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Solved and Unsolved Problems Around One Group

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0 Introduction

Back in 1979, interested in growth, amenability and torsion property of finitely generated groups, I discovered a construction which produced several groups with very similar basic properties. Two of them, G and H, were described in the note [Gri80]. Both groups are infinite finitely generated torsion groups and thus belong to the class of Burnside groups [Adi79, GL02, Gup89].

In 1982 I proved that the first of these groups, G, has growth between polynomial and exponential, hence is amenable but not elementary amenable [Gri84a]. This group therefore provided simultaneous answers to the question of J. Milnor [Mil68] on existence of groups of intermediate growth, and to the question of M.

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Day [Day57]) on existence of amenable but not elementary amenable groups. Main structural properties of the groups G and H were described already in the note [Gri80], and later it started to become clear that these groups are examples of groups in two classes that we now call self-similar groups (or automaton groups) and branch groups.

Automaton groups are groups generated by the states of a finite invertible Mealey automaton. They constitute a subclass of the class of groups of finite automata which was defined in 1963 [Hoř63] and studied during the next two decades, mostly by Ukrainian and Russian mathematicians, with the summary of the results obtained in the first decade published in the book [GP72]. It was already known at that time that they are closely related to subgroups of iterated wreath products introduced by L. Kaloujnin and studied by P. Hall and others. One of the main tools of working with such groups was the method of tables developed by L. Kaloujnin [Kal45, Kal48] and his students V.I. Sushchanskii, Yu.V. Bodnarchuk, and others. Classical methods from computer science (mostly theory of finite automata) were also useful in some investigations (e.g. [Ale72]). The study of the class of automaton groups intensified in the beginning of the 1980-ies after examples appeared in [Gri80] and [GS83], together with new techniques featuring extensive use of self-similarity, projections and contracting.

Originally the groups G and H were defined as groups acting on the interval and on the square, respectively, by measure preserving transformations, but it was also clear that they act on a rooted tree. The construction which appeared in [GS83] used this model, and this point of view now dominates the field. The discovery of Gupta-Sidki p-groups played an extremely important role in the development of the theory of self-similar groups and branch groups; many fundamental properties were established for them.

It was observed already in the pioneering papers [Gri80, GS83] that such groups have unusual subgroup structure which is closely related to the structure of the tree on which the groups act. It was however not until 1997 that this was formalized in the form of a definition, thus bringing us to the notion of a branch group [Gri00a]. The importance of the class of branch groups lies in the fact that they constitute one of three classes into which the class of just infinite groups (i.e., infinite groups with finite proper quotients) naturally splits [Gri00a].

Every infinite finitely generated group has a just infinite quotient. Thus if a group property $\mathcal{P}$ is preserved under homomorphisms (such as torsion, subexponential growth, amenability, finite width, bounded generation, etc.) and there exists an infinite finitely generated group with the property $\mathcal{P}$ then there exists a a just infinite group with the same property. This, in part, explains why, during the last two decades, many examples with remarkable properties were found in the class of groups of branch type.

It was clear from the beginning that the self-similarity phenomenon observed in groups has relations to the self-similarity phenomena arising in other areas of mathematics.

Recent investigations perfectly confirm this. For example, it was observed that transitive actions on compact ultrametric spaces are isomorphic to actions on boundaries of spherically homogeneous rooted trees [GNS00]. This observation provides links to cyclic renormalization and attractors in dynamics [BOERT96], to holomorphic dynamics [Nek03, BGN03, Nek05] , to Grothendick designs [Pil00], to operator algebras [BG00a, Nek04], to fractal geometry [BGN03] and to other areas of mathematics.

New ideas came with the new generation of mathematicians. The idea of self-similarity was developed wonderfully in the work of V. Nekrashevych whose results are collected in the book [Nek05]. Different spaces such as limit space, solenoid, hyperbolicity complex are associated with a self-similar group, which lead naturally to appearance of dynamical systems related to the group. An iterated monodromy group, denoted $IMG(f)$, corresponds to a self-covering map f of a topological space (under certain conditions). In many cases, when a rational map of $\mathbb{C}$ is iterated, the corresponding group is self-similar and the asymptotics of different combinatorial objects that can be associated with the group (first of all, of the associated Schreier graphs) is related to the geometry of the Julia set and other objects arising in dynamics [Nek05]. L. Bathtoldi developed a theory of algebras of branch type and applied it in the study of different types of growth. E. Pervova proved that maximal subgroups in many self-similar groups of branch type have finite index and that there are branch p-groups without the congruence subgroup property. A. Erschler developed a boundary theory of self-similar groups and obtained fascinating results concerning the growth and amenability. B. Virag converted the concept of self-similarity in groups into the self-similarity of random walks, thus opening a new page in theory of random walks on groups. Z. Sunik found strong links of self-similar groups to problems in combinatorics. For example he showed that some of the famous Hanoi towers problems can be modelled by branch groups.

Surprisingly, dynamical systems appear in different ways in self-similar and in branch groups. For instance, in [BG00b] they appear in connection with the study of the spectral problem for the discrete Laplace operator, while in [Nek03, BGN03, Nek05] they come from holomorphic dynamics. Also, the work [Lys85] and the geometry of Schreier graphs considered in [BG00a, BGN03] naturally led to consideration of substitutional dynamical systems related to self-similar groups.

The ideas of self-similarity and branching penetrated also into the theory of profinite groups [Gri00a]. There are indications that they may be useful in number theory [Bos00] and in Galois theory [AHM].

The number of results in the field and new connections to other areas of mathematics is growing so fast that a few books are already necessary to cover all the material. The number of open questions grows equally fast.

In this paper I discuss some topics related to the group G and both to the class of self-similar (or automaton) groups and the class of of branch groups. I recommend the paper [CSMS01] and Chapter VIII of the beautiful book [dlH00] by Pierre de la Harpe as sources for easy and quick introduction to the group G. The group H happened to be less lucky and there are very few publications related to it, one of which is [Vov00].

Many relevant topics are not included in this article, such as Burnside groups, associated algebras, random walks, zeta functions, to name but a few. I recommend [Sid98, Gri00a, CSMS01, GNS00, dlH00, BGŠ03, Nek05] as publications where much of additional material can be found.

The structure of the paper is as follows.

We start with the original definition of the group G followed by the definition via its action on the binary tree, and finally we present the group as a group generated by states of a five state automaton. We also define the class of self-similar groups and discuss its basic properties. One of important notions of the theory of groups acting on rooted trees is the notion of a portrait which we define in the first Section and which appears later, with modifications, in Section 4.

In Section 2 we define the family G_ω which was constructed in [Gri84a] as a generalization of the main example. The construction was created to show that the cardinality of the rates of growth of groups of intermediate growth is uncountable. As a byproduct it suggested introducing a topology on the space of finitely generated groups which we discuss in this Section as well.

In Section 3 we discuss properties of the word length function defined via canonical set of generators, the main of which is the contracting property. Self-similar contracting groups constitute very important subclass and often appear in applications.

In Section 4 and Section 5 we discuss algorithmic problems, especially the word problem and the conjugacy problem. The algorithms presenting solutions of these problems are of branch type and are quite elegant. But in contrast to the algorithm for solution of the word problem the algorithm solving the conjugacy problem is hard for practical realization. Surprisingly, the last fact is very useful from the point of view of modern commercial cryptography, which is looking for groups with easy word problem but difficult conjugacy problem.

At some point L-presentations come into the picture and we discuss a method of finding them. Also, based on the construction G_ω we discuss how to estimate the measure of complexity of the word problem in terms of Kolmogorov complexity.

There are very few results in direction of solving the isomorphism problem, and we mention some of them at the end of Section 5.

Section 6 deals with subgroups of G and features a brief introduction to branch groups. It contains comprehensive information about subgroups of G and touches a little bit on topics such as the congruence subgroup property and maximal subgroups.

Section 7 is about the sequence of finite quotients of G arising as restrictions of the action on levels of the tree. This sequence contains not only a lot of information about the group but it also contains all the information about the profinite closure of G. Topological groups appear here for the first time and don't stay for long. However, branch profinite groups and self-similar profinite groups are playing more and more important role in the latest investigations. Analysis of the structure of portraits of elements of G leads us to the notion of profinite group of finite type, the portraits of elements of which can be described by a finite set of forbidden configurations. This bears analogy with sub-shifts of finite type [Kit98]. Here group theory meets ergodic theory once again.

In Sections 8 and 9 we discuss some topics related to growth and amenability. Though it has been known for more than two decades that G has intermediate growth and is amenable, very little is known about the asymptotics of its growth, about growth of the associated Fölner function, and about the structure of the Cayley graph of the group. We recall some known facts, formulate some recent results and propose a number of open questions.

