuration of 5 disks on 3 pegs*

can be moved to the top of another peg, provided the newly obtained placement of disks represents a configuration. Initially all n disks are on peg 0 and the goal is to move all the disks to another peg.

Words of length n over X encode the configurations of n disks on k pegs. Namely, the word $x_1 \dots x_n$ over X encodes the unique configuration of n disks in which disk number i, $i = 1, \dots, n$, is placed on peg x_i (once the content of each peg is known the order of disks on the pegs is determined by their size). The automorphism $a_{(ij)}$ represents a move between pegs i and j. Indeed, if the symbol i appears before the symbol j (or j does not appear at all) in w then the disk on top of peg i is smaller than the disk on top of peg j (or peg j is empty) and a proper move between these two pegs moves the disk from peg i to peg j, thus changing the first appearance of i in w to j, which is exactly what $a_{(ij)}$ does. If none of the symbols i or j appears, this means that there are no disks on either of the pegs and the automorphism $a_{(ij)}$ acts trivially on such words. In terms of $\mathcal{H}^{(k)}$, the initial configuration of the Hanoi Towers Problem is encoded by the word 0^n and the goal is to find a group element $h \in \mathcal{H}^{(k)}$ written as a word over the generators $a_{(ij)}$ such that $h(0^n) = x^n$, where $x \neq 0$.

For example, the configuration in Figure 10 is encoded by the ternary word 10221 and the three generators $a_{(01)}$, $a_{(02)}$ and $a_{(12)}$ of $\mathcal{H}^{(3)}$, representing moves between the corresponding pegs, produce the configurations

$$a_{(01)}(10221) = 00221, \qquad a_{(02)}(10221) = 12221, \qquad a_{(12)}(10221) = 20221.$$

Let i, j, ℓ be three distinct symbols in X, Since $(ij)(j\ell)(ij)(i\ell)$ is the trivial permutation in S_k the element $h = a_{(ij)}a_{(j\ell)}a_{(ij)}a_{(i\ell)}$ is in the first level stabilizer.

Direct calculation shows that

$$\varphi_i(h) = a_{(ij)}, \qquad \varphi_j(h) = a_{(j\ell)}a_{(i\ell)}, \qquad \varphi_\ell(h) = a_{(ij)}, \qquad \varphi_t(h) = h, \text{for } t \notin \{i, j, \ell\},$$

which implies that every generator of $\mathcal{H}^{(k)}$ belongs to every projection $\varphi_i(\mathsf{St}(\mathcal{L}_1))$, for all $i = 0, \dots, k-1$.

PROPOSITION 3.4. *The group $\mathcal{H}^{(k)}$ is a spherically transitive, recurrent, self-similar group of k-ary tree automorphisms.*

For latter use, in order to simplify the notation, we set

$$a_{(01)} = a, \qquad a_{(02)} = b, \qquad a_{(12)} = c$$

for the generators of the Hanoi Towers group $\mathcal{H}^{(3)}$ on 3 pegs.

3.3. *Problems*

We propose several algorithmic problems on automaton groups. All of them (and many more) can be found in [**43**].

PROBLEM 1. Is the conjugacy problems solvable for all automaton groups?

PROBLEM 2. Does there exist an algorithm that, given a finite automaton $\mathcal{A}$ and a state s in $\mathcal{A}$, decides
(a) if s acts spherically transitively?
(b) if s has finite order?

PROBLEM 3. Does there exist an algorithm that, given a finite automaton $\mathcal{A}$, decides
(a) if $G(\mathcal{A})$ acts spherically transitively?
(b) if $G(\mathcal{A})$ is a torsion group?
(c) if $G(\mathcal{A})$ is a torsion free group?

4. *Iterated monodromy groups*

The notion of iterated monodromy groups was introduced by Nekrashevych. The monograph [**71**] treats the subject in great detail (for earlier work see [**70**, **9**]). We provide only a glimpse into this area.

Let M be a path connected and locally path connected topological space and let $f : M_1 \to M$ be a k-fold covering of M by an open, path connected subspace M_1 of M. Let t be a base point of M. The set of preimages

$$T = \bigcup_{n=0}^{\infty} f^{-n}(t)$$

can be given the structure of a k-regular rooted tree in which t is the root, the points from $L_n = f^{-n}(t)$ constitute level n and each vertex is connected by an edge to its k preimages. Precisely speaking, some preimages corresponding to different levels may coincide in M_1 and one could be more careful and introduce pairs of the form (x, n), where $x \in f^{-n}(t)$ to represent the vertices of T, but we will not use such notation.

Example 5. Consider the map $f : \mathbb{C}^* \to \mathbb{C}^*$ given by $f(z) = z^2$ (here $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$). This is a 2-fold self-covering of $\mathbb{C}^*$. If we choose $t = 1$ we see that the level n in the binary tree T consist of all 2^n-th roots of unity, i.e.

$$L_n = \{ \ e^{\frac{2\pi i}{2^n} \cdot m} \mid m = 0, \ldots, 2^n - 1 \ \}.$$

We can encode the vertex $e^{\frac{2\pi i}{2^n} \cdot m}$ at level n by the binary word of length n representing m in the binary system , with the first digit being the least significant one (see Figure 11).

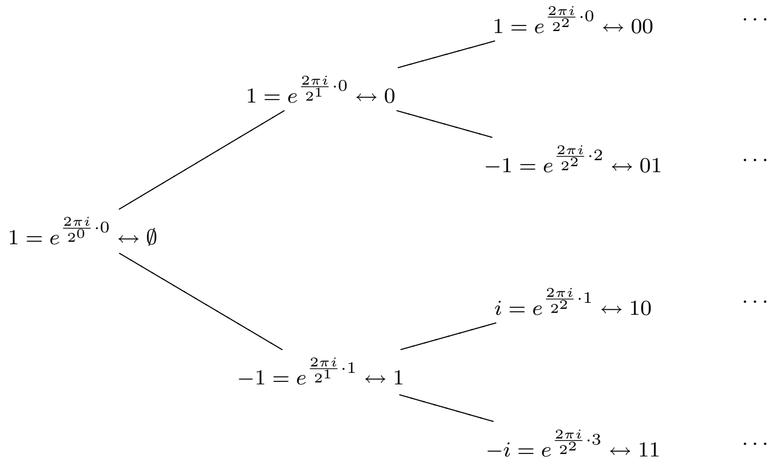


Figure 11. *The tree T corresponding to $f(z) = z^2$*

Define an action of the fundamental group $\pi(M) = \pi(M, t)$ on the tree T of preimages of t as follows. Define $M_n = f^{-n}(M)$. The n-fold composition map $f^n : M_n \to M$ is a k^n-fold covering of M. For a path α in M that starts at t and a vertex u in $L_n = f^{-n}(t) \subseteq M_n$ denote by $\alpha_{[u]}$ the unique lift (under f^n) of α that starts at u. Let γ be a loop based at t. For u in L_n the endpoint of $\gamma_{[u]}$ must also be a point in L_n. Denote this point by $\gamma(u)$ and define a map $L_n \to L_n$ by $u \mapsto \gamma(u)$. This map permutes the vertices at level n in T and is called the n-th *monodromy action* of γ.

Proposition 4.1 (Nekrashevych). *The map $T \to T$ given by*

$$u \mapsto \gamma(u)$$

is a tree automorphism, which depends only on the homotopy class of γ.

The action of $\pi(M)$ on T by tree automorphisms, called the *iterated monodromy action* of π, is not necessarily faithful.

Definition 8. Let N be the kernel of the monodromy action of $\pi(M)$ on T. The *iterated monodromy group* of the k-fold cover $f : M_1 \to M$, denoted $IMG(f)$, is the group $\pi(M)/N$.

The action of $IMG(f)$ on T is faithful. The classical monodromy actions on the levels of T are modeled within the action of $IMG(f)$ on T.

Example 6. Continuing our simple example involving $f(z) = z^2$, let γ be the loop $\gamma : [0,1] \to \mathbb{C}^*$ based at $t = 1$ given by

$$\theta \mapsto e^{2\pi i\theta},$$

i.e. γ is the unit circle traversed in the positive (counterclockwise) direction. For a vertex u in L_n (a 2^n-th root of unity) the path $\gamma_{[u]}$ is just the path starting at u, moving in the positive direction along the unit circle and ending at the next 2^n-th root of unity (see Figure 12 for the case $n = 2$). Thus

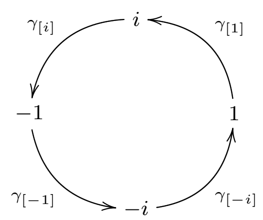


Figure 12. *The monodromy action of $\pi(\mathbb{C}^*) = \mathbb{Z}$ on level 2 in T, for $f(z) = z^2$*

$$\gamma\left(e^{\frac{2\pi i}{2^n}\cdot m}\right) = e^{\frac{2\pi i}{2^n}\cdot(m+1)}$$

and we see that on the corresponding binary tree $\mathcal{T}$ the action of γ is the binary odometer action

$$\gamma = (01)\ (1,\gamma).$$

Therefore the action of $\pi(C^*)$ on T is faithful and $IMG(z \mapsto z^2) \cong \mathbb{Z}$.

It is convenient to set a tree isomorphism $\Lambda : \mathcal{T} \to T$, where $\mathcal{T}$ is the k-ary rooted tree over the alphabet $X = \{0, \dots, k-1\}$, with an induced action of $IMG(f)$ on $\mathcal{T}$. For this purpose, set $\Lambda(\emptyset) = t$, choose a bijection $\Lambda : X \to L_1$, let $\ell(\emptyset)$ be the trivial loop 1_t based at t and fix paths $\ell(0), \dots, \ell(k-1)$ from t to the k points $\Lambda(0), \dots, \Lambda(k-1)$ (the preimages of t). Assuming that, for every word v of length n, a point $\Lambda(v)$ at level n in T and a path $\ell(v)$ in M connecting t to $\Lambda(v)$ are defined, we define, for $x \in X$, the path $\ell(xv)$ by

$$\ell(xv) = \ell(x)_{[\Lambda(v)]}\ell(v)$$

and set $\Lambda(xv)$ to be the endpoint of this path (the composition of paths in $\pi(M)$ is performed from right to left).

Definition 9. Let γ be a loop in $\pi(M)$. Define an action of γ on $\mathcal{T}$ by

$$\gamma(u) = \Lambda^{-1}\gamma\Lambda(u),$$

for a vertex u in $\mathcal{T}$. This action is called the *standard action* of the iterated monodromy group $IMG(f)$ on $\mathcal{T}$.

We think of Λ as a standard isomorphism between $\mathcal{T}$ and T and, in order to simplify notation, we do not distinguish the vertices in $\mathcal{T}$ from the points in T they represent.

Theorem 4.2 (Nekrashevych). *The standard action of $IMG(f)$ on $\mathcal{T}$ is faithful and self-similar. The root permutation of γ is the 1-st monodromy action of γ on $L_1 = X$. The section of γ at u, for u a vertex in $\mathcal{T}$, is the loop*

$$\gamma_u = \ell(\gamma(u))^{-1}\ \gamma_{[u]}\ \ell(u).$$

Example 7. We calculate now the iterated monodromy group $IMG(f)$ of the double cover map $f : \mathbb{C} \setminus \{0, -1, 1\} \to \mathbb{C} \setminus \{0, -1\}$ given by $z \mapsto z^2 - 1$.

Choose the fixed point $t = \frac{1-\sqrt{5}}{2}$ as the base point. The fundamental group $\pi(\mathbb{C} \setminus \{0, -1\})$ is the free group on two generators represented by the loops a and b, where a is the loop based at t moving around -1 in positive direction along the circle centered at -1 (see Figure 13) and b is the loop based at t moving around 0 in the positive direction along the circle center at 0. We have $L_1 = f^{-1}(t) = \{t, -t\}$ and we choose $\Lambda(0) = t$, $\Lambda(1) = -t$. Set $\ell(0)$ to be the trivial loop 1_t at t and $\ell(1)$ to be the path from t to $-t$ that is moving along the top part of the loop b (in direction opposite to b). Let c be the loop based at t^2 traversed in the positive

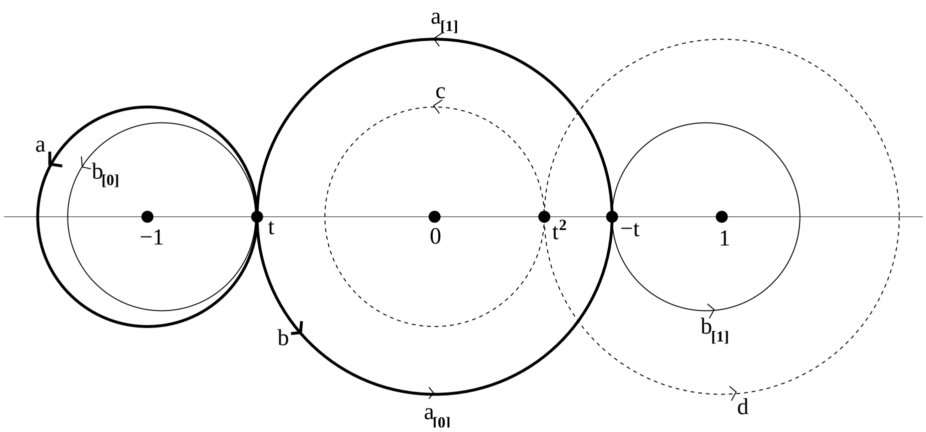


Figure 13. *Calculation of the action of $IMG(z \mapsto z^2 - 1)$*

direction along the circle centered at 0 and d the loop based at t^2 traversed along the circle centered at 1. Since $t^2 - 1 = t$, c is exactly one unit to the right of a and d is exactly one unit to the right of b. It is easy to see that $a_{[0]}$ is the path from t to $-t$ moving along the bottom part of the loop b and $a_{[1]} = \ell(1)^{-1}$ is the path from $-t$ to t moving along the top part of b. This is because applying $z \mapsto z^2$ to either one of these paths produces the loop c. Further, $b_{[0]}$ is the loop based at t that is entirely in the interior of the loop a, whose radius is chosen in such a way that applying $z \mapsto z^2$ yields the loop d. Similarly, $b_{[1]}$ is the loop based at $-t$ that is entirely in the interior of the loop d, whose radius is chosen in such a way that applying $z \mapsto z^2$ yields the loop d ($b_{[0]}$ and $b_{[1]}$ are symmetric with respect to the origin).