Some topics related to presentations of G and other groups acting on rooted trees are discussed in Section 10. Most of the attention is paid to the representation determined by the action of the group on the boundary of a tree with the uniform measure which is invariant under the whole group of tree automorphisms. Also we touch on Kaplansky Conjecture on Jacobson radical and explain why G is a candidate to solve it.

In next section we introduce two C^*-algebras which can naturally be attached to any group acting on a rooted tree and which are closures in norm topology of unitary representations considered in the previous section. The idea to use such algebras in the study of self-similar groups appeared in [BG00a], and further progress was made in [GŻ01] and [Nek04]. The C^*-algebras arising in this way also have self-similar properties, and this is discussed in detail in [Nek05]. We believe that these algebras may open a new page in the theory of operator algebras while the information about their properties will be a powerful tool in the study of self-similar groups.

Schreier graphs are the topic of Section 12. Although Schreier's construction is classics, Schreier graphs have not played a large role in group theory until recently.

The works [BG00a, BGN03] and [Nek05] show that in some problems Schreier graphs are more important than the Cayley graphs, and many questions arise about combinatorics, geometry and analysis on such graphs. Schreier graphs related to self-similar groups are often substitutional graphs which are limits of sequences of finite graphs obtained by iterating a substitutional rule.

Topics discussed in Section 12 and Section 13 include spectral properties, amenability, growth, growth of diameters, and expanding properties of Schreier graphs.

In the last Section we consider the spectral problem for discrete Laplace operator on Schreier graphs, for the group G, for the lamplighter group and for the Basilica group. Our wish is to have a solution of the spectral problem for all self-similar groups. Unfortunately there are only very few cases so far where such a solution has been obtained. We describe roughly the method which was used in all of the solved cases. It relates the spectral problem to the problem of description of invariant subsets for multidimensional rational maps, and to the question of weak form integrability of such maps.

An interesting object that appears in the spectral problem is the so called KNS-spectral measure introduced in [BG00a]. Developing new methods for solution of the spectral problem for other self-similar groups is a matter of extreme importance, and we formulate a number of questions in this direction.

The article contains a large number of open problems. Most of them are formulated rigorously, but sometimes we choose rather to express a wish for certain directions to be pursued or for certain circles of problems to be taken into consideration. Some of them are original but most already appeared in [GNS00, BGŠ03, BGN03, dlH00, dlH04, kou02] and other articles and books.

We hope the reader will not be disappointed.

0.1 Acknowledgments

The author would like to express his gratitude to Tatiana Smirnova-Nagnibeda and Zoran Sunik for for their unselfish contribution and fruitful discussions on the topics touched in the text. In addition, thanks to V. Nekrashevychg, E. Pervova and L. Bartholdi for their valuable input improving the quality, correctness and the readability of the text.

0.2 Notation

Throughout the paper we will fix G as the notation for the first group from [Gri80], while the letter H and other letters will be used for different purposes in different parts of the article.

0.3 Terminology

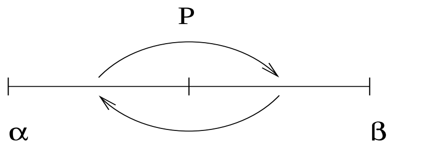
The subject of self-similar groups is young and the terminology is not yet fixed. In this article, by a self-similar group I mean the group generated by the states of a finite invertible automaton. This is different from [Nek05] where finiteness of the automaton is not assumed. There are also some other, less important, differences in terminology.

0.4 Support

1 The group G and self-similarity

We start with different definitions of the main object of this article, the group G. The first definition that we will give here is the original one which appeared in [Gri80]. Denote by Δ the set obtained from $[0,1]$ by removing the set R of dyadic rational points $\frac{k}{2^m}$, $k, m \in \mathbb{Z}$. (All intervals in this section are of the form $(\alpha,\beta)\backslash R$.) The group G acts on Δ by Lebesgue measure preserving transformations (we consider the right action). Let us agree that the letter I written above an interval (α,β) denotes the identity transformation on (α,β) while the letter $\mathbf{P}$ denotes the permutation of two halves:

$$x^{\mathbf{P}} = \begin{cases} x + \dfrac{\beta-\alpha}{2} & \text{if } 0 < x < \alpha + \dfrac{\beta-\alpha}{2} \\ x - \dfrac{\beta-\alpha}{2} & \text{if } \alpha + \dfrac{\beta-\alpha}{2} < x < \beta \end{cases}$$



The group G is the group generated by four transformations a, b, c, d of Δ defined as with the operation

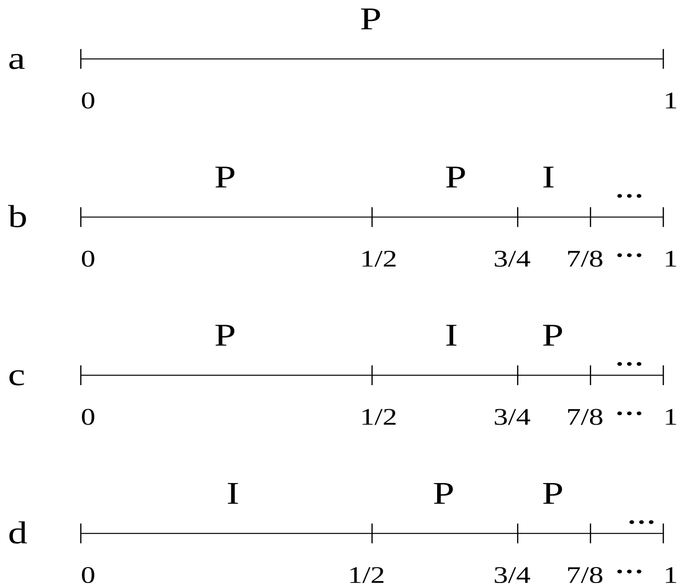


Figure 1.1:

of composition of transformations. The transformations b, c, d are defined by infinite periodic sequences

$$\begin{matrix} \mathbf{P} & \mathbf{P} & \mathbf{I} & \mathbf{P} & \mathbf{P} & \mathbf{I} & \cdots \\ \mathbf{P} & \mathbf{I} & \mathbf{P} & \mathbf{P} & \mathbf{I} & \mathbf{P} & \cdots \\ \mathbf{I} & \mathbf{P} & \mathbf{P} & \mathbf{I} & \mathbf{P} & \mathbf{P} & \cdots \end{matrix} \tag{1.1}$$

which we can abbreviate as $(\mathbf{PPI})^\infty$, $(\mathbf{PIP})^\infty$, $(\mathbf{IPP})^\infty$, and the endpoints of the intervals are of the form $1 - \frac{1}{2^n}$, $n = 1, 2, \ldots$.

It is obvious that a, b, c, d satisfy the following relations

$$\begin{aligned} a^2 = b^2 = c^2 = d^2 = 1 \\ bc = cb = d \\ bd = db = c \\ cd = dc = b \end{aligned} \tag{1.2}$$

where $1 \in G$ is the identity element, so G is indeed a 3-generated group. But for most of our considerations we will use all four generators. G has many other relations, for instance,

$$1 = (ad)^4 = (ac)^8 = (ab)^{16} = (adacac)^4 = \cdots$$

and is not a finitely presented group. Later we will discuss in more details the set of relators for G.

We now proceed to give another definition of the group G. To do so, let us identify the points $x \in \Delta$ with infinite binary sequences

$$x \leftrightarrow x_0 x_1 \dots x_n \dots,$$

$x_i \in \{0, 1\}$, $i = 0, 1, \dots$, where $0.x_0x_1\dots$ is the binary expansion of the number $x \in \Delta$. The space $\{0, 1\}^{\mathbb{N}}$ of such sequences equipped with the Tychonoff topology (the topology of coordinate-wise convergence) is homeomorphic to the Cantor set.

The action of G on Δ transforms into the action by homeomorphisms on $\{0, 1\}^{\mathbb{N}}$ defined recursively as

$$\begin{array}{ll} (xw)^a = \bar{x}w & \\ (0w)^b = 0w^a & (1w)^b = 1w^c \\ (0w)^c = 0w^a & (1w)^c = 1w^d \\ (0w)^d = 0w & (1w)^d = 1w^b \end{array} \tag{1.3}$$

where $x \to \bar{x}$ is the permutation on $\{0, 1\}$

$$\varepsilon\colon \begin{cases} 0 & \to & 1 \\ 1 & \to & 0 \end{cases}$$

$w \in \{0, 1\}^{\mathbb{N}}$ and xw stands for concatenation of x and w.