The loop a acts on the first level of $\mathcal{T}$ by permuting the vertices 0 and 1 and b acts trivially. For the sections we have

$$\begin{aligned}
a_0 &= \ell(a(0))^{-1}\ a_{[0]}\ \ell(0) = \ell(1)^{-1}\ a_{[0]}\ \ell(0) = a_{[1]}\ a_{[0]}\ 1_t = b\\
a_1 &= \ell(a(1))^{-1}\ a_{[1]}\ \ell(1) = \ell(0)^{-1}\ a_{[1]}\ \ell(1) = 1_t\ a_{[1]}\ a_{[1]}^{-1} = 1\\
b_0 &= \ell(b(0))^{-1}\ b_{[0]}\ \ell(0) = \ell(0)^{-1}\ b_{[0]}\ \ell(0) = 1_t\ a\ 1_t = a\\
b_1 &= \ell(b(1))^{-1}\ b_{[1]}\ \ell(1) = \ell(1)^{-1}\ b_{[1]}\ \ell(1) = a_{[1]}\ 1_{-t}\ a_{[1]}^{-1} = 1
\end{aligned}$$

Thus $IMG(z \to z^2 - 1)$ is the self-similar group generated by the automaton

$$\begin{aligned}
a &= (01)\ (b,\ 1)\\
b &= \qquad (a,\ 1)
\end{aligned}$$

and we see that $IMG(z \to z^2 - 1)$ is the Basilica group $\mathcal{B}$.

Basilica group belongs to the class of iterated monodromy groups of post-critically finite rational functions over the Riemann Sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The groups in this class are described as follows. For a rational function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree k let C_f be the set of critical points and $P_f = \cup_{n=1}^{\infty}(f^n(C_f))$ be the *post-critical set*. If the set P_f is finite, f is said to be *post-critically finite*. Set $M = \hat{\mathbb{C}} \setminus P_f$ and $M_1 = \hat{\mathbb{C}} \setminus f^{-1}(P_f)$. Then $f : M_1 \to M$ is a k-fold covering and $IMG(f)$ is, by definition, the iterated monodromy group of this covering.

Example 8. Consider the rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$

$$f(z) = z^2 - \frac{16}{27z}.$$

Denote $\Omega = \{\omega_0, \omega_1, \omega_2\}$, where $\omega_0 = 1$, $\omega_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\omega_2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ are the third roots of unity. The critical set of f is $C_f = -\frac{2}{3}\Omega \cup \{\infty\}$. Direct calculation shows that

$$f\left(-\frac{2}{3}\omega_i\right) = \frac{4}{3}\bar{\omega}_i = f\left(\frac{4}{3}\omega_i\right).$$

The post-critical set $P_f = \frac{4}{3}\Omega \cup \{\infty\}$ is finite (f conjugates the points in this set), and we have $M = \hat{\mathbb{C}} \setminus (\frac{4}{3}\Omega \cup \{\infty\})$ and $M_1 = \hat{\mathbb{C}} \setminus (-\frac{2}{3}\Omega \cup \frac{4}{3}\Omega \cup \{0, \infty\})$. The fundamental group $\pi(M)$ is free of rank 3. It is generated by the three loops a, b and c based at $t = 0$ as drawn in the upper left corner in Figure 14. In the figure, the critical points in $-\frac{2}{3}\Omega$ are represented by small empty circles, while the post-critical points in $\frac{4}{3}\Omega$ are represented by small black disks. The 3 inverse images of the base point $t = 0$ are the points in the set $\frac{2\sqrt[3]{2}}{3}\Omega$ and they are denoted by $\Lambda(0)$, $\Lambda(1)$ and $\Lambda(2)$ as in Figure 14. The base point and its three pre-images are represented by small shaded circles in the figure. The paths $\ell(0)$, $\ell(1)$ and $\ell(2)$ are chosen to be along straight lines from the base point $t = 0$. Calculation of the vertex

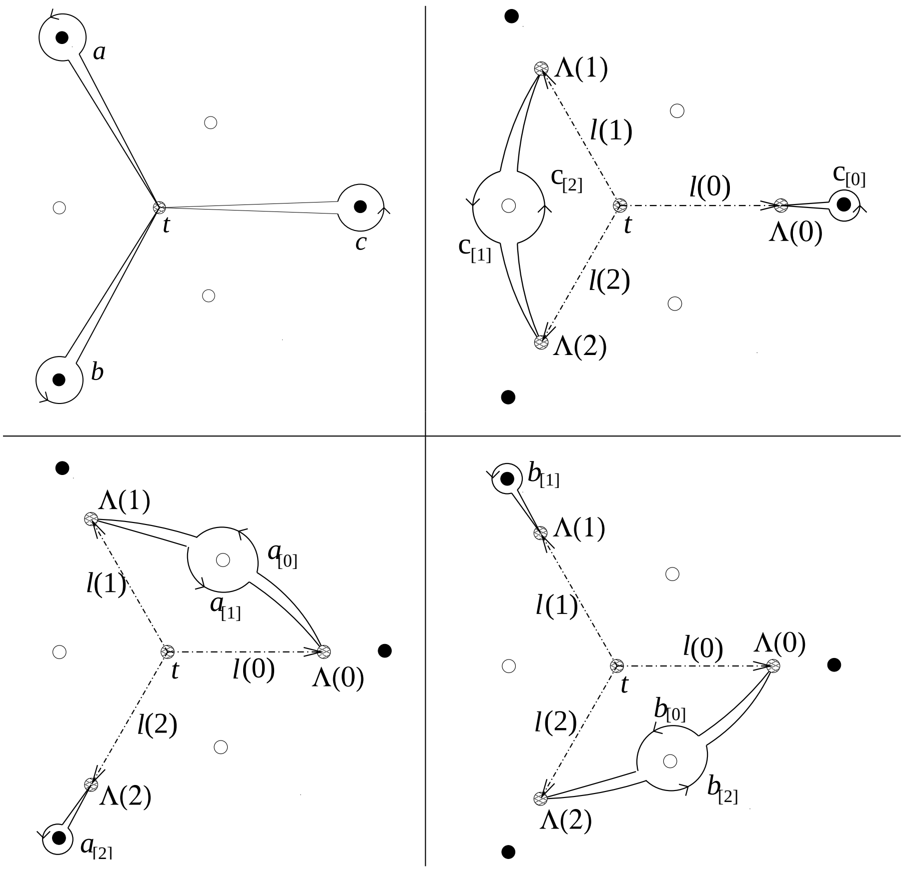


FIGURE 14. *Calculation of the action of* $IMG\left(z \mapsto z^2 - \frac{16}{27z}\right)$

permutations and the sections of the generators a, b and c of $IMG(f)$ then gives

$$\begin{aligned} a &= (01)\ (1, 1, b) \\ b &= (02)\ (1, a, 1) \\ c &= (12)\ (c, 1, 1). \end{aligned}$$

The resemblance between $IMG(f)$ and $\mathcal{H}^{(3)}$ is obvious. In fact, for the map $\bar{f} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ given by

$$z \to \bar{z}^2 - \frac{16}{27\bar{z}},$$

we have $IMG(\bar{f}) = \mathcal{H}^{(3)}$. Further, if we define

$$\begin{aligned} g &= (12)\ (h, h, h) \\ h &= \quad\ (g, g, g), \end{aligned}$$

then g conjugates $IMG(f)$ to $\mathcal{H}^{(3)}$ in $\mathsf{Aut}(\mathcal{T})$. The Julia set of the map f (or $\bar{f}$) is given in Figure 15.

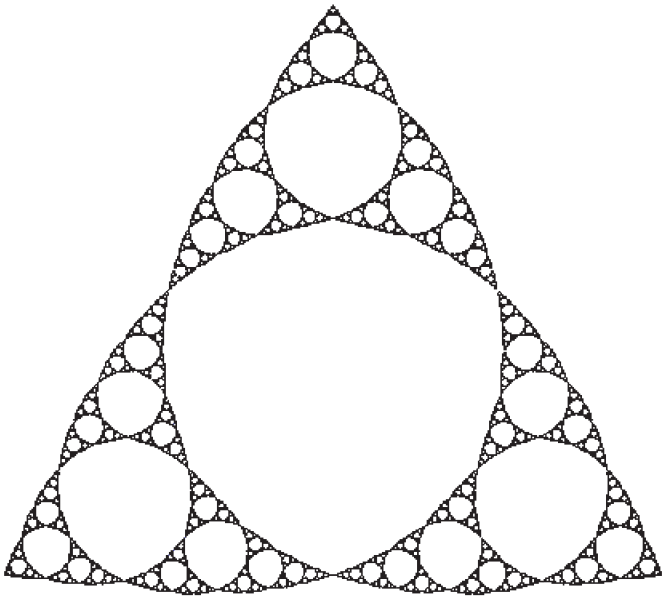


FIGURE 15. *Sierpiński gasket, Julia set of* $z \mapsto z^2 - \frac{16}{27z}$, *limit space of* $\mathcal{H}^{(3)}$

This set is homeomorphic to the Sierpiński Gasket, which is the well known plane analogue of the Cantor middle thirds set. We will return to this set in Section 8, when we discuss Schreier graphs of self-similar groups.

5. *Branch groups*

5.1. *Geometric definition of a branch group*

Here we consider some important subgroups of a group acting on a rooted tree leading to the geometric definition of a branch group. We extend our considerations to spherically homogeneous rooted trees (rather than regular).

For a sequence $\bar{k} = \{k_n\}_{n=1}^{\infty}$ of integers with $k_n \geq 2$, define *spherically homogenous rooted tree* $\mathcal{T} = \mathcal{T}^{(\bar{k})}$ to be the tree in which level n consists of the elements in $\mathcal{L}_n = X_{k_1} \times \cdots \times X_{k_n}$ (recall that X_k is the standard alphabet on k letters $X_k = \{0, \ldots, k-1\}$) and each vertex u in $\mathcal{L}_n$ has k_{n+1} children, namely ux, for $x \in X_{k_{n+1}}$. The sequence $\bar{k}$ is the *degree sequence* of $\mathcal{T}$.

The definition of a vertex and level stabilizer is analogous to the case of regular rooted trees. We define now rigid vertex and level stabilizers.

DEFINITION 10. Let G be a group acting on a spherically homogeneous rooted tree $\mathcal{T}$.

The *rigid stabilizer* of a vertex u in $\mathcal{T}$ is

$$\mathsf{RiSt}_G(u) = \{\ g \in G \mid \text{ the support of } g \text{ is contained in the subtree } \mathcal{T}_u\ \}.$$

The *rigid level stabilizer* of level n in $\mathcal{T}$ is

$$\mathsf{RiSt}_G(\mathcal{L}_n) = \left\langle \bigcup_{u \in \mathcal{L}_n} \mathsf{RiSt}_G(u) \right\rangle = \prod_{u \in \mathcal{L}_n} \mathsf{RiSt}_G(u).$$

If the action of G on $\mathcal{T}$ is spherically transitive (transitive on every level) then all (rigid) vertex stabilizers on the same level are conjugate.

For an arbitrary vertex u of level n we have

$$\mathsf{RiSt}_G(u) \leq \mathsf{RiSt}_G(\mathcal{L}_n) \leq \mathsf{St}_G(\mathcal{L}_n) \leq \mathsf{St}_G(u).$$

The index of $\mathsf{St}_G(\mathcal{L}_n)$ in G finite for every n (bounded by $k_1!(k_2!)^{k_1} \dots (k_n!)^{k_1 \dots k_{n-1}}$). We make important distinctions depending on the (relative) size of the rigid level stabilizers.

Definition 11. Let G act spherically transitively on a spherically homogeneous rooted tree $\mathcal{T}$.

We say that the action of G is

(a) of *branch type* if, for all n, the index $[G : \mathsf{RiSt}_G(\mathcal{L}_n)]$ of the rigid level stabilizer $\mathsf{RiSt}_G(\mathcal{L}_n)$ in G is finite.

(b) of *weakly branch type* if, for all n, the rigid level stabilizer $\mathsf{RiSt}_G(\mathcal{L}_n)$ is non-trivial (and therefore infinite).

(c) of *diagonal type* if, for some n, the rigid level stabilizer $\mathsf{RiSt}_G(\mathcal{L}_n)$ is finite (and therefore trivial after some level).

Definition 12. A group is called a *branch* (*weakly branch*) group if it admits a faithful branch (weakly branch) type action on a spherically homogeneous tree.

When we say that G is a branch group, we implicitly think of it as being embedded as a spherically transitive subgroup of the automorphism group of $\mathcal{T}$.

Example 9. A rather trivial, but important, example of a branch group is the full group of tree automorphisms $\mathsf{Aut}(\mathcal{T}^{(\overline{k})})$ of the spherically homogeneous tree defined by the degree sequence $\overline{k}$. Indeed, in this case, for any vertex u at level n,

$$\mathsf{RiSt}_G(u) \cong \mathsf{Aut}\left(\mathcal{T}^{(\sigma^n(\overline{k}))}\right),$$

where $\mathsf{Aut}\left(\mathcal{T}^{(\sigma^n(\overline{k}))}\right)$ is the spherically homogeneous rooted tree defined by the n-th shift $\sigma^n(\overline{k})$ of the degree sequence $\overline{k}$ and

$$\mathsf{RiSt}(\mathcal{L}_n) = \mathsf{St}(\mathcal{L}_n).$$

In particular, for a regular k-ary tree $\mathcal{T}$

$$\mathsf{RiSt}_G(u) \cong \mathsf{Aut}(\mathcal{T}) \qquad \text{and} \qquad \mathsf{RiSt}(\mathcal{L}_n) = \mathsf{St}(\mathcal{L}_n).$$

Another example is given by the group of finitary automorphisms of $\mathcal{T}^{(k)}$. This group consists of those automorphisms that have only finitely many non-trivial sections.

When the tree is k-regular another important example is the group of finite state automorphisms. This group consists of those automorphisms that have only finitely many distinct sections.

DEFINITION 13. Let G be a self-similar spherically transitive group of regular k-ary tree and let K and $M_0, \dots, M_{k-1}$ be subgroups of G. We say that K *geometrically contains* $M_0 \times \cdots \times M_{k-1}$ if $\psi^{-1}(M_0 \times \cdots \times M_{k-1})$ is a subgroup of K.

In the above definition ψ is the map $\psi : \mathsf{St}_G(\mathcal{L}_1) \to G \times \cdots \times G$ defined in Section 2.

DEFINITION 14. A self-similar spherically transitive group of automorphisms G of the regular k-ary tree $\mathcal{T}$ is *regular branch group* over its normal subgroup K if K has finite index in G and $K \times \cdots \times K$ (k copies) is geometrically contained in K as a subgroup of finite index.

The group G is *regular weakly branch* group over its non-trivial subgroup K if $K \times \cdots \times K$ (k copies) is geometrically contained in K.

THEOREM 5.1. *The Hanoi Towers group $H = \mathcal{H}^{(3)}$ is a regular branch group over its commutator subgroup H'. The index of H' in H is 8, $H/H' = C_2 \times C_2 \times C_2$ and the index of the geometric embedding of $H' \times H' \times H'$ in H' is 12.*

Proof. Since the order of every generator in $\mathcal{H}^{(3)}$ is 2, every square in H is a commutator. The equalities

$$\begin{aligned}(abac)^2 &= ([b,c],\ 1,\ 1),\\ (babc)^2 &= ([a,c],\ 1,\ 1),\\ (acacba)^2 &= ([a,b],\ 1,\ 1),\end{aligned}$$

show that H' geometrically contains $H' \times H' \times H'$. Indeed, since $\mathcal{H}^{(3)}$ is recurrent, for any element h_0 in H there exists an element h in $\mathsf{St}_H(\mathcal{L}_1)$ such that $h = (h_0, *, *)$, where the $*$'s denote elements that are not of interest to us. Since H' is normal we than have that whenever $g = (g_0, 1, 1)$ is in H' so is $g^h = (g_0^{h_0}, 1, 1)$.