This action is self-similar in the sense that for any $g \in G$ and $x \in \{0, 1\}$ there are $h \in G$ and $y \in \{0, 1\}$ such that for any $w \in \{0, 1\}^{\mathbb{N}}$

$$(xw)^g = yw^h. \tag{1.4}$$

The relations (1.3) also define an action of G on the set $\{0, 1\}^*$ of finite words over the alphabet $\{0, 1\}$. We will view $\{0, 1\}^*$ as a free monoid generated by the symbols 0 and 1 (the empty word corresponds to the identity element). This monoid is in natural bijection with the vertices of the binary rooted tree T_2 embedded in the plane as shown in Figure 1.2:

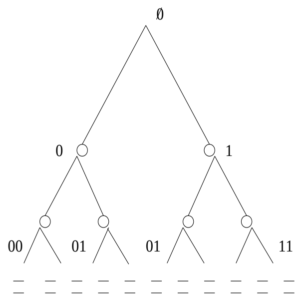


Figure 1.2:

The set V of vertices splits into a disjoint union

$$V = \bigsqcup_{n=0}^{\infty} V_n,$$

where the root vertex corresponds to the empty set; and for $n \geq 1$, V_n is the set of vertices at the n-th level (i.e., the set of vertices at combinatorial distance n from the root vertex). The vertices of the the n-th level are ordered from left to right in the lexicographic order on $\{0,1\}^*$ (with the standard agreement that $0 < 1$).

Any element $g \in G$ induces a bijection on V. Moreover, this bijection preserves the structure of the tree (this can been easily seen if one considers G as a group generated by the states of a finite automaton, as below, or it can be seen directly from (1.3)). In this way we get an action of G by automorphisms of the rooted binary tree. The root vertex is the fixed point and the levels V_n are the invariant sets for the action of G.

Let now $T = T_d$, $d \geq 2$, be the d-regular rooted tree whose vertices can be identified with the elements of the free monoid generated by any set X of cardinality d. Let $X = \{x_1, \ldots, x_d\}$ and let $\mathcal{G}$ be the full group of automorphisms of the tree (we consider the right action on the tree). There is a natural embedding (in fact isomorphism)

$$\psi\colon \mathcal{G} \to \mathcal{G} \wr_X S_d \tag{1.5}$$

with $\wr_X$ denoting the permutational wreath product, so that the right-hand side in (1.5) is the semidirect product

$$(\underbrace{\mathcal{G} \times \cdots \times \mathcal{G}}_{d}) \rtimes S_d$$

where S_d acts on the direct product by permuting the factors. If $g \in \mathcal{G}$ and

$$\psi(g) = (g_1, \ldots, g_d)\alpha \tag{1.6}$$

$g_1, \ldots, g_d \in \mathcal{G}, \alpha \in S_d$, then α is the permutation of X induced by the action of g on the first level of the tree, and $g_1, \ldots, g_d$ are the projections of g on subtrees with roots at the first level. Using the canonical identification of any of these subtrees with T allows to view the projections g_i as elements of $\mathcal{G}$.

For instance, the map ψ acts on the generators of G by

$$\psi\colon \begin{cases} a \to (1,1)\epsilon \\ b \to (a,c)e \\ c \to (a,d)e \\ d \to (1,b)e \end{cases}$$

or in simpler notation

$$\psi\colon \begin{cases} a \to \varepsilon \\ b \to (a,c) \\ c \to (a,d) \\ d \to (1,b) \end{cases} \tag{1.7}$$

where e and ε are the trivial and the nontrivial element, respectively, in the symmetric group S_2.

The embedding (1.5) leads to the notion of a portrait $P(g)$ of an automorphism $g \in \mathcal{G}$. By a portrait we mean a labelling of the vertices of the tree by elements of the symmetric group S_d, defined as follows.

Using the decomposition (1.5) we label the root vertex by the element α. For each projection g_i we do the same. Namely we decompose g_i as

$$\psi(g_i) = (g_{i1}, \ldots, g_{id})\alpha_i$$

and label the i-th vertex of the first level by α_i. Then we repeat this for projections g_{ij} for the second level etc. There is a bijection between the elements of Aut T and portraits and the set of portraits is the set S_d^V of functions $V \to S_d$ (or labellings), where $V = V(T)$ is the set of vertices. The elements of S_d^V will also be called configurations. For more on portraits see [BGŠ03].

Using the relations (1.3) we get the following portraits for generators of G (e and ε are elements of S_2). In each of the portraits $P(b), P(c), P(d)$, labels σ only appear at distance 1 from the rightmost ray of the tree. Labels at distance 1 from the rightmost ray form periodic sequences of period three, as in (1.1). Other vertices of T_2 that do not appear in Figure 1.3 are labelled by the identity element.

Presentation of elements by portraits allows us to visualize them and is useful in many situations.

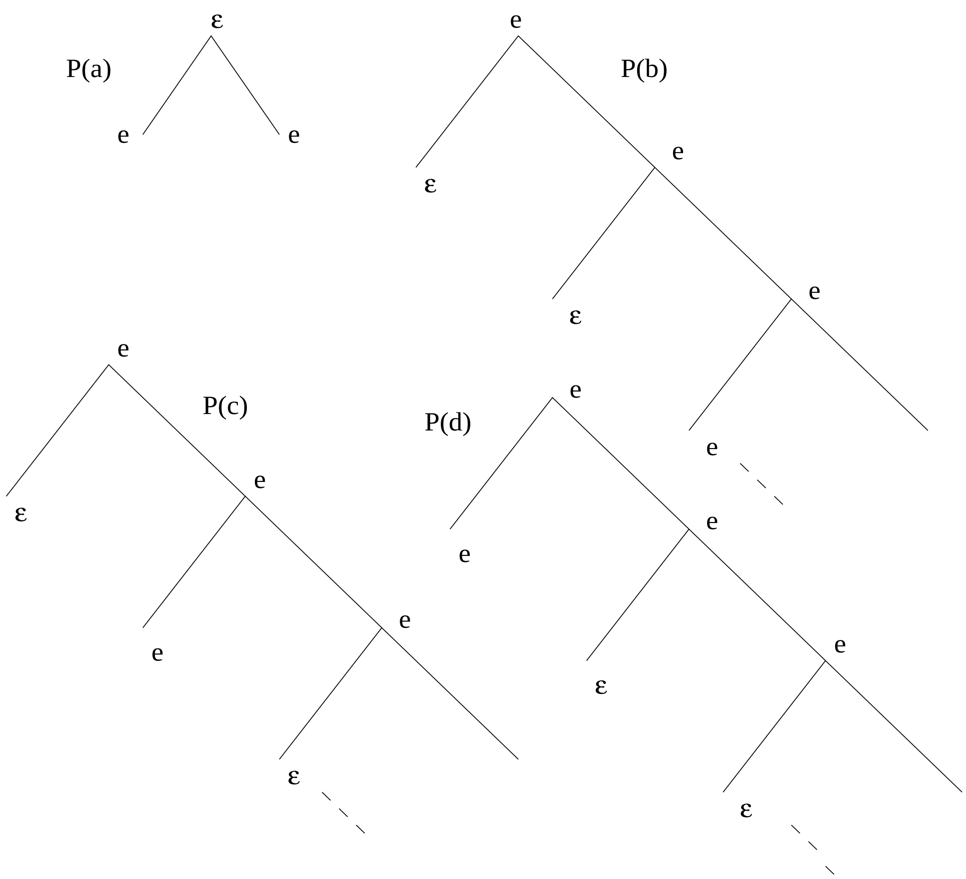


Figure 1.3:

Quite often we will rewrite (1.6) in the form

$$g = (g_1, \ldots, g_d)\alpha \tag{1.8}$$

and call this the decomposition of g.

The self-similarity of G that was mentioned above (see the relation (1.4)) can be interpreted here in the following way.

Let $g \in G$ and let $P = P(g)$ be the portrait of g.

For any vertex $u \in V(T)$ let P_u be the restriction of P on the subtree T_u with root u; and let g_u be the automorphism of T_u given by P_u. If we identity T_u with T using the canonical isomorphism, then g_u is again an element of G called the projection (or the slice) of g at vertex u. For the generators this is obvious and for the elements of G it follows easily from the formulas

$$(gf)_u = g_u f_{u^g},$$
$$(g^{-1})_u = (g_{u^{g^{-1}}})^{-1}.$$

In the same way, self-similarity of any group L acting on a regular rooted tree essentially means that any projection of any element $g \in L$ at any vertex u is again an element of L. This remark will be refined below in Definition 1.1 to allow a distinction between weakly self-similar groups and self-similar groups.

The data contained in Figure 1.3 can be described by the diagram (a) given in Figure 1.4. If $q \in \{a, b, c, d, id\}$ is any node of the diagram, $u \in \{0,1\}^*$ is a word, and s is the final node of the path ℓ_u in the diagram starting at q and determined by the word u, then the label (e or ε) of the node s coincides with the label of the vertex u in the portrait of the element q. The node id corresponds to the identity element of G given by the portrait where all the labels are the identity. We will call the subset $\{id, a, b, c, d\} \subset G$ the core (or nucleus) of G.

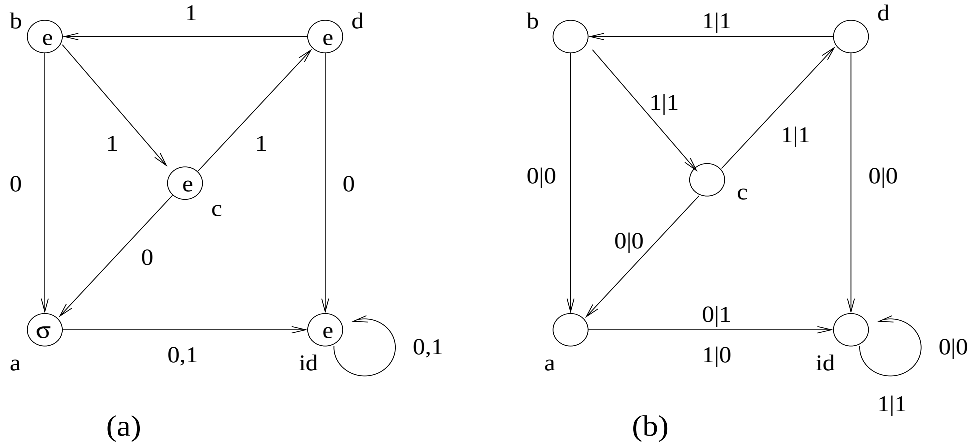
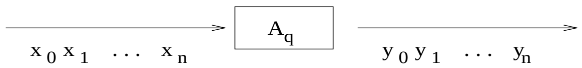


Figure 1.4:

The diagram in Figure 1.4 determines a noninitial Mealey automaton A over the aplhabet $\{0,1\}$ with five states a, b, c, d, id, with transition function given by the oriented edges and with exit function given by the labelling of the vertices by elements of the symmetric group S_2. Specifying which state q is initial we get an initial automaton A_q which can be viewed as a self-mapping $\tilde{A}_q$ of the set $\{0,1\}^* \cup \{0,1\}^{\mathbb{N}}$ consisting of finite and infinite strings



Given a sequence $x_0x_1\ldots x_n\ldots$ the automaton acts on the first symbol by its label $\sigma_q \in S_2$ and changes its initial state to the state given by the end of the arrow going out of q and labelled by x_0. Now the automaton is in a new state, reads the next symbol x_1, transforms it and moves to the next state according to the described rule; and continues to operate in this way. Composition of two maps given by two finite initial automata is again a transformation given by a finite automaton, and the inverse map can also be given by a finite automaton. Thus finite initial (invertible) automata constitute a group with a simple rule for composition and inversion. We recommend [GNS00] as a source of information about groups of finite automata (or automaton groups). An equivalent description of the automaton A is given by Figure 1.4 (b) which presents the diagram of A.

We have seen that the group G can be considered as a group generated by the states of the automaton from Figure 1.4 (the state id corresponds to the identity element and can be deleted from the generating set). Similarly, any group generated by a finite automaton acts on a regular rooted tree and this action is self-similar because any slice of any generator is again a generator. And vice versa, any group L acting on a rooted tree in a self-similar way can be generated by the states of some automaton (not necessary finite). Namely, having a generating set S of L one can consider the set of projections of the elements of S as the set of states of the automaton and construct the diagram of the automaton using the rewriting rules

$$s = (s_1, \ldots, s_m)\alpha, \tag{1.9}$$

$s \in S$, analogous to 1.8.

Definition 1.1. *(i) A group L is called self-similar if it is isomorphic to a group generated by all states of a finite invertible Mealey automaton.*
(ii) A group L is called weakly self-similar if it is isomorphic to a group generated by the states of an invertible Mealey automaton.
(iii) A closed subgroup L in $Aut\,T$, T a regular rooted tree, is called self-similar if it is a closure of a self-similar subgroup of $Aut\,T$.

A self-similar group L defined by an automaton over the alphabet X of cardinality d embeds into the wreath product $L \wr_X S_d$ through a map induced by the map ψ in (1.5). This embedding is completely determined by the Moore diagram of the corresponding automaton. In case of G the embedding is described in (1.7).

The groups G and H happen to be the first examples considered in the literature of groups generated by all the states of an automaton, so they are the first (nontrivial) examples of self-similar groups.

Groups generated by the states of an automaton form an extremely interesting and important class of groups related to many topics (see for instance [GNS00]), and many open questions about them await solution.

Problem 1.1. *(i) What groups are weakly self-similar ?*
(ii) What groups are self-similar?

As an example let us mention that it was shown only recently that a free group of finite rank can be generated by the states of a finite automaton [GM03], [VV05]. For instance, Mariya Vorobets and Yaroslav Vorobets show in [VV05] that the automaton given in Figure 1.5 generates a free group of rank 3 (thus answering a question of S. Sidki).

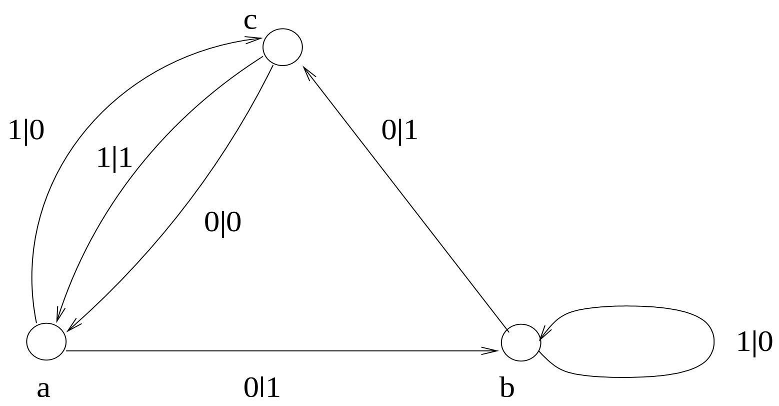


Figure 1.5:

Perhaps the simplest example of an interesting group generated by a finite automaton is the lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ which is generated by the automaton in Figure 1.6.

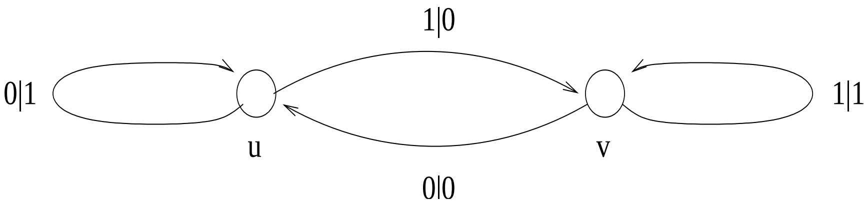


Figure 1.6:

One more interesting example of an automaton group (the so-called Basilica group) will be discussed in Section 9.

There are two obvious restrictions which self-similar groups have to conform to. First, they have to be residually finite as the full group of automorphisms of a rooted tree is residually finite (the approximating sequence of finite groups is Aut $T/st(n)$, where $st(n)$ is the stabilizer of the level n of the tree). Also, a group generated by a finite automaton has solvable word problem (this follows from the minimization algorithm for finite automata, see Section 4).

2 Groups G_ω and the topology in the space of groups

The following construction generalizes the main example. Let $\Omega = \{0,1,2\}^{\mathbb{N}}$ be the space of sequences over the alphabet $\{0,1,2\}$. The following bijection

$$0 \longleftrightarrow \begin{pmatrix} P \\ P \\ I \end{pmatrix}$$

$$1 \longleftrightarrow \begin{pmatrix} P \\ I \\ P \end{pmatrix}$$

$$2 \longleftrightarrow \begin{pmatrix} I \\ P \\ P \end{pmatrix}$$

between symbols and columns will be used in the construction. Namely, for a sequence $\omega = \omega_0\omega_1 \ldots \in \Omega$ replace each ω_i, $i = 0, 1, \ldots$, by the corresponding vector and get a vector

$$\begin{pmatrix} U_\omega \\ V_\omega \\ W_\omega \end{pmatrix}$$

consisting of three words $U_\omega, V_\omega, W_\omega$ over the alphabet $\{I, P\}$. For instance, the triple corresponding to the sequence $\xi = 012\ 012\ldots$ is

$$\begin{aligned} U_\xi &= PPI\ PPI\ldots \\ V_\xi &= PIP\ PIP\ldots \\ W_\xi &= IPP\ IPP\ldots \end{aligned}$$

Similarly, the triple corresponding to the sequence $\eta = 01\ 01\ldots$ is

$$\begin{aligned} U_\eta &= PPP\ PPP\ldots \\ V_\eta &= PIP\ IPI\ldots \\ W_\eta &= IPI\ PIP\ldots \end{aligned}$$

Using the words $U_\omega, V_\omega, W_\omega$ construct the transformations b_ω, c_ω, d_ω of the interval Δ defined in a way analogous to the case of the sequence ξ (see Figure 1.1). Define the group G_ω as the group generated by the transformation a from 1.1 and by the transformations b_ω, c_ω, d_ω, i.e. $G_\omega = \langle a, b_\omega, c_\omega, d_\omega\rangle$.