Thus H' geometrically contains $H' \times 1 \times 1$. The spherical transitivity of H then implies that the copy of H' at vertex 0 can be conjugated into a copy of H' at any other vertex at level 1. Thus H' geometrically contains $H' \times H' \times H'$.

For the other claims see [**48**]. We just quickly justify that the index of $H' \times H' \times H'$ in H' must be finite, without calculating the actual index.

Since all the generators of H have order 2, the index of the commutator subgroup H' in H is finite (not larger than 2^3). On the other hand $H' \times H' \times H'$ is a subgroup of the stabilizer $\mathsf{St}(\mathcal{L}_1)$ which embeds via ψ into $H \times H \times H$. The index of $H' \times H' \times H'$ in $H \times H \times H$ is finite, which then shows that the index of $H' \times H' \times H'$ in $\mathsf{St}(\mathcal{L}_1)$ is finite. Since the index of $\mathsf{St}(\mathcal{L}_1)$ in H is 6 we have that the index of $H' \times H' \times H'$ in H (and therefore also in H') is finite. □

Figure 16 provides the full information on the indices between the subgroups of H mentioned in the above proof.

EXAMPLE 10. The group $\mathcal{G}$ is a regular branch group over the subgroup $K = [a,b]^{\mathcal{G}}$.

$$\begin{tikzcd}
 & H \arrow[dl, dash, "2^3"'] \arrow[dr, dash, "6"] & & H \times H \times H \\
H' \arrow[dr, dash, "12"'] & & \mathsf{St}(\mathcal{L}_1) \arrow[ur, hook, dashed, "\psi", "2^5"'] \arrow[dl, dash, "2^4"] & \\
 & H' \times H' \times H' & &
\end{tikzcd}$$

FIGURE 16. *Some subgroups near the top of* $\mathcal{H}^{(3)}$

The group $IMG(z \mapsto z^2 + i)$ is also a regular branch group. On the other hand, Basilica group $\mathcal{B} = IMG(z \mapsto z^2 - 1)$ is a regular weakly branch group over its commutator, but it is not a branch group.

We offer several immediate consequences of the branching property.

THEOREM 5.2 (Grigorchuk [**47**]). *Let G be a branch group of automorphisms of $\mathcal{T}$. Then G is centerless. Moreover the centralizer*

$$C_{\mathsf{Aut}(\mathcal{T})}(G)$$

of G in $\mathsf{Aut}(\mathcal{T})$ is trivial.

THEOREM 5.3 (Grigorchuk [**47**]). *Let N be a non-trivial normal subgroup of a weakly branch group G of automorphisms of $\mathcal{T}$. Then there exists level n such that*

$$(\mathsf{RiSt}_G(\mathcal{L}_n))' \leq N.$$

COROLLARY 5.4. *Any proper quotient of a branch group is virtually abelian.*

Quotients of branch groups can be infinite abelian groups. For example, this is the case for $\mathsf{Aut}(\mathcal{T})$ whose abelianization is infinite. It was an open question if quotients of finitely generated branch groups can be infinite and this was answered affirmatively in [**25**].

THEOREM 5.5 (Bartholdi, Grigorchuk [**10**]). *For any branch group G of automorphisms of tree the stabilizer $P = \mathsf{St}_G(\xi)$ of an infinite ray ξ in $\partial\mathcal{T}$ is weakly maximal in G, i.e. P has infinite index and is maximal with respect to this property.*

The following is a consequence of a more general result of Abért stating that if a group G acts on a set X and all stabilizers of the finite subsets of X are different, then G does not satisfy any group identities (group laws).

THEOREM 5.6 (Abért [**1**]). *No weakly branch group satisfies any group identities.*

Proof. Consider the action of G on the boundary $\partial\mathcal{T}$. It is enough to show that for any finite set of rays $\Xi = \{\xi_1, \dots, \xi_m\}$ in $\partial\mathcal{T}$ and any ray ξ in $\partial\mathcal{T}$ that is not in Ξ there exists an element g in G that fixes the rays in Ξ but does not fix ξ. Such an element g can be chosen from the rigid stabilizer $\mathsf{RiSt}_G(u)$ of any vertex u along ξ that is not a vertex on any of the rays in Ξ. $\square$

The above result, with a more involved proof, can also be found in [**62**].

THEOREM 5.7 (Abért [**2**]). *No weakly branch group is linear.*

A weaker version of the non-linearity result, applying only to branch groups, was proved earlier by Delzant and Grigorchuk.

THEOREM 5.8 (Delzant, Grigorhcuk [**25**]). *A finitely generated branch group G has Serre's property (FA) (i.e. any action without inversions of G on a tree has a fixed point) if and only if G is not indicable.*

Under some conditions the branch type action of a branch group is unique up to level deletion/insertion, i.e. we have rigidity results. Let $\mathcal{T}$ be a spherically homogenous rooted tree and $\overline{m}$ an increasing sequence of positive integers (whose complement in $\mathbb{N}$ is infinite). Define a spherically homogeneous rooted tree $\mathcal{T}'$ obtained by deleting the vertices from $\mathcal{T}$ whose level is in $\overline{m}$ and connecting two vertices in $\mathcal{T}'$ if one is descendant of the other in $\mathcal{T}$ and they belong to two consecutive undeleted levels in $\mathcal{T}$. An action of a group G on $\mathcal{T}$ induces an action on $\mathcal{T}'$. We say that the tree $\mathcal{T}'$ is obtained from $\mathcal{T}$ by level deletion and the action of G on $\mathcal{T}'$ is obtained from the action on $\mathcal{T}$ by level deletion.

THEOREM 5.9 (Grigorchuk, Wilson [**45**]). *Let G be a branch group of automorphisms of $\mathcal{T}$ such that*

(1) the degree sequence $\overline{k}$ consists of primes,

(2) each vertex permutation of each element g in G acts as a transitive cycle of prime length on the children below it, and

(3) for each pair of incomparable vertices u and v (neither u is in $\mathcal{T}_v$ nor v is in $\mathcal{T}_u$), there exists an automorphism g in $\mathsf{St}_G(u)$ that is active at v, i.e., the root permutation of the section g_v is nontrivial.

Then

(a) for any branch type action of G on a tree $\mathcal{T}''$, there exists an action of G on $\mathcal{T}'$ obtained by level deletion such that there exists a G-equivariant tree isomorphism between $\mathcal{T}''$ and $\mathcal{T}'$.

(b)

$$\mathsf{Aut}(G) = N_{\mathsf{Aut}(\mathcal{T})}(G).$$

The following result provides conditions on topological as well as combinatorial rigidity of weakly branch groups. It uses the notion of a saturated isomorphism. Let G and H be level transitive subgroups of $\mathsf{Aut}(\mathcal{T})$. An isomorphism $\phi : G \to H$ is *saturated* if there exists a sequence of subgroups $\{G_n\}_{n=0}^{\infty}$ such that G_n and $H_n = \phi(G_n)$ are subgroups of $\mathsf{St}_G(\mathcal{L}_n)$ and $\mathsf{St}_H(\mathcal{L}_n)$, respectively, and the action of both G_n and H_n is level transitive on every subtree $\mathcal{T}_u$ hanging below a vertex u on level n in $\mathcal{T}$.

THEOREM 5.10 (Lavrenyuk, Nekrashevych [**61**]; Nekrashevych [**71**]). *Let G and H be two weakly branch groups acting faithfully on spherically homogeneous rooted trees $\mathcal{T}_1$ and $\mathcal{T}_2$.*

(a) Any group isomorphism $\phi : G \to H$ is induced by a measure preserving homeomorphism $F : \mathcal{T}_1 \to \mathcal{T}_2$, i.e., there exists a measure preserving homeomorphism $F : \partial\mathcal{T}_1 \to \partial\mathcal{T}_2$ such that

$$\phi(g)(F(w)) = F(g(w)),$$

for all g in G and w in $\partial\mathcal{T}_1$.

(b) If $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}$ and $\phi : G \to H$ is a saturated isomorphism, then ϕ is induced by a tree automorphism $F : \mathcal{T} \to \mathcal{T}$.

COROLLARY 5.11 (Lavrenyuk, Nekrashevych [**61**]). *(a) For any weakly branch group G of automorphisms of $\mathcal{T}$ the automorphism group of G is the normalizer of G in the group of homeomorphisms of the boundary $\partial\mathcal{T}$ of $\mathcal{T}$, i.e.,*

$$\mathsf{Aut}(G) = N_{Homeo(\partial\mathcal{T})}(G).$$

(b) For any saturated weakly branch group G of automorphisms of $\mathcal{T}$ the automorphism group of G is the normalizer of G in $\mathsf{Aut}(\mathcal{T})$, i.e.,

$$\mathsf{Aut}(G) = N_{\mathsf{Aut}(\mathcal{T})}(G).$$

A spherically transitive group G of tree automorphism is saturated if it has a characteristic sequence of subgroups $\{G_n\}_{n=0}^{\infty}$ such that, for all n, G_n is a subgroup of $\mathsf{St}_G(\mathcal{L}_n)$ and G_n acts transitively on all subtrees $\mathcal{T}_u$ hanging below a vertex u on level n.

We note that the first complete description of the automorphism group of a finitely generated branch group was given by Sidki in [**80**]. More recent examples can be found in [**44**] and [**13**].

5.2. *Algebraic definition of a branch group*

We give here an algebraic version of a definition of a branch group. It is based on the subgroup structure of the group.

DEFINITION 15. A group G is *algebraically branch group* if there exists a sequence of integers $\overline{k} = \{k_n\}_{n=0}^{\infty}$ and two decreasing sequences of subgroups $\{R_n\}_{n=0}^{\infty}$ and $\{V_n\}_{n=0}^{\infty}$ of G such that

(1) $k_n \geq 2$, for all $n > 0$, $k_0 = 1$,

(2) for all n,

$$R_n = V_n^{(1)} \times V_n^{(2)} \times \cdots \times V_n^{(k_1 k_2 \dots k_n)}, \tag{5.1}$$

where each $V_n^{(j)}$ is an isomorphic copy of V_n,

(3) for all n, the product decomposition (5.1) of R_{n+1} is a refinement of the corresponding decomposition of R_n in the sense that the j-th factor $V_n^{(j)}$ of R_n, $j = 1, \dots, k_1 k_2 \dots k_n$ contains the j-th block of k_{n+1} consecutive factors

$$V_{n+1}^{((j-1)k_{n+1}+1)} \times \cdots \times V_{n+1}^{(jk_{n+1})}$$

of R_{n+1},

(4) for all n, the groups R_n are normal in G and

$$\bigcap_{n=0}^{\infty} R_n = 1,$$

(5) for all n, the conjugation action of G on R_n permutes transitively the factors in (5.1),

and

(6) for all n, the index $[G : R_n]$ is finite.

A group G is *weakly algebraically branch group* if there exists a sequence of integers $\overline{k} = \{k_n\}_{n=0}^{\infty}$ and two decreasing sequences of subgroups $\{R_n\}_{n=0}^{\infty}$ and $\{V_n\}_{n=0}^{\infty}$ of G satisfying the conditions (1)-(5).

Thus the only difference between weakly algebraically branch and algebraically branch groups is that in the former we do not require the indices of R_n in G to be finite. The diagram in Figure 17 may be helpful in understanding the requirements of the definition.

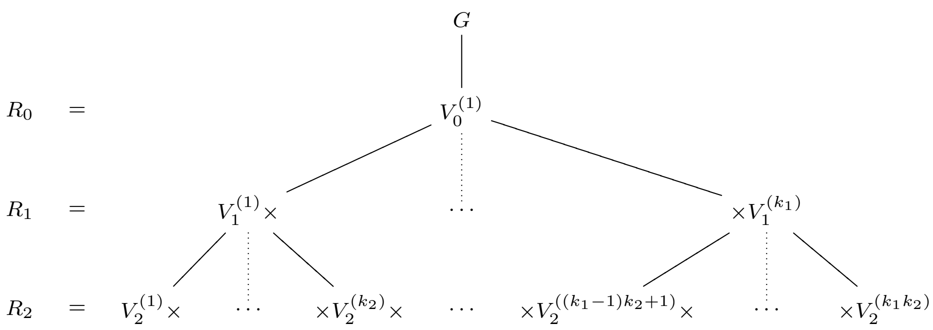


FIGURE 17. *Branch structure of an algebraically branch group*

PROPOSITION 5.12. *Every (weakly) branch group is algebraically (weakly) branch group.*

Proof. Let G admits a faithful branch type action on a spherically homogeneous rooted tree $\mathcal{T}$. By letting $R_n = \mathsf{RiSt}_G(\mathcal{L}_n)$ and $V_n = \mathsf{RiSt}_G(v_n)$, where v_n is the vertex on level n on some fixed ray $\xi \in \partial\mathcal{T}$ we obtain an algebraically (weakly) branch structure for G. □

Every algebraically branch group G acts transitively on a spherically homogeneous rooted tree, namely the tree determined by the branch structure of G (G acts on its subgroups by conjugation and, by definition, this action is spherically transitive on the tree in Figure 17). However, this action may not be faithful. In particular, it is easy to see that a direct product of an algebraically branch group and a finite abelian group produces an algebraically branch group with non-trivial center. According to Theorem 5.2 such a group cannot have a faithful branch type action.

Remark 1. We note that every infinite residually finite group has a faithful transitive action of diagonal type on some spherically homogeneous tree (even if the group is a branch group and has a branch type action on another tree). Indeed, let $\{G_n\}_{n=0}^{\infty}$ be a strictly decreasing sequence of normal subgroups of G such that (i) $G_0 = G$, (ii) the index $[G_{n-1}, G_n] = k_n$ is finite for all $n > 0$ and (iii) the intersection $\cap_{n=1}^{\infty} G_n$ is trivial. The set of all left cosets of all the groups in the sequence G_n can be given the structure of a spherically homogenous tree, called the coset tree of $\{G_n\}$, in which the vertices at level n are the left cosets of G_n and the edges are determined by inclusion (each coset of G_{n-1} splits as a disjoint union of k_n left cosets of G_n, which are the k_n children of G_{n-1}). The group G acts on the coset tree by left multiplication and this action preserves the inclusion relation. Thus we have an action of G on the coset tree by tree automorphisms. The action is transitive on levels (since G acts transitively on the set of left cosets of any of its subgroups) and it is faithful because the only element that can fix the whole sequence $\{G_n\}$ is an element in the intersection $\cap_{n=0}^{\infty} G_n$, which is trivial. Let g be an element in the rigid stabilizer of the vertex G_1 in the coset tree. Since G_1 must be fixed by g we have $g \in G_1$. Let h be an element in $G \setminus G_1$. For all $n \geq 1$ we must have $ghG_n = hG_n$. This is because, for all $n \geq 1$, the coset hG_n is contained in hG_1 and is therefore not in the subtree of the coset tree rooted at G_1. However, $ghG_n = hG_n$ if and only if $h^{-1}gh \in G_n$, and the normality of G_n implies that $g \in G_n$. Since the intersection $\cap_{n=1}^{\infty} G_n$ is trivial we obtain that $g = 1$ and therefore the rigid stabilizer of G_1 is trivial. The spherical transitivity of the action then shows that $\mathsf{RiSt}_G(\mathcal{L}_1)$ is trivial.