The following relations hold

$$\begin{aligned} &a^2 = b_\omega^2 = c_\omega^2 = d_\omega^2 = 1 \\ &b_\omega c_\omega = c_\omega b_\omega = d_\omega \\ &b_\omega d_\omega = d_\omega b_\omega = c_\omega \\ &c_\omega d_\omega = d_\omega c_\omega = b_\omega \end{aligned}$$

so all groups G_ω are 3-generated (in some degenerate cases even 2-generated, such as in the case of the sequence $0\ 0\ldots 0\ldots$), but we prefer to consider them as 4-generated groups with the system $\{a, b_\omega, c_\omega, d_\omega\}$ assumed as the canonical system of generators.

Let

$$\begin{aligned} \Lambda &= \{\omega \in \Omega : \text{ each symbol 0,1,2 occurs in } \omega \text{ infinitely many times}\} \\ \Xi &= \{\omega \in \Omega : \text{ at least two symbols occur in } \omega \text{ infinitely many times}\}. \end{aligned}$$

Then $\Lambda \subset \Xi \subset \Omega$. The groups G_ω are infinite groups and there is uncountably many pairwise non-isomorphic groups among them, since for each $\omega \in \Omega$ there is at most a countable set of sequences ζ such that $G_\omega \simeq G_\zeta$

[Gri84a]. Indeed there is not more than 6 such sequences [Nek05]. If $\omega \in \Omega - \Xi$ then G_ω is virtually abelian, if $\omega \in \Xi$ the group G_ω has degree of growth that is intermediate between polynomial and exponential (for more on growth see Section 8), and if $\omega \in \Lambda$ then G_ω is a torsion 2-group.

If ω is a periodic sequence then G_ω embeds in the group of finite automata. For instance for the sequence $\eta = (01)^\infty$ the group G_η is the group generated by the states of the automaton E. Let $\tau\colon\ \Omega \to \Omega$ be the

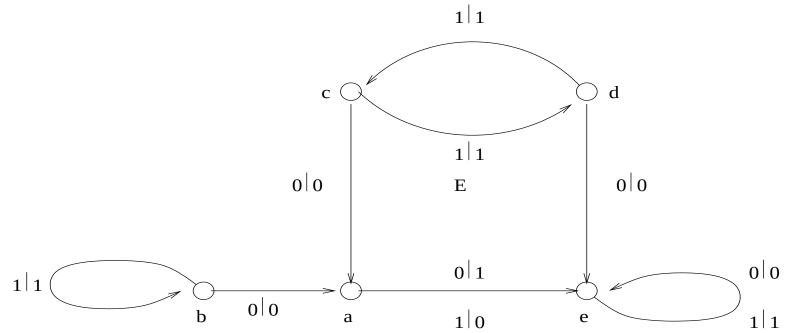


Figure 2.1: Automaton E

shift map given by $(\tau\omega)_n = \omega_{n+1}$. The sequence $\{G_{\tau^n\omega}\}_{n=0}^{\infty}$ is called the linking class of G_ω. In case ω is periodic it consists of finitely many groups while for non-periodic ω it consists of infinitely many non-isomorphic groups, which follows from the solution to the isomorphism problem by Nekrashevych [Nek05] (see Section 5). The groups from one linking class share many properties and the study of a group G_ω is usually impossible without studying the whole linking class.

The study of the groups G_ω, $\omega \in \Omega$ inspired the invention of a topology on the set X_k of k-generated groups, $k \geq 2$, in [Gri84a]. More precisely, the elements of the space X_k are k-generated marked groups, i.e. pairs (M, S_M) consisting of a group M and a generating system $S_M = \{s_1, \dots, s_k\}$ of k-elements with a given order (the change of the order changes the point). A (metrizable) topology in the space X_k is generated by a system of open sets $U(M, S, r)$, $r = 1, 2, \dots$, consisting of those points $(L, S_L) \in X_k$ for which the Cayley graphs $\Gamma(M, S_M)$ and $\Gamma(L, S_L)$ of the groups M and L constructed with respect to the system of generators S_M and S_L are isomorphic in the neighborhoods of the identity element of radius r (here by isomorphism of graphs we mean a graph isomorphism which also preserves the labelling of the edges by generators and their inverses). Other interpretation of the elements of the space X_k can be obtained by replacing (M, S_M) by the corresponding Cayley graph $\Gamma(M, S_M)$ or by a pair (F_k, N) where F_k is a fixed free group with basis $s_1, \dots, s_k$ and N is a normal subgroup such that F_k/N is isomorphic to M under the isomorphism that sends the generators of F_k to the corresponding generators of M.

For each $k \geq 2$, the space X_k is a totally disconnected compact metric space. The space X_k naturally embeds in X_{k+1}, simply by adding one more generator to the generating set. Thus each "point" (F_k, N) in X_k is presented by the pair $(F_{k+1}, \widetilde{N})$ where $\widetilde{N}$ is the normal subgroup in F_{k+1} generated by N and s_{k+1}. We can now consider the space $X = \bigcup\limits_{k=2}^{\infty} X_k$ with a natural topology that has the property that its restriction to each X_k gives back the original topology of this space. Thus X is a locally compact totally disconnected space. It can be identified with the space of normal subgroups of the free group F_∞ of infinite rank with a basis $s_1, s_2, \dots,$, which contains all generators a_n, $n \geq n_0$ for some n_0, with Chabauty topology.

The group $\mathcal{A}_k$ of automorphisms of F_k is generated by the elementary Nielsen transformations (interchange of generators $s_1, \dots, s_k$, replacement of one generator s_i by its inverse s_i^{-1} and replacement of one generator s_i by $s_i s_j$, $j \neq i$) and naturally acts on the space X_k: if $\varphi \in A_n$ then $(F_m, N)^\varphi = (F_m, N^\varphi)$. The group F_m/N is isomorphic to F_m/N^φ but the action of $\mathcal{A}_n$ on X_n is not transitive on classes of isomorphic groups. Nevertheless simple arguments [Cha91] based on use of Tietze transformations show that if two points (M, S_M), $(L, S_L) \in X_k$ correspond to isomorphic groups (i.e. $M \cong L$) then there is $\varphi \in \mathcal{A}_{2k+1}$ s.t. $(M, S_M)^\varphi = (L, S_L)$ if (M, S_M), (L, S_L) are viewed as elements of the space X_{2k+1}.

Therefore the group $\mathcal{A}_\infty$ of Nielsen automorphisms of the free group F_∞ of infinite rank with basis $s_1, \dots, s_n, \dots$ ($\mathcal{A}_\infty$ is generated by elementary Nielsen transformations) acts on X in such a way that this action is transitive on classes of isomorphic groups (that is if (M, S_M), $(L, S_L) \in X$ and $M \cong L$ then there is $\varphi \in \mathcal{A}_\infty$ s.t. $(M, S_M)^\varphi = (L, S_L)$).

Thus the space X encompasses the whole world of finitely generated groups and has many symmetries given by the action of $\mathcal{A}_\infty$. A number of questions arise about X.

Problem 2.1. *What is the topological type of the space X_k, $k = 2, 3, \dots$?*

Recall that there is a complete system of invariants for metrizable totally disconnected compact spaces. Up to homeomorphism there is only one such space without isolated points, namely the Cantor set. If there are i_1 isolated points ($(1 \le i_1 \le \infty)$) delete them. The space that is left is still compact and totally disconnected. If it has no isolated points, i.e. it ia a Cantor set, we stop. Otherwise the we record the number i_2 of isolated points, delete them, and so on. Continuing this procedure one gets a sequence $i_1, i_2, \dots$ in which each i_j is a positive integer or ∞).

The sequence $\bar{i} = \{i_j\}$, which can be empty (in case of the Cantor set), finite, or infinite is a complete invariant of totally disconnected compact spaces. The problem above asks what is this invariant for the set X_k and in particular what is the cardinality d of the sequence $\{i_j\}$ (which is called the Cantor-Bendixson rank [Kec95]).

A point of a totally disconnected set is called isolated point of rank j if it appears as isolated point in the j-th step of the above procedure of deletion of isolated points (so the isolated points of rank 1 are just ordinary isolated points).

Problem 2.2. *Describe the set of isolated points of rank j, $j = 1, 2, \dots, d$ of the set X_k.*

Let us have a look at the isolated points of rank one. Let (M, S_M) be an isolated point of the space X_k presented by a pair (F_k, N) so $M \cong F_k/N$. Then M is finitely presented group as otherwise N would be a union $\bigcup_{i=1}^{\infty} N_i$ of a strictly increasing sequence of normal subgroups and therefore the point (F_k, N) would be the accumulating point of the sequence (F_k, N_i).

Let us say that a group M has the finite approximation property if for any finite subset $E \subset M$ there is a nontrivial normal subgroup $H \lhd M$ such that the image $\overline{E}$ of E in quotient group M/H has the same cardinality as E.

Theorem 2.1. *A point (M, S_M) is a isolated point in X_k if and only if M is a finitely presented group without the finite approximation property.*

Proof. Assume M is finitely presented, let (M, S_M) be identified with the pair (F_k, N), and let $\{(M_n, S_{M_n})\}_{n=1}^{\infty}$ converge to (M, S_M). Let $m = \max_{i \in I} |R_i|$ be the maximum length of a relator in some fixed finite presentation of M

$$M = \langle s_1, \dots, s_k \mid R_i, i \in I \rangle.$$

As the Cayley graphs $\Gamma(M, S_M)$ and $\Gamma(M_n, S_{M_n})$ have the same neighborhoods of radius m for all $n \ge N_0$ (N_0 sufficiently large number) the relations $R_i = 1$ also hold in the groups $M_n, n \ge N_0$ and therefore all M_n with $n \ge N_0$ are quotients of M.

Assume M is not isolated and $\{(M_n, S_n)\}_{n=1}^{\infty}$ is an approximating sequence consisting of proper homomorphic images M_n of M. Thus $M_n = M/H_n$ with $H_n \ne 1$, for all n. Let $B_M(r)$ be the ball of radius r centered at the identity element in $\Gamma(M, S_M)$. For each r there is n_0 such that $B_M(r) \cong B_{M_n}(r)$ for $n \ge n_0$. Any finite set $E \subset M$ can be included in $B_M(r)$ for sufficiently large r. This implies the finite approximation property for M as $H_n \ne \{1\}$, for all n. By contraposition, if M does not have the finite approximation property than is is an isolated point.

On the other hand, if M has the finite approximation property then (M, S_M) is accumulating point of the set $\{(M/H, \overline{S}_M)\}$ consisting of all proper homomorphic images of M and corresponding images of systems of generators because using as E the set $B_M(r)$ and finding the corresponding normal subgroup $H \lhd M$ we get an isomorphism $B_M(r) \simeq B_{M/H}(r)$. ☐

Examples of isolated points are finite groups, finitely presented simple groups and, more generally, finitely presented groups with nontrivial intersection of all nontrivial normal subgroups (also known as monolithic groups or subdirectly irreducible groups).

Problem 2.3. *Is there a description of finitely presented groups without the finite approximation property in terms of simple groups?*

Examples of isolated points of rank 2 are the finitely presented residually finite just infinite groups (recall that a group is just infinite if it is infinite but every proper quotient is finite). In [Gri00b] the class of just infinite groups is split in three classes: branch groups (defined in Section 6), almost hereditarily just infinite groups, and almost simple groups. Hereditarily just infinite group is a residually finite group in which every normal subgroup of finite index is just infinite (for instance $SL(n,\mathbb{Z}), n \geq 3$). "Almost" hereditarily just infinite (simple) means that there is a subgroup of finite index which is a direct power of several copies of some group which is hereditarily just infinite (simple). All finitely presented almost simple groups are isolated points of rank 1.

At the moment there are no known examples of finitely presented branch groups and no known examples of infinitely presented hereditarily just infinite groups.

One of the challenges related to the introduced spaces X_k is in finding closed subsets consisting of groups with specific properties. This was our original idea realized in [Gri84a] based on the construction of the groups G_ω. Namely, two groups G_ω and G_ζ have isomorphic neighborhoods of identity of radius $r = 2^t$ in the Cayley graphs with respect to the canonical systems of generators, where t is the coordinate where the sequences ω and ζ become different. This rule works for sequences from the subset $\Xi \subset \Omega$ and follows from the solution of the word problem in the groups G_ω (for more on this see Section 4). The groups G_ω, $\omega \in \Omega - \Xi$ are virtually abelian and for them the above rule does not work for them. Let us modify our construction replacing the groups G_ω, $\omega \in \Omega - \Xi$, by the limits

$$\widetilde{G}_\omega = \lim_{n\to\infty} G_{\zeta_n} \tag{2.1}$$

where $\{\zeta_n\}_{n=1}^{\infty}$ is any sequence of points $\eta_n \in \Xi$ converging to ω. The limit (2.1) doesn't depend on the choice of sequence $\{\zeta_n\}$. The limit groups $\widetilde{G}_\omega$, $\omega \in \Omega - \Xi$, are virtually metabelian of exponential growth [Gri84a]. After this modification let us return to the notation G_ω and omit writing the tilda. The set $\{G_\omega\colon\ \omega \in \Omega\}$ is a closed subset of X_4 homeomorphic to a Cantor set. It consists of infinitely presented amenable groups only countably many of which (namely G_ω, $\omega \in \Omega - \Xi$) are elementary amenable, (more about amenability and elementary amenable groups is written in Section 9). The groups G_ω, $\omega \in \Xi$ have intermediate growth and the groups G_ω, $\omega \in \Lambda$ are torsion 2-groups.

Problem 2.4. *Find other interesting closed subsets in X_k consisting of groups with some particular (interesting) properties.*

The set defined as the closure of the set of torsion free Gromov's hyperbolic groups is studied in [Cha00]. Some interesting facts about the structure of the space X are summarized in [CG00]. An interesting fact proven in [CG00] is that the class of groups known as fully residually free groups [KM98] and limit groups of Sela [Sel01] consists of accumulating points of pairs (M, S_M) where M is isomorphic to a free group of finite rank. Another fact is the result of Y. Shalom claiming that groups with Kazhdan property (T) constitute a closed subset in X [Sha00].

Given the action of the shift $\tau\colon\ \Omega \to \Omega$ on the set $Z = \{G_\omega\colon\ \omega \in \Omega\}$ one can study the typical properties of groups from Z using a τ-invariant measure. The most natural measure in our situation is the Bernoulli $\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$ measure. Any group property that is preserved in subgroups of finite index and homomorphic images holds for the groups in Z with either probability one or probability zero. For instance, with probability 1 the groups in Z are torsion groups. It is also known that with probability 1 the groups in Z are of intermediate growth, and are branch groups. It would be interesting to find a τ-invariant property (i.e. a property common for all groups from the same linking class) for which it is difficult to say whether the measure is 0 or 1.

An interesting question raised by E. Ghys is the question on typical properties (those that occur with probability 1) of groups viewed as elements of the space X supplied with a probability measure μ. It is unclear what measure μ on X would be natural to consider. For the space X_k, it is natural to consider the uniform measure: the cylinder set of pairs (M, S_M) in X_k having a given neighborhood of radius r has to have a measure $\frac{1}{b(r)}$ where $b(r)$ is the number of possible different neighborhoods of radius r in X_k (by the way, it would be interesting to know the precise asymptotic of $b(r)$ when $r \to \infty$). It is unclear what we should do with the isolated points and how their deletion affects the type of the measure.

We have the action of the group $\mathcal{A}_\infty$ on X and the orbits consist of representatives of the same group. As we want to study statistical properties of finitely generated groups the measure μ has to be quasiinvariant (there is a very little chance to have invariant measure). Our discussion leads to the following question.

Problem 2.5. *a) Is there a $\mathcal{A}_\infty$ quasiinvariant measure defined on the space X?*
b) If such a measure μ exists, which group properties are typical with respect to μ?
c) Which group properties are typical with respect to the uniform measure on X_k?

It is easy to see that the action of A_∞ on X is not totally disconnected so there is no good factorization X/A_∞ in topological sense. However, it is possible that the partition into orbits is measurable and then the factor space X/A_∞ with the induced σ-algebra of sets can be defined.

3 Length functions

In this section we discuss the length properties of the elements of the group G, the most important of which is the contracting property. Some other length functions, such as for instance the depth function, are also discussed here. For details we recommend [Gri80, Gri84a, Gri98, CSMS01, dlH00]

As was already mentioned G is a 3-generated group but most of our considerations will be given with respect to the system of generators

$$\{a, b, c, d\}$$

which we will be considered canonical. The set

$$\{1, a, b, c, d\} \tag{3.1}$$

which represents the states of the automaton A from Figure 1.4 is called the core and the reason will be clear later.

We denote by $|g|$ the length of g with respect to the canonical systems of generators (i.e. the length of the shortest presentation of g as a product of generators). As the generators have order 2 we do not use negative powers and because of relations (1.2) the shortest representative of an element $g \in G$ has the form

$$\flat a * a \cdots * a * a \flat \tag{3.2}$$

where $*$ represents an element from the set $\{b, c, d\}$ and $\flat$ represents an element from the set $\{\emptyset, b, c, d\}$ ($\emptyset$ – empty symbol).

Let

$$\Gamma = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = bcd = 1 \rangle. \tag{3.3}$$

Then $\Gamma \simeq C_2 * (C_2 \times C_2)$ (where the first factor is generated by a and $C_2 \times C_2 = \{1, b, c, d\}$ is the Klein group). Because of the relations (1.2) the group Γ naturally covers the group G in the sense that there is a canonical epimorphism $\Gamma \to G$. It is also clear that (3.2) is the normal form for the elements in the group Γ viewed as a free product.