5.3. *Just infinite branch groups*

Definition 16. A group is *just infinite* if it is infinite and all of its proper quotients are finite.

Remark 2. Equivalently, a group is just infinite if it is infinite and all of its non-trivial normal subgroups have finite index.

Proposition 5.13. *Every finitely generated group has a just infinite quotient.*

Proof. Union of normal subgroups of infinite index in a finitely generated group G has infinite index in G (since the subgroups of finite index in G are finitely generated). Therefore, by Zorn's Lemma, there exists a maximal normal subgroup N of infinite index and the quotient G/N is just infinite. □

Definition 17. A group is *hereditarily just infinite* if all of its subgroups of finite index are just infinite.

Remark 3. Equivalently, a group is hereditarily just infinite if all of its normal subgroups of finite index are just infinite. This is because all subgroups of finite index in a group G contain a normal subgroup of G of finite index.

Another way to characterize hereditarily just infinite groups is by the property that all of their non-trivial subnormal subgroups have finite index.

Theorem 5.14 (Grigorchuk [**47**]). *Let G be a just infinite group. Then either*

(i) G is a branch group
or
G contains a a normal subgroup of finite index of the form $K \times \cdots \times K$, G acts transitively on the factors in $K \times \cdots \times K$ by conjugation, and
(iia) K is residually finite hereditarily just infinite group
or
(iib) K is an infinite simple group.

The above trichotomy result refines the description of just infinite groups proposed by Wilson in [**90**].

5.4. *Minimality*

The notion of largeness was introduced in group theory by Pride in [**78**]. We say that the group G is *larger* than the group H and we write $G \succeq H$ if there exists a finite index subgroup G_0 of G, finite index subgroup H_0 of H and a finite normal subgroup N of H_0 such that G_0 maps homomorphically onto H_0/N. Two groups are *equally large* if each of them is larger than the other. The class of groups that are equally large with G is denoted by $[G]$. Classes of groups can be ordered by the largeness relation $\succeq$. The class of the trivial group [1] consists of all finite groups. This is the smallest class under the largeness ordering. A class of infinite groups $[G]$ is *minimal* (or atomic) if the only class smaller than $[G]$ is the class [1]. An infinite group G is minimal if $[G]$ is minimal class.

Example 11. Obviously, the infinite cyclic group is minimal. Also, any infinite simple group is minimal.

A fundamental question in the theory of largeness of groups is the following.

Question 1. Which finitely generated groups are minimal?

We note that, since each finitely generated infinite group has a just infinite image, every minimal class of finitely generated groups has a just infinite representative. In the light of the trichotomy result in Theorem 5.14 we see that it is of particular interest to describe the minimal branch groups. The following result provides a sufficient condition for a regular branch group to be minimal.

Theorem 5.15 (Grigorchuk, Wilson [**46**]). *Let G be a regular branch group over K acting on the k-ary tree $\mathcal{T}$. Assume that K is a subdirect product of finitely many just infinite groups each of which is abstractly commensurable to G. If $\psi_n^{-1}(K \times \cdots \times K)$ is contained in K' for some $n \geq 1$, then G is a minimal group.*

Corollary 5.16. *The following groups are minimal.*
(1) The group $\mathcal{G}$.
*(2) The Gupta-Sidki p-groups from [**56**].*

The following questions were posed in [**78**] and in [**26**].

QUESTION 2. (1) Are there finitely generated groups that do not satisfy the ascending chain condition on subnormal subgroups?

(2) Are all minimal finitely generated groups finite-by-D_2-by-finite, where D_2 denotes the class of hereditarily just infinite groups?

The group $\mathcal{G}$ provides positive answer to the first question [**39**] and negative answer to the second question. The minimality of $\mathcal{G}$ was established only recently in [**46**]. An example of a minimal branch group answering negatively the second question was constructed by P. Neumann in [**72**]. In fact, [**72**] contains examples answering most of the questions from [**78**] and [**26**].

5.5. *Problems*

Most of the following problems appear in [**11**].

PROBLEM 4. Is the conjugacy problem solvable in all finitely generated branch groups with solvable word problem?

PROBLEM 5. Do there exist finitely presented branch groups?

PROBLEM 6. Do there exist branch groups with Property T?

PROBLEM 7. Let N be the kernel of the action of an algebraically branch group G by conjugation on the spherically homogeneous rooted tree determined by the branch structure of G. What can be said about N?

PROBLEM 8. Are there finitely generated torsion groups that are hereditarily just infinite?

PROBLEM 9. Can hereditarily just infinite group have the bounded generation property?

PROBLEM 10. Is every maximal subgroup of a finitely generated branch group necessarily of finite index?

PROBLEM 11. Which finitely generated just infinite branch groups are minimal?

The following problem is due to Nekrashevych [**71**].

PROBLEM 12. For which post-critically finite polynomials f is the iterated monodromy group $IMG(f)$ a branch group?

6. *Growth of groups*

6.1. *Word growth*

Let G be a group generated by a finite symmetric set S (symmetric set means that $S = S^{-1}$). The *word length* of an element g in G with respect to S is defined as

$$|g|_S = \min\{\ n \mid g = s_1 \dots s_n,\ \text{for some } s_1, \dots, s_n \in S\ \}.$$

The ball of radius n in G is the set

$$B_S(n) = \{\ g \mid |g|_S \le n\ \}$$

of elements of length at most n in G. The *word growth function* of G with respect to S is the function $\gamma_S(n)$ counting the number of elements in the ball $B_S(n)$, i.e.,

$$\gamma_S(n) = |B_S(n)|,$$

for all $n \ge 0$. The word growth function (often called just growth function) depends on the chosen generating set S, but growth functions with respect to different generating sets can easily be related.

PROPOSITION 6.1. *Let S_1 and S_2 be two finite symmetric generating sets of G. Then, for all $n \ge 0$,*

$$\gamma_{S_1}(n) \le \gamma_{S_2}(Cn),$$

where $C = \max\{\ |s|_{S_2} \mid s \in S_1\ \}$.

For two non-decreasing functions $f, g : \mathbb{N} \to \mathbb{N}^+$, where $\mathbb{N}$ is the set of non-negative integers and $\mathbb{N}^+$ is the set of positive integers, we say that f is dominated by g, and denote this by $f \preceq g$, if there exists $C > 0$ such that $f(n) \le g(Cn)$, for all $n \ge 0$. If f and g mutually dominate each other we denote this by $f \sim g$ and say that f and g have the same *degree of growth*.

Thus any two growth functions of a finitely generated group G have the same degree of growth, i.e., the degree of growth is invariant of the group G. For example, the free abelian group $\mathbb{Z}^m$ of rank m has degree of growth equal to n^m. On the other hand, for $m \ge 2$, the degree of growth of the free group of rank m is exponential, i.e., it is equal to e^n. In general we have the following possibilities. The growth of a finitely generated group G can be

$$\begin{array}{ll}
\text{polynomial:} & \lim_{n\to\infty} \sqrt[n]{\gamma(n)} = 1,\ \gamma(n) \preceq n^m,\ \text{for some } m \ge 0 \\
\text{intermediate:} & \lim_{n\to\infty} \sqrt[n]{\gamma(n)} = 1,\ n^m \preceq \gamma(n),\ \text{for all } m \ge 0 \\
\text{exponential:} & \lim_{n\to\infty} \sqrt[n]{\gamma(n)} = c > 1,\ \gamma(n) \sim c^n
\end{array}$$

The class of groups of polynomial growth is completely described. Recall that, by definition, a group G is virtually nilpotent if it has a nilpotent subgroup of finite index.

THEOREM 6.2. *A finitely generated group G has polynomial growth if and only if it is virtually nilpotent.*

In that case, $\gamma(n) \sim n^m$, where m is the integer

$$m = \sum_{i=1}^{n} i \cdot rank_{\mathbb{Q}}(G_{i-1}/G_i),$$

$1 = G_n \le \dots G_1 \le G_0 = G$ is the lower central series of G and $rank_{\mathbb{Q}}(H)$ denotes the torsion free rank of the abelian group H.

The formula for the growth of a nilpotent group appears in [**17**] as well as in [**55**]. The other direction, showing that polynomial growth implies virtual nilpotence, is due to Gromov [**54**].

In [**68**] Milnor asked if groups of intermediate growth exist. There are many classes of groups that do not contain groups of intermediate growth. Such is the class of linear groups (direct consequence of Tits Alternative [**84**]), solvable groups [**93**, **67**], elementary amenable groups [**22**], etc.

The first known example of a group of intermediate growth is the group $\mathcal{G}$.

THEOREM 6.3 (Grigorchuk [**38**, **39**]). *The group $\mathcal{G}$ has intermediate growth.*

The group $\mathcal{G}$ was constructed in [**36**] as an example of a finitely generated infinite 2-group. Other examples followed in [**39**, **40**, **30**] and more recently in [**14**, **11**, **28**, **18**]. All known examples of groups of intermediate growth are either branch self-similar groups or are closely related to such groups.

The best known upper bound on the growth of a group of intermediate growth is due to Bartholdi.

THEOREM 6.4 (Bartholdi [**6**]). *The growth function of $\mathcal{G}$ satisfies*

$$\gamma(n) \preceq e^{n^\alpha},$$

where $\alpha = \frac{\log 2}{\log 2 - \log \eta} \approx 0.767$ *and* $\eta \approx 0.81$ *is the positive root of the polynomial* $x^3 + x^2 + x - 2$.

We define now the class of groups of intermediate growth introduced in [**39**]. Each group in this class is defined by an infinite word $\bar{\omega}$ in $\Omega = \{0,1,2\}^{\mathbb{N}}$. Set up a correspondence

$$0 \leftrightarrow \begin{bmatrix}1\\1\\0\end{bmatrix}, \qquad 1 \leftrightarrow \begin{bmatrix}1\\0\\1\end{bmatrix}, \qquad 2 \leftrightarrow \begin{bmatrix}0\\1\\1\end{bmatrix} \tag{6.1}$$

and, for a letter ω in $\{0,1,2\}$, define $\omega(b)$, $\omega(c)$ and $\omega(d)$ to be the top entry the middle entry and the bottom entry, respectively, in the matrix corresponding to ω. Let a be the binary tree automorphism defined by

$$a = (01)\ (1,1).$$

Thus a only changes the first letter in every binary word. For a word $\bar{\omega} = \omega_1\omega_2\dots$ in Ω define binary tree automorphisms $b_{\bar{\omega}}$, $c_{\bar{\omega}}$ and $d_{\bar{\omega}}$ by

$$\begin{aligned} b_{\bar{\omega}} &= (a^{\omega_1(b)}, b_{\sigma(\bar{\omega})}),\\ c_{\bar{\omega}} &= (a^{\omega_1(c)}, c_{\sigma(\bar{\omega})}),\\ d_{\bar{\omega}} &= (a^{\omega_1(d)}, d_{\sigma(\bar{\omega})}) \end{aligned}$$

where $\sigma(\bar{\omega})$ is the shift of $\bar{\omega}$ defined by $\sigma(\bar{\omega}) = \omega_2\omega_3\dots$, i.e, $(\sigma(\bar{\omega}))_n = (\bar{\omega})_{n+1}$, for $n = 1,2,\dots$. The only possible non-trivial sections of $b_{\bar{\omega}}$, $c_{\bar{\omega}}$ and $d_{\bar{\omega}}$ appear at the vertices along the infinite ray $111\dots$ and the vertices at distance 1 from this ray. Define $G_{\bar{\omega}}$ to be the group

$$G_{\bar{\omega}} = \langle a, b_{\bar{\omega}}, c_{\bar{\omega}}, d_{\bar{\omega}} \rangle$$

of binary tree automorphisms.

EXAMPLE 12. Let $\bar{\omega}$ be the periodic sequence $\bar{\omega} = (01)^\infty$. The corresponding sequence of matrices is

$$\begin{bmatrix}1\\1\\0\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}\dots$$

Since the top row of entries is invariant under the shift σ we have that $b_{\sigma(\bar{\omega})} = b_{\bar{\omega}}$. Therefore

$$b_{\bar{\omega}} = (a, b_{\bar{\omega}}).$$

On the other hand, the shift of the middle row of entries is equal to the bottom row of entries and vice versa. Therefore we have $c_{\sigma(\bar{\omega})} = d_{\bar{\omega}}$ and $d_{\sigma(\bar{\omega})} = c_{\bar{\omega}}$ and

$$c_{\bar{\omega}} = (a, d_{\bar{\omega}}),$$
$$d_{\bar{\omega}} = (1, c_{\bar{\omega}})$$

Thus $G_{(01)^\infty}$ is a self-similar group defined by the 5 state binary automaton in Figure 18. The similarity with the automaton generating $\mathcal{G}$ is not accidental. In

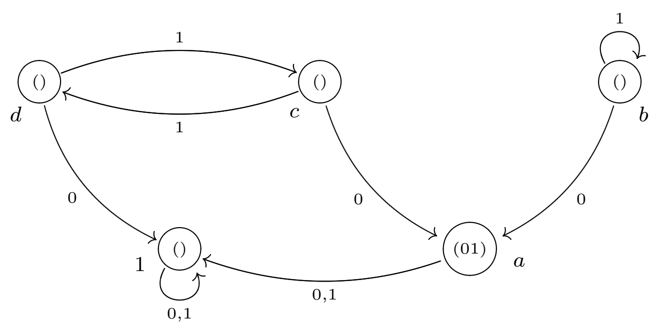


FIGURE 18. *The binary automaton generating the group* $G_{(01)^\infty}$

fact, in this context, the group $\mathcal{G}$ is defined by the periodic sequence $012012\dots$.

THEOREM 6.5 (Grigorchuk [**39**]). *(a) For every word $\bar{\omega}$ in Ω, $G_{\bar{\omega}}$ is a spherically transitive group of binary automorphisms such that the upper companion group of $G_{\bar{\omega}}$ at any vertex at level n is $G_{\sigma^n(\bar{\omega})}$.*

(b) For every word $\bar{\omega}$ in Ω that is not ultimately constant, $G_{\bar{\omega}}$ is not finitely presented.

(c) For every word $\bar{\omega}$ in Ω in which all letters appear infinitely often, $G_{\bar{\omega}}$ is a just infinite branch 2-group.

(d) For every word $\bar{\omega}$ in Ω that is not ultimately constant, $G_{\bar{\omega}}$ has intermediate growth.

The groups in this class exhibit very rich range of (intermediate) growth behavior. For example, the set of degrees of growth of these groups contain uncountable chains and anti-chains (under the comparison of degrees of growth given by dominance).

The examples from [**39**] were generalized to examples of groups of intermediate growth acting on p-ary trees, for p a prime, in [**40**]. These groups are defined by infinite words over $\{0, 1, \ldots, p\}$. There is a general upper bound on the growth in the case of groups defined by homogeneous sequences. A sequence $\bar{\omega}$ in $\Omega = \{0, 1, \ldots, p\}^{\mathbb{N}}$ defining a group $G_{\bar{\omega}}$ is *r-homogeneous* if every symbol in $\{0, 1, \ldots, p\}$ appears in every subword of $\bar{\omega}$ of length r.

THEOREM 6.6. *Let $\bar{\omega}$ be an r-homogeneous sequence.*

*(a) (Muchnik, Pak [**69**]) In case $p = 2$, the growth of $G_{\bar{\omega}}$ satisfies*

$$\gamma(n) \preceq e^{n^{\alpha}},$$

where $\alpha = \frac{\log 2}{\log 2 - \log \eta}$ and η is the positive root of the polynomial $x^r + x^2 + x - 2$.

*(b) (Bartholdi, Šunić [**14**]) For arbitrary prime p, the growth of $G_{\bar{\omega}}$ satisfies*

$$\gamma(n) \preceq e^{n^{\alpha}},$$

where $\alpha = \frac{\log p}{\log p - \log \eta}$ and η is the positive root of the polynomial $x^r + x^{r-1} + x^{r-2} - 2$.

An estimate analogous to the one above holds in the wider context of the so called spinal groups (see [**14**, **11**]).

Groups of intermediate growth appear also as iterated monodromy groups of post-critically finite polynomials.

THEOREM 6.7 (Bux, Perez [**18**]). *The group $IMG(z \to z^2 + i)$ has intermediate growth.*

Most of the proofs that a group has sub-exponential growth are based on a variation of the following contraction themes (see [**39**, **14**, **69**, **18**]).

PROPOSITION 6.8. *(a) If G is a self-similar group of k-ary tree automorphisms generated by a finite set S and there exist η in $(0, 1)$, $\alpha \in (0, 1]$ and a constant C such that, for all n, the ratio between the number of elements g in the ball $B_S(n)$ that satisfy*

$$\sum_{x=0}^{k-1} |g_x|_S \leq \eta |g|_S + C$$

and $\gamma_S(n)$ is at least α, then G has sub-exponential growth.

(b) If G is a self-similar group of k-ary tree automorphisms generated by a finite set S and there exist η in $(0, 1)$ and a constant C such that, for all $g \in G$,

$$\sum_{x=0}^{k-1} |g_x|_S \leq \eta |g|_S + C,$$

then the growth of G satisfies

$$\gamma_S(n) \preceq e^{n^{\alpha}},$$

where $\alpha = \frac{\log k}{\log k - \log \eta}$.

(c) Let $\Phi = \{G_\lambda\}_{\lambda \in \Lambda}$ be a family of groups acting on spherically homogeneous rooted trees, let Φ be closed for upper companion groups, let each member G_λ of Φ be generated by a finite set S_λ and let there be a uniform bound on the number of generators in S_λ. Let η be a number in $(0, 1)$. Assume that for each λ there exists

a level m_λ and a constant $C_\lambda \geq 0$ such that, for all elements g in G_λ,

$$\sum_{u \in \mathcal{L}_{m_\lambda}} |g_u|_{S_{\lambda_u}} \leq \eta |g|_{S_\lambda} + C_\lambda,$$

Then all groups in Φ have sub-exponential growth.

A quite a different approach, both for the lower and upper bounds on the growth, based on the Poisson boundary, is used by Erschler in her work.

THEOREM 6.9 (Erschler [**27**]). *The growth of the group G_ω acting on the binary tree and defined by the sequence $\bar{\omega} = (01)^\infty$ satisfies*

$$e^{\frac{n}{\ln^{2+\epsilon} n}} \leq \gamma(n) \leq e^{\frac{n}{\ln^{1-\epsilon} n}}$$

for any $\epsilon > 0$ and any sufficiently large n.

In many cases, a general lower bound exists for the growth of groups of intermediate growth.

THEOREM 6.10. *(a) (Grigorchuk [**41**]) The degree of growth of any residually-p group that is not virtually nilpotent is at least $e^{\sqrt{n}}$.*

*(b) (Lubotzky, Mann [**64**]) The degree of growth of any residually nilpotent group that is not virtually nilpotent is at least $e^{\sqrt{n}}$.*

Improvements over these general bounds are given for the particular case of $\mathcal{G}$ in [**63**] and [**7**].

Finally, we mention that there are many examples of automaton groups that have exponential growth. Such examples are the lamplighter groups [**51**, **15**, **82**], most ascending HNN extensions of free abelian groups [**15**] (including the Baumslag-Solitar solvable groups $BS(1, m)$ for $m \neq \pm 1$), free groups [**33**, **88**], etc. There are self-similar groups of exponential growth within the class of branch automaton groups. For example, $\mathcal{H}^{(k)}$ has exponential growth, for all $k \geq 3$.

6.2. *Uniformly exponential growth*

Let G be a group of exponential growth and let

$$\epsilon_G = \inf\{\ \sqrt[n]{\gamma_S(n)} \mid S \text{ a finite symmetric generating set of } G\ \}.$$

We say that G has uniformly exponential growth if $\epsilon_G > 1$. In 1981 Gromov asked if all groups of exponential growth have uniformly exponential growth. Affirmative answer has been obtained for the classes of hyperbolic groups [**60**], one-relator groups [**20**, **42**], solvable groups (Wilson and Osin [**77**], independently), linear groups over fields of characteristic 0 [**29**], etc. In [**92**] John Wilson showed that the answer is negative in the general case.

THEOREM 6.11 (Wilson [**92**]). *There exist 2-generated branch groups of exponential growth that do not have uniformly exponential growth.*

Further examples of this type appear in [**91**] and [**8**].

6.3. *Torsion growth*

Let G be a finitely generated torsion group and let S be a finite generating symmetric set for G. For $n \in \mathbb{N}$, denote by $\tau(n)$ the largest order of an element in G of length at most n with respect to S. The function τ is called the *torsion growth function* of G with respect to S. The degree of the torsion growth function of G does not depend on the chosen finite generating set, i.e., if τ_1 and τ_2 are two torsion growth functions of G with respect to two finite symmetric generating sets S_1 and S_2, then $\tau_1 \sim \tau_2$.

There exist polynomial estimates for the torsion growth of some groups G_ω from [**39**] and [**40**].

Theorem 6.12 (Bartholdi, Šunić [**14**]). *(a) Let $G_{\bar{\omega}}$ be a p-group defined by a r-homogeneus sequence $\bar{\omega}$. The torsion growth function satisfies*

$$\tau(n) \preceq n^{(r-1)\log_2(p)}.$$

(b) The torsion growth function of $\mathcal{G}$ satisfies

$$\tau(n) \preceq n^{\frac{3}{2}}.$$

6.4. *Subgroup growth*

We describe here a result of Segal [**79**] that uses branch groups to fill a conjectured gap in the spectrum of possible rates of subgroup growth.

For a finitely generated group G the number of subgroups of index at most n, denoted $s(n)$, is finite. The function s counting the subgroups up to a given index is called the *subgroup growth function* of G. All subgroup growth considerations are usually restricted to residually finite groups, since the lattice of subgroups of finite index in G is canonically isomorphic to the lattice of subgroups of finite index in the residually finite group G/N, where N is the intersection of all groups of finite index in G.

It is shown by Lubotzky, Mann and Segal in [**65**] that a residually finite group has polynomial subgroup growth (there exists m such that $s(n) \leq n^m$, for all sufficiently large n) if and only if it is a virtually solvable group of finite rank. It has been conjectured in [**66**] that if a finitely generated residually finite group does not have polynomial subgroup growth then there exists a constant $c > 0$ such $s(n) \geq n^{\frac{c\log(n)}{(\log\log(n))^2}}$, for infinitely many values of n.

Theorem 6.13 (Segal [**79**]). *There exists a finitely generated just infinite branch group such that, for any $c > 0$, $s(n) \leq n^{\frac{c\log(n)}{(\log\log(n))^2}}$, for all sufficiently large n.*

The construction used by Segal is rather flexible and can be used to answer other problems related to finite images of finitely generated groups.

Theorem 6.14 (Segal [**79**]). *Let S be any set of finite non-abelian simple groups. There exists a finitely generated just infinite branch group G such that the upper composition factors of G (the composition factors of the finite images of G) are precisely the members of S.*

6.5. *Problems*

Problem 13. (a) What is the degree of growth of $\mathcal{G}$?
(b) What is the degree of growth of any group $G_{\bar{\omega}}$, when $\bar{\omega}$ is not ultimately constant?

Problem 14. (a) Is it correct that every group that is not virtually nilpotent has degree of growth at least $e^{\sqrt{n}}$?
(b) Are there groups whose degree of growth is $e^{\sqrt{n}}$?

Problem 15 (Nekrashevych [**71**]). For which post-critically finite polynomials f does the iterated monodromy group $IMG(f)$ have intermediate growth?

7. *Amenability*

In this section we present some basic notions and results concerning amenability of groups and show how branch ($\mathcal{G}$) and weakly branch ($\mathcal{B}$) self-similar groups provide some crucial examples distinguishing various classes of groups related to the notion of amenability.

7.1. *Definition*

The fundamental notion of amenability is due to von Neuman [**87**].

Definition 18. A group G is *amenable* if there exists a finitely additive left-invariant probabilistic measure μ defined on all subsets of G, i.e.,
(i) (μ is defined for all subsets)

$$0 \leq \mu(E) \leq 1,$$

for all subsets E of G,
(ii) (μ is probabilistic)

$$\mu(G) = 1,$$

(iii) (μ is left invariant)

$$\mu(E) = \mu(gE),$$

for all subsets E of G and elements g in G,
(iv) (μ is finitely additive)

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2),$$

for disjoint subsets E_1, E_2 of G

Denote the class of amenable groups by AG. All finite groups are amenable (under the uniform measure, which is the only possible measure in this case). Also, all abelian groups are amenable, but all known proofs rely on the Axiom of Choice (even for the infinite cyclic group).

Remark 4. A group G is amenable if and only if it admits a left invariant mean, i.e., there exists a non-negative left invariant linear functional m on the space of bounded complex valued functions defined on G that maps the constant function 1 to 1. If such a mean exists we may define a measure on G by $\mu(E) = m(f_E)$, where f_E is the characteristic function of the subset E of G.

The next result gives a combinatorial characterization of amenability.

THEOREM 7.1 (Følner). *A countable group G is amenable if and only if there exists a sequence of finite subsets $\{A_n\}$ of G such that, for every g in G,*

$$\lim_{n\to\infty} \frac{|gA_n \cap A_n|}{|A_n|} = 1.$$

DEFINITION 19. Let $\Gamma = (V, E)$ be a graph. The *boundary* of a subset A of the vertex set V, denoted $\partial(A)$, is the set of edges in E connecting a vertex in A to a vertex outside of A.

DEFINITION 20. Let $\Gamma = (V, E)$ be a graph of uniformly bounded degree. The *Cheeger constant* of Γ is the quantity

$$ch(\Gamma) = \inf \left\{ \frac{|\partial(A)|}{|A|} \mid A \text{ a finite subset of } V \right\}$$

DEFINITION 21. An graph Γ of uniformly bounded degree is *amenable* if its Cheeger constant is 0.

REMARK 5. Let G be a finitely generated infinite group with finite generating set S and let $\Gamma = \Gamma(G, S)$ be the Cayley graph of G with respect to S. Then G is amenable if and only if Γ is amenable.

EXAMPLE 13. The infinite cyclic group $\mathbb{Z}$ is amenable. Indeed, for the sequence of intervals $A_n = [1, n]$, $n = 1, 2, \dots$, and the symmetric generating set $S = \{\pm 1\}$, we have

$$\frac{|\partial(A_n)|}{|A_n|} = \frac{4}{n}$$

which tends to 0 as n grows.

As was already mentioned all abelian groups are amenable. Indeed Følner criterion can be used to prove the following more general result.

THEOREM 7.2. *Every finitely generated group of sub-exponential growth is amenable.*

Since all finitely generated abelian (virtually nilpotent) groups have polynomial growth we obtain the following corollary.

COROLLARY 7.3. *All virtually nilpotent groups are amenable.*

7.2. *Elementary classes*

THEOREM 7.4 (von Neumann [**87**]). *The class of amenable groups AG is closed under taking*

(i) subgroups,
(ii) homomorphic images,
(iii) extensions, and
(iv) directed unions.

Call the constructions (i)-(iv) above *elementary constructions*.

DEFINITION 22. The class of *elementary amenable groups*, denoted EA, is the smallest class of groups that contains all finite and all abelian groups, and is closed under the elementary constructions.

THEOREM 7.5. *The free group F_2 of rank 2 is not amenable.*

The fact that F_2 is not amenable can be easily proved by several different arguments. In particular, one may use the doubling condition of Gromov.

THEOREM 7.6 (Gromov doubling condition). *Let Γ be a graph of uniformly bounded degree. Then Γ is not amenable if and only if there exists a map $f : V(\Gamma) \to V(\Gamma)$ such that the pre-image of every vertex has at least 2 elements and the distance between any vertex and its image is uniformly bounded.*

To see now that $F_2 = F(a, b)$ is not amenable just map every vertex $ws^{\pm 1}$ in the Cayley graph of F_2 with respect to $S = \{a, b\}$ to w and leave 1 fixed. Under this map, the distance between every vertex and its image is at most 1 and every vertex has at least 3 pre-images.

Since F_2 is not amenable and AG is closed under subgroups, no group that contains F_2 is amenable. Denote by NF the class of groups that do not contain a copy of the free group F_2 of rank 2 as a subgroup. We have

$$EA \subseteq AG \subseteq NF.$$

Mahlon Day asked in [**24**] if equality holds in either of the two inclusions above. The question if $AG = NF$ is sometimes referred to as von Neumann Problem. Chou showed in 1980 that $EA \neq NF$ by using the known fact that there exist infinite torsion groups and proving the following result.

THEOREM 7.7 (Chou [**22**]). *No finitely generated torsion group is elementary amenable. Therefore*

$$EA \neq NF.$$

A negative solution to the von Neumann Problem was given by Ol'shanskii [**73**] in 1980 and later by Adian [**3**].

THEOREM 7.8 (Ol'shanskii [**73**]). *There exist non-amenable groups that do not contain free subgroups of rank 2. Therefore*

$$AG \neq NF.$$

The examples Ol'shanskii used to show that $AG \subsetneq NF$ are the Tarski monsters he constructed earlier [**75**].