The embedding (1.5) induces the embedding (also denoted ψ)

$$\psi \colon G \to G \wr S_2 \tag{3.4}$$

(the permutational wreath product sign can be replaced by the standard wreath product sign in the case $d = 2$) where ψ is determined by (1.7). We have already agreed to write $g = (g_0, g_1)\alpha$ instead of $\psi(g) = (g_0, g_1)\alpha$, for $g_i \in G$, $\alpha \in S_2$. The following Lemma is a statement about length reduction with respect to the canonical system of generators.

Lemma 3.1. *[Gri80] Let $g \in G, g = (g_0, g_1)\alpha$. Then*

$$|g_i| \le \frac{|g| + 1}{2}. \tag{3.5}$$

If a shortest word of the form (3.2) *representing g has one of two symbols $\flat$ empty then*

$$|g_i| \le \frac{|g|}{2}. \tag{3.6}$$

Corollary 3.1.

$$|g_0| + |g_1| \le |g| + 1. \tag{3.7}$$

On the other hand we have

Lemma 3.2.

$$|g| \le 2(|g_0| + |g_1|) \tag{3.8}$$

The estimate follows from the fact that the image Im ψ is the group $(B \times B) \rtimes \widetilde{D}$ where $B = \langle b \rangle^G$ has index 8 in G and $\widetilde{D} = \langle (a, d), (d, a) \rangle$. For more details see [Gri84a]. The inequalities of the type (3.8) are important for the solution of the conjugacy problem (see Section 5).

Problem 3.1. *Let L be a self-similar group acting on a tree T_d. Is there a constant C s.t.*

$$|g| \le C \sum_{i=1}^{d} |g_i| \tag{3.9}$$

for the decomposition $g = (g_1, \dots, g_d)\alpha$, of any element $g \in L$?

The property of the length given by the inequality (3.5) reflects the contracting property of the group (definition is provided in the next paragraph). We already mentioned that an embedding of type (3.4) holds for any self-similar group L and has the form

$$\psi\colon\ L \to L \wr_X S_d \tag{3.10}$$

where d is the arity of the rooted tree on which the group acts and $X = \{x_1, \dots, x_d\}$ is the corresponding alphabet on which the symmetric group S_d acts. We denote by $|g|$ the length of $g \in L$ with respect to the system of generators given by the states of the corresponding automaton.

Definition 3.1. *A self-similar group L whose action induces an embedding* (3.10) *is called contracting if there are constants $\lambda < 1$ and C such that for any element $g = (g_1, \dots, g_d)\alpha \in L$,*

$$|g_i| \le \lambda |g| + C, \qquad i = 1, \dots, d. \tag{3.11}$$

The smaller the constant λ, the better contraction properties. The definition does not depend on the choice of generating system.

Problem 3.2. *What is the class of contracting self-similar groups? Can it be characterized in algebraic terms?*

Problem 3.3. *Is every contracting self-similar group amenable?*

Recall that a (discrete) group L is amenable if there is an invariant mean on the Banach space $B^\infty(L)$ of bounded functions [Gre69]. All known examples of contracting groups are amenable. We will discuss amenability in more detail in Section 9.

Estimates of the contracting coefficient λ from Definition 3.1 are important in the study of growth of groups, which is discussed in Section 9. While many contracting groups (in particular G) have subexponential growth, there are contracting groups of exponential growth (for instance Basilica groups defined in Section 9).

In the Definition 3.1 and last problem we consider the length of elements with respect to a system of generators (that is given by a set of states of corresponding automaton). One can consider a more general type of length functions on a group G and other (self-similar) groups, namely the functions $\ell\colon\ L \to \mathbb{R}$ satisfying the conditions

$$\begin{cases} \ell(g) = 0 \Leftrightarrow g = 1 & (3.12)\\ \ell(g) = \ell(g^{-1}) & (3.13)\\ \ell(gh) \le \ell(g) + \ell(h) & (3.14) \end{cases}$$

and study its properties. One extra requirement for reasonable length function is to insist on the finiteness of the balls

$$B_1^\ell(n) = \{g \in L\colon\ \ell(g) \le n\}$$

and hence the existence of a growth function

$$\gamma_L^\ell(n) = \#(B_1^\ell(n)). \tag{3.15}$$

where $\#$ denotes cardinality. The last condition should sometimes be strengthened by requiring that $\gamma_L^\ell(n)$ grows no faster than an exponential function.

Growth functions with respect to the word length induced by a generating set are discussed further in Section 8.

Length functions are closely related to weights [Gri96]. We are going to define now another type of length function called depth which first appeared in the work of S. Sidki [Sid87a].

Applying the embedding ψ n times to an element $g \in G$ gives the embedding

$$\psi_n\colon\ G \to G \wr_{X^n} Q_n$$

where the group $Q_n = \underbrace{C_2 \wr \cdots \wr C_2}_{2^n}$ is the group acting on the n-th level of the tree (which can be identified with the set X^n, $X = \{0,1\}$). The group Q_n can also be viewed as a group acting on the part of the tree T up to level n and the action is given by restriction of the action of G. If

$$\psi_n(g) = (g_{0\cdots 0} \dots, g_{i_1\cdots i_n}, \dots, g_{1\cdots 1})\alpha$$

$g_{i_1\dots i_n} \in G,\ \alpha \in Q_n$ then the element g can be represented by a diagram of the form given in Figure 3 where

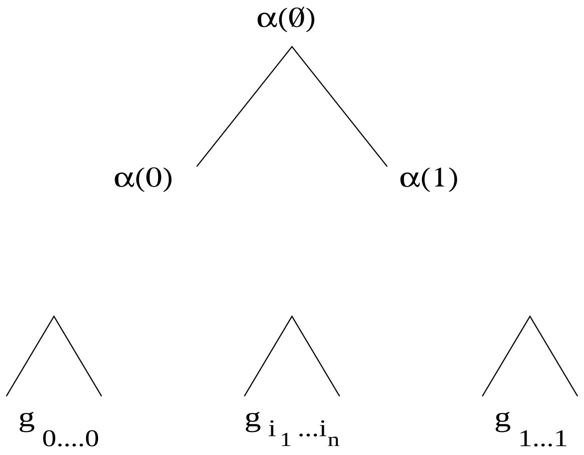


Figure 3.1:

the vertices up to level $n-1$ are labeled by elements from S_2 according to the portrait of α. By Lemma 3.1

$$|g_{i_1\dots i_n}| \le \frac{|g|}{2^n} + 1$$

$\forall i_1, \dots, i_n \in \{0,1\}$.

Thus there is $n \le \log_2 |g| + 1$ s.t. that all elements $g_{i_1\dots i_n}$ belong to the core $\{1, a, b, c, d\}$. The smallest such n is called the depth of the element g and is denoted by $d(g)$. We have

$$d(g) \le \lceil \log_2 |g| \rceil + 1. \tag{3.16}$$

Using the inequality (3.8) it is easy to get the estimate

$$\frac{1}{2}\log_2 |g| \le d(g).$$

Problem 3.4. *(i) Are there constants $\lambda < 1$ and c such that*

$$d(g) \le \lambda \log_2 |g| + c \tag{3.17}$$

holds for any $g \in G$? What is the minimal value of λ?

(ii) Are there constants $\mu > 1/2$ and C such that

$$d(g) \geq \mu \log_2 |g| + C?$$

What is the supremum of values of μ?

(iii) Is there a good description of the portraits of the elements of fixed length in the group G?

There are five elements of depth 0 (namely $1, a, b, c, d$) and

$$d(gh) \leq \max(d(g), d(h)) + 1 \leq d(g) + d(h) + 1.$$

So $\ell(g) = d(g) + 1$ satisfies the inequality (3.14). The balls

$$\{g \in G\colon\ d(g) \leq n\} \tag{3.18}$$

are finite and have the cardinality

$$\leq 5^{1+2+\cdots+2^n} = 5^{2^{n+1}-1}.$$

Problem 3.5. *What is the precise growth function of G with respect to the depth?*

For groups with the contracting property naturally the notion of a core (or nucleus [BGN03], [Nek05]) arises. Namely the inequality (3.11) implies that if $|g| > \frac{C}{1-\lambda}$ then the projections g_i, $i = 1, \ldots, d$ have smaller length. Thus the set of elements for which there is no shortening of the lengths in the projections is contained in the ball

$$\left\{g \in L\colon\ |g| \leq \frac{C}{1-\lambda}\right\}$$

and hence is finite. It is called the core of the group and denoted by $\mathrm{core}(L)$.