THEOREM 7.9 (Adian [**3**]). *The free Burnside groups*

$$B(m,n) = \langle\ a_1, \dots, a_m \mid x^n = 1,\ \textit{for all } x\ \rangle,$$

for $m \geq 2$ and odd $n \geq 665$, are non-amenable.

Both Ol'shanskii and Adian used the following co-growth criterion of amenability in their work.

THEOREM 7.10 (Grigorchuk [**37**]). *Let G be a m-generated group presented as F_m/H where $F_m = F(X)$ is the free group of rank m and H is a normal subgroup of F_m. Denote by $h(n)$ the number of words of length n over $X \cup X^{-1}$ that represent elements in H. Then G is amenable if and only if*

$$\limsup_{n\to\infty} \sqrt[n]{h(n)} = 2m - 1.$$

We quote another result of Chou.

THEOREM 7.11 (Chou [**22**]). *No finitely generated group of intermediate growth is elementary amenable.*

This shows that any example of a group of intermediate growth would answer the question if $EG = AG$ negatively (since all such groups are amenable). Such an example was provided in [**38**, **39**].

THEOREM 7.12. *The group $\mathcal{G}$ is amenable but not elementary amenable group. Thus*

$$EA \neq AG.$$

In fact, the classes EA, AG and NF are distinct even in the context of finitely presented groups.

THEOREM 7.13 (Grigorhchuk [**49**]). *The HNN extension*

$$\tilde{\mathcal{G}} = \langle\ \mathcal{G}, t \mid a^t = aca,\ b^t = d,\ c^t = b,\ d^t = c\ \rangle$$

is a finitely presented amenable group that is not elementary amenable. Thus the class $AG \setminus EA$ contains finitely presented (torsion-by-cyclic) groups.

THEOREM 7.14 (Ol'shanskii, Sapir [**74**]). *The class $NF \setminus AG$ contains finitely presented (torsion-by-cyclic) groups.*

DEFINITION 23. The class of *sub-exponentially amenable groups*, denoted SA, is the smallest class of groups that contains all finitely generated groups of sub-exponential growth and is closed under the elementary constructions.

It is clear that

$$EA \subseteq SA \subseteq AG.$$

The group $\mathcal{G}$ is an example of a group in $SA \setminus EA$.

Theorem 7.15. *(a) (Grigorchuk,Żuk [**52**]) Basilica group is not sub-exponentially amenable.*

*(b)(Bartholdi, Virág [**16**]) Basilica group is amenable.*

Corollary 7.16.

$$EA \subsetneq SA \subsetneq AG \subsetneq NF.$$

We will only prove here that the Basilica group is not sub-exponentially amenable. The proof follows the exposition in [**52**], but we first introduce the notion of elementary classes.

Definition 24 (Chou [**22**], Osin [**76**]). Let $\mathcal{C}$ be a class of groups. For each ordinal α define the *elementary class* $E_\alpha(\mathcal{C})$ as follows. For $\alpha = 0$ define

$$E_0(\mathcal{C}) = \mathcal{C}.$$

For non-limit ordinals of the form $\alpha = \beta + 1$ define $E_\alpha(\mathcal{C}) = E_{\beta+1}(\mathcal{C})$ to be the class of groups that can be obtained either as an extension of a group in $E_\beta(\mathcal{C})$ by a group in $\mathcal{C}$ or as a directed union of a family of groups in $E_\beta(\mathcal{C})$. For limit ordinals α define

$$E_\alpha(\mathcal{C}) = \bigcup_{\beta<\alpha} E_\beta(\mathcal{C}).$$

Finally, define $E(\mathcal{C})$ to be the union of $E_\alpha(\mathcal{C})$ taken over all ordinals.

Theorem 7.17 (Chou [**22**], Osin [**76**]). *If $\mathcal{C}$ is closed under taking homomorphic images and subgroups then $E(\mathcal{C})$ is the smallest class of groups containing $\mathcal{C}$ that is closed under all four elementary constructions (i)-(iv).*

Moreover, all elementary classes $E_\alpha(\mathcal{C})$ are closed under taking homomorphic images and subgroups.

This result is proved for the case of elementary amenable groups in [**22**] and in general in [**76**].

Proof (Proof of Theorem 7.15(a)). Let $\mathcal{SG}$ be the class of groups of sub-exponential growth and $\mathcal{C}$ be the closure of $\mathcal{SG}$ under taking subgroups and homomorphic images. We show that $\mathcal{B}$ does not belong to $E_\alpha(\mathcal{C})$ for any cardinal α.

By way of contradiction, assume that α is the smallest ordinal such that $\mathcal{B} \in E_\alpha(\mathcal{C})$. Since $\mathcal{B}$ has exponential growth, we have $\mathcal{B} \notin E_0(\mathcal{C})$ and since $\mathcal{B}$ is finitely generated it cannot be a directed union of its proper subgroups. Thus $\alpha = \beta + 1$ and there exists a short exact sequence

$$1 \to N \to \mathcal{B} \to Q \to 1$$

with N in $E_\beta(\mathcal{C})$. Since $\mathcal{B}$ is weakly branch over its commutator $\mathcal{B}'$, according to Theorem 5.3, there exists a level n such that

$$N \geq (\mathsf{RiSt}_{\mathcal{B}}(\mathcal{L}_n))' \geq (\mathcal{B}' \times \cdots \times \mathcal{B}')' \geq \mathcal{B}'' \times \cdots \times \mathcal{B}''.$$

Since the elementary class $E_\beta(\mathcal{C})$ is closed for subgroups and homomorphic images [**76**] we conclude that $\mathcal{B}''$ is in $E_\beta(\mathcal{C})$. The following observations (made in [**52**]) can then be used to show that $\mathcal{B}$ must also be in $E_\beta(\mathcal{C})$, leading to a contradiction.

Namely, we have $\mathcal{B}'' \leq \gamma_3(\mathcal{B}) \leq \mathsf{St}_{\mathcal{B}}(L_1)$ (here $\gamma_3(\mathcal{B}) = [[\mathcal{B},\mathcal{B}],\mathcal{B}]$), $\mathcal{B}''$ geometrically decomposes as

$$\mathcal{B}'' = \gamma_3(\mathcal{B}) \times \gamma_3(\mathcal{B}),$$

and we have the following projections

$$\begin{aligned} \varphi_1(\gamma_3(\mathcal{B})) &= \langle \gamma_3(\mathcal{B}), (a^2)^b \rangle \\ \varphi_1(\langle \gamma_3(\mathcal{B}), (a^2)^b \rangle) &= \langle \gamma_3(\mathcal{B}), (a^2)^b, b \rangle \\ \varphi_0(\langle \gamma_3(\mathcal{B}), (a^2)^b, b \rangle) &= \mathcal{B}. \end{aligned}$$

□

In fact the four classes EA, SA, AG and NF are separated even within the class of finitely presented groups, since there exists an HNN-extension

$$\begin{aligned} \tilde{\mathcal{B}} &= \langle\ \mathcal{B}, t \mid a^t = b,\ b^t = a^2\ \rangle \\ &= \langle\ a, t \mid [[a, a^t], a^t] = 1,\ a^{tt} = a^2\ \rangle \end{aligned}$$

of $\mathcal{B}$ that is finitely presented and amenable.

The group $\tilde{\mathcal{B}}$ has a balanced presentation on 2 generators and 2 relations, just as Thompson group

$$F = \langle\ a, b \mid b^{aa} = b^{ab},\ b^{aaa} = b^{abb}\ \rangle.$$

While it is known that F is in NF it is a long standing question if F is amenable.

Question 3. Is Thompson group F amenable?

We observe that F cannot be realized as a group of k-ary rooted tree automorphisms for any finite k, since F is not residually finite (the commutator of F is not trivial and is contained in all normal subgroups of F). However, F can be realized as a group of homeomorphisms of the boundary $\partial\mathcal{T}$ of the binary rooted tree. Moreover the action of F on $\partial\mathcal{T}$ can be given by a finite asynchronous automaton [**43**].

7.3. *Tarski numbers*

Definition 25. A finitely generated group G has a *paradoxical decomposition* if there exist a decomposition

$$G = A_1 \cup \cdots \cup A_m \cup B_1 \cup \cdots \cup B_n,$$

of G into a disjoint union of $m+n$ nonempty sets (with $m, n \geq 1$) and there exist elements a_i, $i = 1, \ldots, m$, and b_j, $j = 1, \ldots, n$, in G such that

$$G = a_1A_1 \cup \cdots \cup a_mA_m = b_1B_1 \cup \cdots \cup b_nB_n.$$

The smallest $m+n$ in a paradoxical decomposition of G is called the *Tarski number* of G. Groups that have no paradoxical decomposition have infinite Tarski number.

Denote the Tarski number of a group G by $\tau(G)$. The Tarski number of any group cannot be 3 or less. A proof of the following result of Tarski is provided in [**89**]. Another proof, based on Hall-Radon matching theorem, is provided in [**19**].

THEOREM 7.18 (Tarski). *A finitely generated group G is amenable if and only if it has a paradoxical decomposition.*

THEOREM 7.19. *A group G contains a copy of the free group F_2 if and only if its Tarski number is 4.*

THEOREM 7.20 (Ceccherini-Silberstein, Grigorchuk, de la Harpe [**19**]). *(a) There exists a 2-generated non-amenable torsion-free group G whose Tarski number satisfies*

$$5 \leq \tau(G) \leq 34.$$

(b) For $m \geq 2$ and odd $n \geq 665$, the Tarski number of the free Burnside group $B(m,n)$ satisfies the inequalities

$$5 \leq \tau(B(m,n)) \leq 14.$$

7.4. *Other topics related to amenability*

A large class of automaton groups in NF was constructed by Sidki. For a k-ary tree automorphism g define $\alpha_g(n)$ (see [**81**]), called the *activity number* at level n, to be the number of nontrivial sections at level n, $n = 0, 1, 2, \dots$. If a k-ary tree automorphism is generated by a state of a finite automaton, then the growth of α_g is either polynomial or exponential. Denote by Pol_k the set of k-ary tree automorphisms g for which α_g grows polynomially. This set forms a subgroup of $\mathsf{Aut}(\mathcal{T}^{(k)})$.

THEOREM 7.21 (Sidki [**81**]). *The group Pol_k does not contain free subgroups of rank 2, for any finite $k \geq 2$.*

It is easy to check if the growth of α_s is exponential or polynomial for a state s of a finite automaton $\mathcal{A}$. Each state t of $\mathcal{A}$ can be classified as active or non-active depending on whether π_t is non-trivial or trivial, respectively. The activity number $\alpha_s(n)$ is equal to the number of paths in the directed graph representing $\mathcal{A}$ starting at s and ending in an active state. A simple criterion, due to Ufnarovskiĭ [**85**], may be used to see that a k-ary automaton group $G(\mathcal{A})$ is in Pol_k if and only if every state of the automaton $\mathcal{A}$ appears as a vertex in at most one directed cycle from which an active state can be reached.

DEFINITION 26. An automaton $\mathcal{A}$ is *bounded* if all of its states have bounded activity growth.

EXAMPLE 14. All the states in the automaton generating Basilica group (see Example 2) have bounded activity growth. In fact, it is clear that $\alpha_a(n), \alpha_b(n) \leq 1$ for any n. Thus Basilica group is generated by a bounded automaton.

Similarly, the activity growth of all the states generating $\mathcal{G}$ is bounded.

On the other hand, the activity growth of every nontrivial state generating $\mathcal{H}^{(4)}$ is exponential (the two loops at each non-trivial state in the automaton in Figure 9 provide exponentially many paths leading to an active state).

THEOREM 7.22 (Bartholdi, Kaimanovich, Nekrashevych, Virág [**5**]). *All groups generated by bounded automata are amenable.*

The proof of the above result is based on self-similar random walks techniques developed by Bartholdi and Virág in [**16**], "Münchausen trick" of Kaimanovich [**58**] and embedding techniques of Brunner, Nekrashevych and Sidki reducing the problem, for each arity k, to a specific self-similar automaton group, called the *mother group*.

Greenleaf [**34**] asked if non-amenable groups can have amenable actions. An action of G on X is amenable if X admits a G-invariant mean. A group has the property ANA if it is non-amenable but admits a faithful transitive amenable action. It is shown in [**86**] that F_2 has the property ANA. On the other hand, it is known that groups with Kazhdan Property (T) never have the property ANA.

THEOREM 7.23 (Grigorchuk, Nekrashevych, [**35**]). *Every finitely generated, residually finite, non-amenable group embeds into a finitely generated, residually finite, non-amenable group that has faithful, transitive, amenable actions.*

Groups with the property ANA and other related properties were also studied by Monod and Glasner in [**32**], where many new amenable actions of non-amenable groups are introduced.

7.5. *Problems*

PROBLEM 16 ([**19**]). (a) What is the range of Tarski numbers?
(b) Give an example of a non-amenable group with explicitly determined Tarski number different from 4.

PROBLEM 17. Is Pol_k amenable?

PROBLEM 18. Are there hereditarily just infinite groups that are amenable but not elementary amenable?

PROBLEM 19. Are there non-amenable branch groups in NF?

PROBLEM 20. Are there non-amenable automaton groups that do not contain the free group F_2 of rank 2?

8. *Schreier graphs related to self-similar groups*

Let G be a finitely generated spherically transitive group of k-ary automorphisms, ξ be a ray in $\partial\mathcal{T}$ and, for $n \geq 0$, ξ_n be the unique level n vertex on ξ. In other words, ξ is an infinite word over X and, for $n \geq 0$, ξ_n is its prefix of length n. Further, let P_n be the stabilizer $P_n = P_{\xi_n} = \mathsf{St}_G(\xi_n)$ and P be the stabilizer $P = P_\xi = \mathsf{St}_G(\xi)$. The sequence of subgroups $\{P_n\}_{n=0}^{\infty}$ is decreasing to $P = \cap_{n=0}^{\infty} P_n$.

For a fixed symmetric generating set S the *Schreier graph* $\Gamma_n = \Gamma_n(G, P_n, S)$ is the graph whose vertices are the left cosets of P_n and in which, for each pair of a coset gP_n and a generator s in S, there is an edge from gP_n to sgP_n labeled by s.

Since the action of G is spherically transitive the graph Γ_n, for $n \geq 0$, is connected graph on k^n vertices that is isomorphic (independently of ξ) to the Schreier graph of the action of G on level n. The vertices of the Schreier graph of the action at level n are the vertices at level n and, for each vertex u and a generator s, there is an edge from u to $s(u)$ labeled by s. The isomorphism between the two graphs is given by $gP_{\xi_n} \leftrightarrow u$ if and only if $g(\xi_n) = u$.

On the other hand, since the group G is countable and $\partial\mathcal{T}$ is not, the action of G on $\partial\mathcal{T}$ is not transitive and the Schreier graph Γ_ξ is isomorphic to the connected component of the Schreier graph of the action of G on $\partial\mathcal{T}$ representing the G orbit of the ray ξ.

The following proposition is rather obvious in the light of the fact that tree automorphisms preserve prefixes.

Proposition 8.1. *For all $n \geq 0$, the map $\Gamma_{n+1} \to \Gamma_n$ given by*

$$wx \mapsto w$$

is a k-fold graph covering. The map $\Gamma \to \Gamma_n$ given by

$$\zeta \mapsto \zeta_n,$$

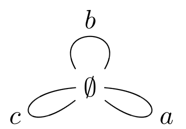
where ζ is an infinite word in the G orbit of ξ and, for $n \geq 0$, ζ_n is the prefix of ζ of length n, is also a graph covering.

Example 15. (Schreier graphs of $\mathcal{H}^{(3)}$) Let $G = \mathcal{H}^{(3)}$. The Schreier graph $\Gamma_3 = \Gamma_{000}$ corresponding to the action of $\mathcal{H}^{(3)}$ at level 3 is given in Figure 19.

The graphs Γ_n, as n grows, look more and more like the Sierpiński gasket (see Figure 15). We will see that this is not a random phenomenon and the reason for this is that the Sierpiński gasket is the Julia set of the map $f : z \to \bar{z}^2 - \frac{16}{27\bar{z}}$, whose iterated monodromy group is exactly $\mathcal{H}^{(3)}$.

We offer two recursive ways to build the graph Γ_{n+1} from Γ_n. Both are based on the similarity between these graphs.

The graph Γ_0 corresponding to the action at the root is given by



For $n \geq 0$, in order to build Γ_{n+1}, first build 3 copies of Γ_n, denoted by $\Gamma_{n,0}$, $\Gamma_{n,1}$ and $\Gamma_{n,2}$. The copy $\Gamma_{n,x}$ differs from Γ_n only by the fact that each vertex label u in Γ_n is replaced by ux in $\Gamma_{n,x}$. To get Γ_{n+1} delete, for each pair $x, y \in X_3$, $x \neq y$, the loops at $z^n x$ and $z^n y$ in $\Gamma_{n,x}$ and $\Gamma_{n,y}$, respectively, and replace them by a single edge labeled by a_{xy} connecting $z^n x$ and $z^n y$ (here z denotes the third letter in X_3 different from both x and y).

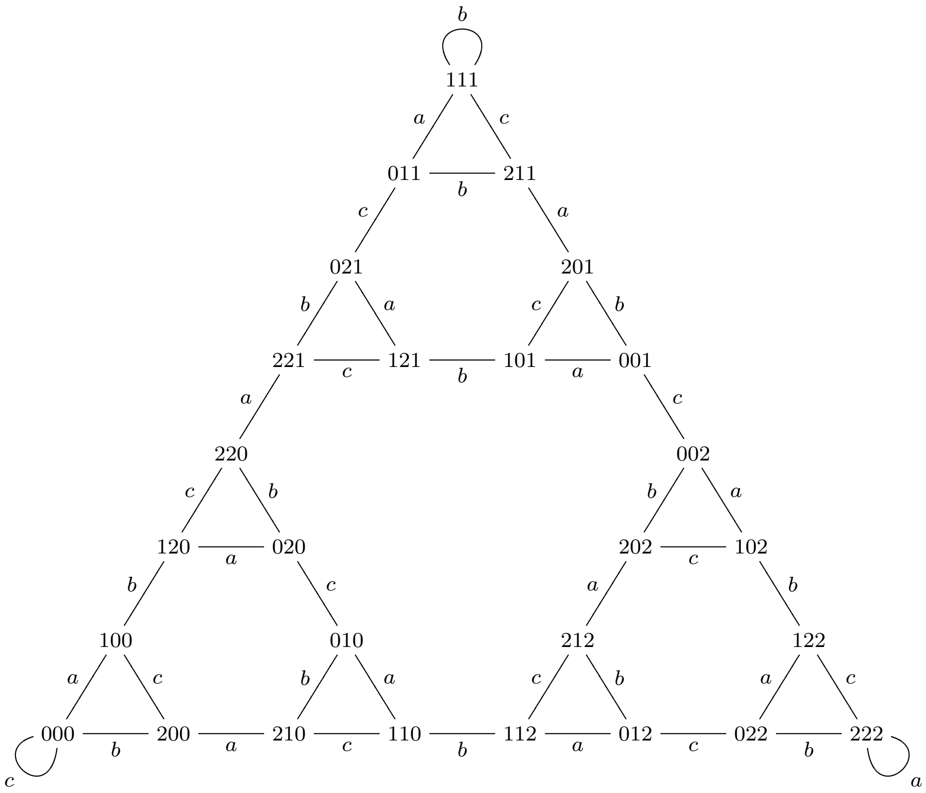
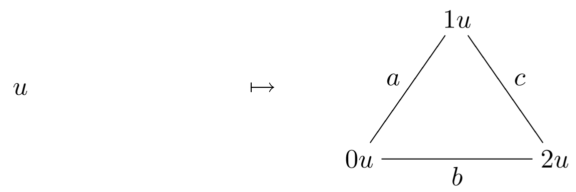


FIGURE 19. *The Schreier graph of $\mathcal{H}^{(3)}$ at level 3*

The second recursive way to build Γ_{n+1} from Γ_n is by graph substitution. The axiom is the graph describing Γ_0 corresponding to level 0 and the rules are



$$u \xrightarrow{\ a_{xy}\ } v \qquad \mapsto \qquad zu \xrightarrow{\ a_{xy}\ } zv$$

for each vertex and each edge in Γ_n. Given these rules Γ_{n+1} is built from Γ_n by replacing each occurrence of a vertex u in Γ_n by a triangle according to the first rule and replacing each occurrence of a labeled edge in Γ_n by a labeled edge given by the second rule connecting the indicated vertices in Γ_{n+1} (again, here z denotes the third letter in X_3 different from both x and y).

Example 16. (Schreier graphs of $\mathcal{G}$) The Schreier graphs of $\mathcal{G}$ corresponding to the first 3 levels are given in Figure 20

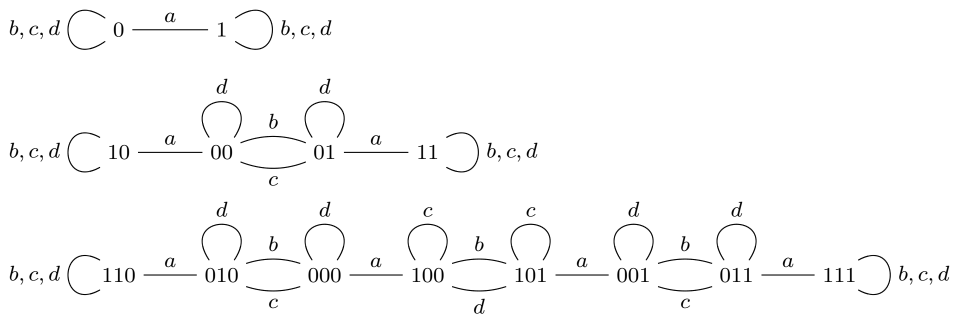
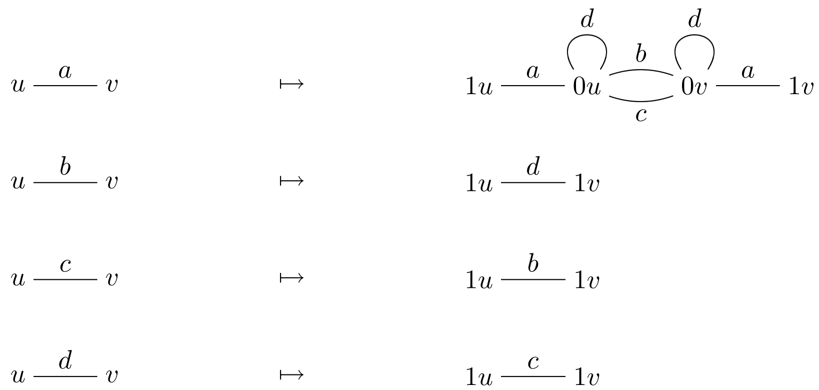


Figure 20. *The Schreier graphs of $\mathcal{G}$ corresponding to levels 1, 2 and 3*

The graphs Γ_n, $n \geq 1$, can be obtained by graph substitution as follows. The axiom is the graph Γ_1 and the rules are



Another description of Γ_{n+1} in terms of Γ_n is as follows. Build two copies $\Gamma_{n,0}$ and $\Gamma_{n,1}$ of Γ_n by adding 0 and 1, respectively, on the right of each vertex label in Γ_n. If $n = 3m+1$ delete the loops labeled by b and c at $1^{n-1}00$ and $1^{n-1}01$ in $\Gamma_{n,0}$ and $\Gamma_{n,1}$ and replace them by two edges labeled by b and c connecting $1^{n-1}00$ and $1^{n-1}01$. In a similar manner, if $n = 3m+2$ replace the loops labeled by b and d by two edges labeled by b and d connecting $1^{n-1}00$ and $1^{n-1}01$ and if $n = 3m$ do the same with the loops labeled by c and d.

Example 17. (The Schreier graphs of $\mathcal{B}$) The Schreier graph Γ_5 of $\mathcal{B}$ is given in Figure 21 (no loops are drawn on any vertex) The resemblance of the Schreier graphs of $\mathcal{B}$ to the Julia set of the polynomial $z \mapsto z^2 - 1$ (see Figure 4) is due to the fact that $IMG(z \mapsto z^2 - 1) = \mathcal{B}$.

8.1. *Contracting actions and limit spaces*

The results in this subsection touch on the phenomenon that Schreier graphs of many self-similar groups exhibit self-similarity features. In particular, when the

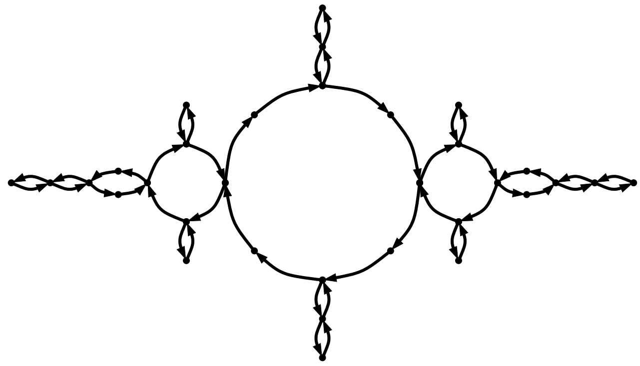


FIGURE 21. *Schreier graphs of Basilica group $\mathcal{B}$ at level 5*

group in question happens to be the iterated monodromy group of a post-critically finite rational map f on $\hat{\mathbb{C}}$ then the Schreier graphs approximate the Julia set of f.

DEFINITION 27. A self-similar group G, generated by a finite generating set S, is *contracting*, if there exists a constant $C \geq 0$ and integer $n \geq 1$ such that, for all elements g in G with $|g| \geq C$ and all their level n sections g_u, $u \in \mathcal{L}_n$,

$$|g_u| < |g|.$$

We note that all contracting groups are automaton groups. Indeed the above definition implies that the set of sections of every element in G is finite. Therefore one can easily define a finite automaton for each generator in S. Because of the self-similarity, the group generated by all the states in these automata is still just G.

We also note that the question whether a self-similar finitely generated group is contracting or not does not depend on the chosen generating set, i.e., being a contracting self-similar group is property of the group and not of its Cayley graph.

Denote by $X^{-\omega}$ the space of words over X that are infinite to the left. This space is canonically homeomorphic to $\partial\mathcal{T}$.

DEFINITION 28. Let G be a finitely generated self-similar contracting group. Define a relation of *asymptotic equivalence* $\asymp$ on $X^{-\omega}$ by

$$\dots x_3x_2x_1 \asymp \dots y_3y_2y_1$$

if and only if there exists a sequence $\{g_n\}_{n=0}^{\infty}$ of elements in G taking only finitely many different values in G such that, for $n \geq 0$,

$$g_n(x_n \dots x_1) = y_n \dots y_1.$$

The *limit space* of G, denoted $\mathcal{J}_G$, is the space $X^{-\omega}/\asymp$.

The following proposition shows that the Schreier graphs $\Gamma_n(G, S)$ approximate the limit space $\mathcal{J}_G$.

Proposition 8.2. *Let G be a finitely generated contracting self-similar group. Then*

$$\dots x_3x_2x_1 \asymp \dots y_3y_2y_1$$

if and only if there exists a constant $C \geq 0$ such that, for all $n \geq 0$, the distance between $x_n \dots x_1$ and $y_n \dots y_1$ in $\Gamma_n(G,S)$ is no greater than C.

Recall that the Julia set of a post-critically finite rational map f on $\hat{\mathbb{C}}$ is the closure of the set of repelling cycles of f.

Theorem 8.3 (Nekrashevych). *Let f be a post-critically finite rational map on $\hat{\mathbb{C}}$. Then the action of $IMG(f)$ is contracting and the limit space $\mathcal{J}_{IMG(f)}$ is homeomorphic to the Julia set of f.*

Thus the similarities between Schreier graphs of some self-similar groups and Julia sets of some post-critically finite rational maps that we already observed (compare Figure 15 and Figure 19, as well as Figure 4 and Figure 21) is due to the fact that the group G defining the Schreier graphs is also the iterated monodromy group of the corresponding map.

8.2. *Cayley and Schreier spectra*

Recall that, given a graph Γ, one can define a *Markov operator* M acting on the Hilbert space $\ell^2(\Gamma, deg)$ of square integrable functions with weight determined by the vertex degrees by

$$Mf(x) = \frac{1}{deg(x)} \sum_{y \sim x} f(y),$$

where the sum is taken over all neighbors of x. The Markov operator M is a self-adjoint operator of norm ≤ 1. If Γ is m-regular (i.e. all vertex degrees are equal to m) then M is a multiple of the *adjacency operator* (or matrix) usually used in discrete analysis. The operator M corresponds to the simple random walk on Γ in which the random walker is moving from a vertex x to any of its neighbors with equal probability. *Spectrum* of the graph Γ is the spectrum of the corresponding Markov operator M. To each vertex one can associate the spectral measure μ_v, whose moments coincide with the corresponding n-step return probabilities $p^n_{v,v} = \langle M^n\delta_v, \delta_v\rangle = \int_{-1}^{1} \lambda^n d\mu_v(\lambda)$, for $n = 0, 1, 2, \dots$. If Γ is vertex transitive then μ_v does not depend on v. This is the case, for example, when Γ is the Cayley graph of a group with respect to some finite system of generators. When we speak of a spectrum of a group we mean the spectrum of the Cayley graph with respect to some fixed finite symmetric system of generators. By *Cayley spectrum* of an automaton group $G = G(\mathcal{A})$ we mean the spectrum of the Cayley graph of G with respect to the standard generating set $S \cup S^{-1}$, where S is the set of states of $\mathcal{A}$, and by *Cayley spectral measure* $\mu_{\mathcal{A}}$ we mean the spectral measure of G.

It is quite difficult problem to study the spectrum of non-abelian groups because of a lack of well developed theory of representations of such groups. Many fundamental problems of mathematics have interpretations that relate them to particular questions about spectra and spectral properties. For instance, an example of a torsion free group with a gap in the spectrum (for some system of generators) would provide a counterexample to the famous Kadison-Kaplansky Conjecture on

Idempotents [**59**] (and consequently to Baum-Connes [**23**] and Novikov Conjecture [**31**]).

Groups generated by finite automata (or realizations of known groups by finite automata) lead to solution of difficult problems through methods and ideas based on self-similarity. The original ideas go back to the paper [**4**] where the first examples of regular graphs with Cantor spectra were given. The realization of the lamplighter group $L_2 = (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ as automaton group was used in [**51**] to calculate the spectrum of L_2, which turned out to be the first example of a group with pure point spectrum. It also led to a counterexample to the Strong Atiyah Conjecture on L^2-Betti numbers.

THEOREM 8.4 (Grigorchuk, Linell, Schick, Żuk [**50**]). *There exists a 7-dimensional manifold $\mathcal{M}$ such that all torsion elements in the fundamental group $G = \pi(\mathcal{M})$ have order dividing 2, but for which the second L^2-Betti number is $\frac{7}{3}$.*

Another way to attach a spectrum to a self-similar group G generated by a finite symmetric set S is as follows. Consider the Schreier graph $\Gamma_\xi = \Gamma(G, P_\xi, S)$. The graph Γ_ξ depends on ξ but in case of a spherically transitive action of G on $\mathcal{T}$ all graphs Γ_ξ are locally isomorphic and the spectrum (as a set) does not depend on ξ. In such a situation we call the spectrum of the graph Γ_ξ the *Schreier spectrum* of G and denote it by $\mathsf{spec}(\Gamma)$.

In some cases the Schreier spectrum coincides with the Cayley spectrum and such cases are of special interest since it is usually easier to calculate the Schreier spectrum. For instance this happens whenever P_ξ is trivial or cyclic (as happened for the lamplighter group [**51**]).

We note that the Schreier graphs Γ_ξ may be far from vertex transitive (and may even have trivial automorphism group as in the case of $\mathcal{G}$). Therefore one has to pay attention to the possibility of having spectral measures μ_v depending on v. There is hope that all μ_v would at least be in the same measure class, but there are no results in this direction. The so called *KNS (Kesten - von Neumann - Serre) spectral measure* ν appears in the study of μ_v, as introduced in [**4**] and studied further in different situations in [**51**, **53**, **83**].

DEFINITION 29. Let G be a self-similar group of k-ary tree automorphisms generated by a symmetric set S and let ξ be a point on the boundary $\partial\mathcal{T}$. For an interval I in $[-1, 1]$ define the *KNS spectral measure* by

$$\nu(I) = \lim_{n\to\infty} \frac{\#_n(I)}{k^n},$$

where $\#_n(I)$ denotes the number of eigenvalues of Γ_n in the interval I.

At the moment there are more complete calculations involving the Schreier spectrum than the Cayley spectrum. We provide the full description in case of $\mathcal{G}$, the Gupta-Sidki 3 group and $\mathcal{H}^{(3)}$. The calculations are based on the following result.

THEOREM 8.5 (Bartholdi, Grigorchuk [**4**]). *Let either the Schreier graph $\Gamma = \Gamma(G, \xi, S)$ or the parabolic subgroup P_ξ be amenable. Then*

$$\mathsf{spec}(\Gamma) = \overline{\bigcup_{n=0}^{\infty} \mathsf{spec}(\Gamma_n)}.$$

THEOREM 8.6 (Bartholdi, Grigorhuk [**4**]). *(a) The n-th level spectrum $\mathsf{spec}(\Gamma_n)$ of $\mathcal{G}$ with respect to $\{a,b,c,d\}$ is, as a set, equal to*

$$\left\{1 \pm \sqrt{5 + 4\cos\theta} \mid \theta \in \frac{2\pi\mathbb{Z}}{2^n}\right\} \setminus \{0, -2\}.$$

The Schreier spectrum of $\mathcal{G}$ is equal to

$$[-2, 0] \cup [2, 4].$$

(b) The Schreier spectrum of the Gupta-Sidki 3-group G with respect to $\{a, a^{-1}, b, b^{-1}\}$, given by

$$\begin{aligned} a &= (012)(1,\ 1,\ 1) \\ b &= \qquad (a, a^{-1}, b) \end{aligned}$$

is equal to the closure of the set

$$\left\{ \begin{array}{c} 4, \\ -2, \\ 1, \\ 1 \pm \sqrt{\frac{9 \pm 3}{2}}, \\ 1 \pm \sqrt{\frac{9 \pm \sqrt{45 \pm 4 \cdot 3}}{2}}, \\ 1 \pm \sqrt{\frac{9 \pm \sqrt{45 \pm 4 \cdot \sqrt{45 \pm 4 \cdot 3}}}{2}}, \\ \dots \end{array} \right\}.$$

The spectrum is a Cantor set symmetric with respect to 1.

THEOREM 8.7 (Grigorchuk, Šunić [**48**]). *The n-th level spectrum of $\mathcal{H}^{(3)}$ with respect to $\{a,b,c\}$, as a set, has $3 \cdot 2^{n-1} - 1$ elements and is equal to*

$$\{3\} \ \cup \ \bigcup_{i=0}^{n-1} f^{-i}(0) \ \cup \ \bigcup_{j=0}^{n-2} f^{-j}(-2),$$

where f is the polynomial function $f(x) = x^2 - x - 3$.

The multiplicity of the 2^i level n eigenvalues in $f^{-i}(0)$, $i = 0, \dots, n-1$, is $\frac{3^{n-1-i}+3}{2}$ and the multiplicity of the 2^j eigenvalues in $f^{-j}(-2)$, $j = 0, \dots, n-2$, is $\frac{3^{n-1-i}-1}{2}$.

The Schreier spectrum of $\mathcal{H}^{(3)}$, as a set, is equal to

$$\overline{\{3\} \ \cup \ \bigcup_{i=0}^{\infty} f^{-i}\{0, -2\}} = \overline{\bigcup_{i=0}^{\infty} f^{-i}(0)}.$$

It consists of the set of isolated points $I = \bigcup_{i=0}^{\infty} f^{-i}(0)$ and its set of accumulation points J, which is the Julia set of the polynomial $f(x) = x^2 - x - 3$ and is a Cantor set.

The KNS spectral measure is discrete and concentrated on the the set of eigenvalues in $\cup_{i=0}^{\infty} f^{-i}\{0,-2\}$. *The KNS measure of the eigenvalues in* $f^{-i}\{0,-2\}$ *is* $\frac{1}{6\cdot 3^i}$, $i = 0, 1, \ldots$.

The spectra of Γ_n are calculated by using operator recursion induced by the self-similarity of the group in question. We illustrate this approach in the case of $\mathcal{H}^{(3)}$.

The action of $\mathcal{H}^{(3)}$ on level n of the ternary tree induces permutational representations of dimension 3^n, recursively defined by

$$a_0 = b_0 = c_0 = [1]$$

$$a_{n+1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & a_n \end{bmatrix}$$

$$b_{n+1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & b_n & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$c_{n+1} = \begin{bmatrix} c_n & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

where 0 and 1 are the zero and the identity matrix, respectively, of size $3^n \times 3^n$.

The matrix $\Delta_n = a_n + b_n + c_n$ is the adjacency matrix of Γ_n and it satisfies the recursive relation

$$\Delta_0 = [3]$$
$$\Delta_{n+1} = \begin{bmatrix} c_n & 1 & 1 \\ 1 & b_n & 1 \\ 1 & 1 & a_n \end{bmatrix}$$

For $n \geq 1$ and real numbers x and y define $\Delta_n(x, y)$ to be the $3^n \times 3^n$ matrix given by

$$\Delta_n(x,y) = \begin{bmatrix} c - x & y & y \\ y & b - x & y \\ y & y & a - x \end{bmatrix}.$$

Let $D_n(x, y) = \det(\Delta_n(x, y))$. We first find the set of points in the plane for which the matrix $D_n(x, y)$ is not invertible. Call this set the *auxiliary spectrum*. The spectrum we are interested in is the intersection of the auxiliary spectrum with the line $y = 1$.

We first determine a recursive formula for $D_n(x,y)$. The matrix

$$\Delta_n(x,y) = \begin{bmatrix} c-x & 0 & 0 & y & 0 & 0 & y & 0 & 0 \\ 0 & -x & 1 & 0 & y & 0 & 0 & y & 0 \\ 0 & 1 & -x & 0 & 0 & y & 0 & 0 & y \\ y & 0 & 0 & -x & 0 & 1 & y & 0 & 0 \\ 0 & y & 0 & 0 & b-x & 0 & 0 & y & 0 \\ 0 & 0 & y & 1 & 0 & -x & 0 & 0 & y \\ y & 0 & 0 & y & 0 & 0 & -x & 1 & 0 \\ 0 & y & 0 & 0 & y & 0 & 1 & -x & 0 \\ 0 & 0 & y & 0 & 0 & y & 0 & 0 & a-x \end{bmatrix}$$

is row/column equivalent to

$$\begin{bmatrix} c-x' & y' & y' & 0 & 0 & 0 & 0 & 0 & 0 \\ y' & b-x' & y' & 0 & 0 & 0 & 0 & 0 & 0 \\ y' & y' & a-x' & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & P_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & P_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

where

$$x' = \frac{x^4 - x^3 - x^2 + x + (x^2 - x^3)y + (2x - 3x^2)y^2 + xy^3 + 2y^4}{(x-1-y)(x^2-1+y-y^2)},$$

$$y' = \frac{y^2(x-1+y)}{(x-1-y)(x^2-1+y-y^2)}.$$

and

$$P_4P_5P_6y = (x^2-(1+y)^2)^{3^{n-2}}(x^2-1+y-y^2)^{2\cdot 3^{n-2}}.$$

Direct calculation shows that

$$D_1(x,y) = -(x-1-2y)(x-1+y)^2$$

and, for $n \geq 2$,

$$D_n(x,y) = (x^2-(1+y)^2)^{3^{n-2}}(x^2-1+y-y^2)^{2\cdot 3^{n-2}} D_{n-1}(F(x,y)),$$

where $F(x,y)$ is the two-dimensional rational map given by $F(x,y) = (x',y')$. Define a transformation $\Psi : \mathbb{R}^2 \to \mathbb{R}$ by

$$\Psi(x,y) = \frac{x^2 - 1 - xy - 2y^2}{y}$$

and a transformation $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x^2 - x - 3.$$

It is easy to check that

$$\Psi(F(x,y)) = f(\Psi(x,y)),$$

i.e., the diagram

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "F"] \arrow[d, "\Psi"'] & \mathbb{R}^2 \arrow[d, "\Psi"] \\ \mathbb{R} \arrow[r, "f"'] & \mathbb{R} \end{tikzcd}$$

commutes. Thus the two-dimensional rational map F is semi-conjugate, in a non-trivial way, to the one-dimensional map f and this is precisely what makes all calculations possible.

For $n \geq 2$, the auxiliary spectrum consists of 3 lines and $3 \cdot 2^{n-2} - 2$ hyperbolas. For example, for $n = 5$ the auxiliary spectrum is given in Figure 22.

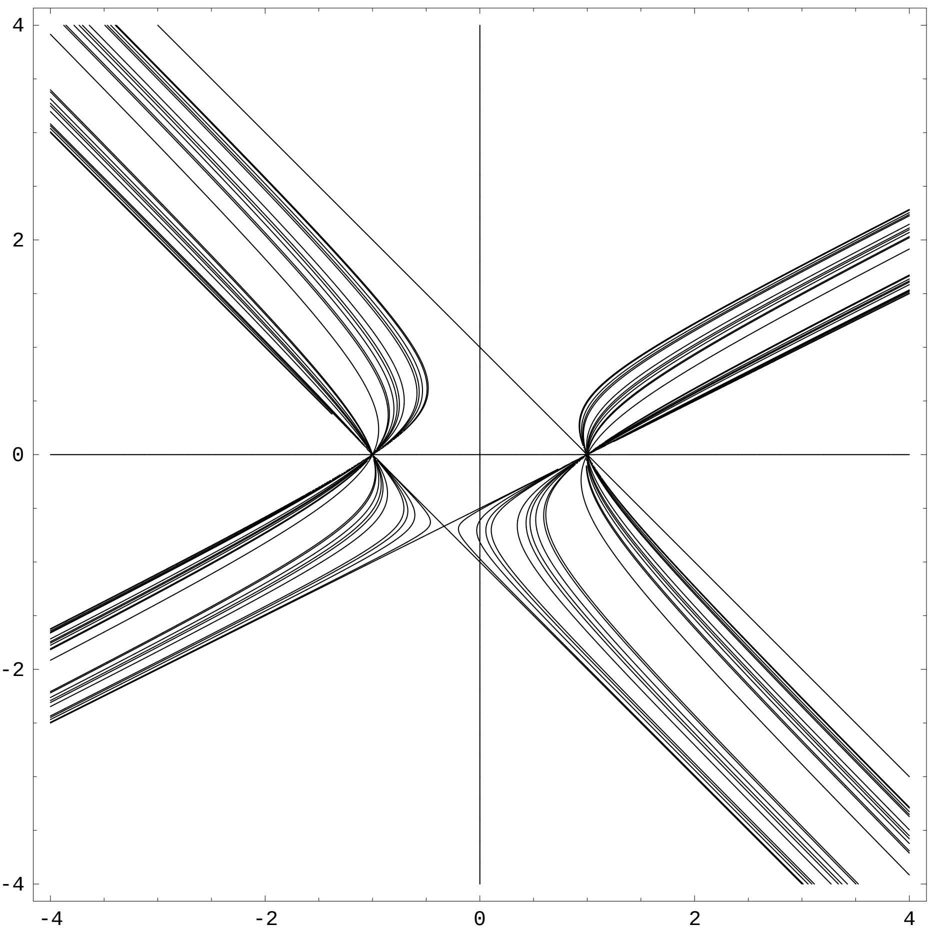


FIGURE 22. *The auxiliary spectrum of* Γ_5 *for* $\mathcal{H}^{(3)}$

8.3. *Problems*

Problem 21 ([**43**]). (a) Does there exist an algorithm that, given a finite automaton $\mathcal{A}$, and a recursively defined ray ξ, decides if the parabolic subgroup P_ξ of $G(\mathcal{A})$ is trivial?

(b) Does there exist an algorithm that, given a finite automaton $\mathcal{A}$, decides if there exists a parabolic subgroup P_ξ in $G(\mathcal{A})$ that is trivial?

(c) Does there exist an algorithm that, given a finite automaton $\mathcal{A}$ and a recursively defined ray ξ, decides if Γ has polynomial growth?

Problem 22. Describe the possible types of growth of the Schreier graph Γ for automaton groups.

Problem 23 (Nekrashevych [**71**]). Are all contracting groups amenable?

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