The core depends on the system of generators. For the group G and the canonical system of generators

$$\mathrm{core}(G) = \{1, a, b, c, d\}.$$

The notion of the core leads to the notion of the core portrait $P_{\mathrm{core}}(g)$, $g \in L$. Namely when defining $P_{\mathrm{core}}(g)$ start with the same procedure as in the definition of the portrait $P(g)$ but stop the development of the portrait at those vertices u of the tree for which the corresponding projection g_u of the element g belongs to $\mathrm{core}(L)$. Attach to such a vertex u the label g_u. This procedure leads to a finite rooted tree (with degree of vertices $\leq d$) whose internal vertices are labeled by elements of the symmetric group S_d (according to the portrait $P(g)$) while the leaves are labeled by corresponding elements of the core. For instance the core portrait of the element $adacac$ in G is

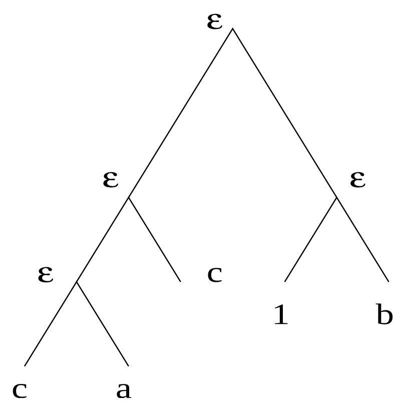


Figure 3.2:

The core portraits are useful in many situations, for instance in solving the conjugacy problem.

4 The word problem and L-presentations

The group G and the whole family G_ω have nice and interesting algorithmic properties and we start our discussion with the word problem (WP), whose solution is given by a new (at least in group theory) type of algorithm. We call it branch algorithm because of it branch structure. Let us mention first of all that the word problem for G can be solved by the algorithm which works for all groups generated by finite automata (and in particular for all self-similar groups) as G is one of the representatives of this class (as was described in Section 1). This algorithm works as follows. Given a word in the generating set one multiplies the corresponding automata using the composition rule and then to minimizes the product using the classical algorithm of minimization [Eil74].

The word represents the identity element if and only if the minimal automaton is 1-state automaton with trivial exit function (that doesn't changes the input symbols). The rough upper estimate of complexity of such an algorithm (both in time and in space) is exponential as the number of states of composition of automata is equal to the product of numbers of states of factors. But indeed we don't know any example of an automaton group with exponential complexity of word problem. We even don't have an examples with complexity higher than polynomial. All known examples of automaton groups are either solvable groups of special type, or groups of matrices over $\mathbb{Z}$ or groups of branch type with good contracting properties. For all of them the WP has polynomial complexity. Thus we formulate

Problem 4.1. *a) What is the maximal possible complexity of WP for self-similar groups?*

b) In particular, is there a self-similar group with superpolynomial complexity of WP? Is there a self-similar group with exponential complexity?

The WP can be formulated for any finitely generated group (and even for infinitely generated if we do this carefully). The next theorem gives an information about a solvability of WP for the whole class G_ω.

Theorem 4.1 ([Gri83, Gri84a]). *The word problem is solvable in the group G_ω if and only if the sequence ω is recursive.*

We will describe the algorithm for the case of the group G modifying the exposition done in [Bar98]. As was mentioned in Section 3 a word $w = w(a, b, c, d)$ of the form (3.2) represents the normal form of elements in the group Γ given by the representation (3.3) For any word $u = u(a, b, c, d)$ we denote by $r(u)$ the reduction of u in Γ which can be done by rewriting u using the following rules

$$\begin{aligned} &\text{(i)} \quad x^2 \to 1 \\ &\text{(ii)} \quad xy \to z \end{aligned} \tag{4.1}$$

where $x, y, z \in \{b, c, d\}$ and for the rule of the second type x, y, z have to be different elements.

Now we are going to describe two rewriting processes ϕ_0, ϕ_1. Namely $\phi_i(w), i \in 0, 1$ is the word obtained from w by associating with each of the letters b, c, d occurring in w a symbol in an accordance with the following rule:

$$\varphi_i\colon \quad \begin{cases} b \to a \\ c \to a \\ d \to 1 \end{cases} \tag{4.2}$$

if the number of occurrences of a in w preceding the present occurrence of the symbol in question is even for $i = 0$ or odd for $i = 1$. In a similar way

$$\varphi_i\colon \quad \begin{cases} b \to c \\ c \to d \\ d \to b \end{cases} \tag{4.3}$$

if the number of occurrences of a in w proceeding the present occurrence of the symbol in question is odd for $i = 0$ or even for $i = 1$.

ALGORITHM. To verify the relation $w(a, b, c, d) = 1$ proceed as follows.

(1) Calculate $|w|_a$ the number of occurrences of a in w. If $|w|_a$ is odd, then $w \neq 1$. If $|w|_a$ is even, then reduce w using the rules (4.1) (that is, calculate $r(w)$). If $r(w)$ is empty word then $w = 1$, if $r(w)$ is nonempty, then pass to step (2).

(2) Calculate $w_i = \phi_i(r(w)), i = 0, 1$, and return to step (1), but now verify two relations in G, namely, $w_i = 1, i = 0, 1$ keeping in mind that

$$(w = 1) \Leftrightarrow (w_i = 1, i = 0, 1)$$

One can visualize the algorithm as a branching rewriting process as in Figure 4, which stops at some level n if either some of the words $w_{i_1, i_2, \ldots, i_n}, i_1, \ldots, i_n = 0, 1$ have odd number of occurrences of a (and then $w \neq 1$) or all these words are empty (and then $w = 1$).

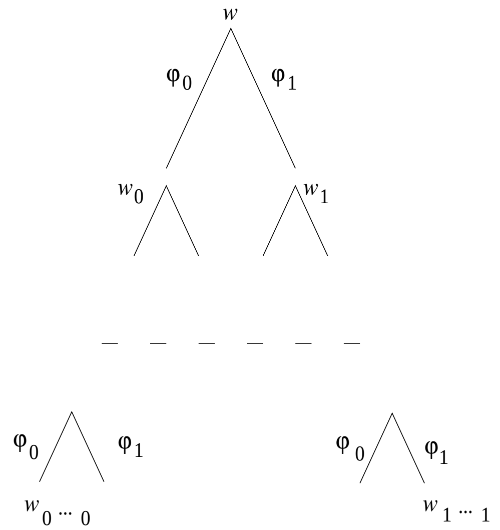


Figure 4.1:

The process terminates at most at level $[\log_2 |w|] + 1$ because of the reduction of the length

$$|w_i| \leq \frac{|w| + 1}{2} \tag{4.4}$$

(this inequalities are analogous to (3.5) in case w is reduced in Γ). We see that when we apply parallel computations this process takes a logarithmic time to solve WP. Rough estimates give $n \log(n)$ for the time and cn estimate for the space needed, where n is the length of the input (the length of the word w) and c is some constant.

Problem 4.2. *a) What is the actual time needed for the solution of WP for G.*
b) Is it linear (when using a Turing machine with one tape)?

For the group G_ω the word problem is solvable by similar branch algorithm, only now the rewriting rules $\varphi_i^{(k)}$, $i = 0, 1$ used for k-th level of the tree depend on the k-th digit of oracle ω and the (Turing) degree of unsolvability of WP for G_ω is the same as the degree of unrecursiveness of oracle ω [Gri83, Gri84a, Gri85].

In modern combinatorial group theory an important role is played by formal languages. Let us indicate a few interesting questions which arise around the group G.

A group M is better understood if there is a good normal form for its elements. By this we mean the one-to-one correspondence $g \to W_g$ between the elements of M and words over an alphabet $S \cup S^{-1}$ where s is a generating set. "Good" means that the subset $\mathcal{L} = \{W_g : \ g \in M\}$ of the set $\{S \cup S^{-1}\}^*$ of all words viewed as a formal language should belong to the class of well understood languages, for instance to be regular, context free, indicable or at least context sensitive language [Rev91, HU79]. In view of problems of

Section 8 (like Problems 8.2, 8.3) it is better to deal with a geodesic normal forms, that is when the length of element with respect to generating system S coincides with the length of corresponding word:

$$|g|_S = |W_g|, \quad g \in M.$$

In this case the growth series $\Gamma_M^S(z) = \sum_{n=0}^{\infty} \gamma_M^S(n) z^n$ of the group M (see Section 8) coincides with the growth series $\Gamma_{\mathcal{L}}(z) = \sum_{n=0}^{\infty} \ell(n) z^n$ of the language $\mathcal{L}$ ($\ell(n)$ (here $\ell(n)$ is the number of words of length $\leq n$ in $\mathcal{L}$).

There are many examples of groups with regular geodesic normal form (for instance Coxeter groups and hyperbolic groups). But the geodesic normal form of a group of intermediate growth can not belong to this class and even cannot be a context free language as context free languages have either polynomial or exponential growth [BG02b, Inc01]. It was shown in [GM99] that there are indexed languages of intermediate growth.

Problem 4.3. *a) Is there an indexed geodesic normal form for the elements of G.*

b) What is the simplest class of languages that has a representative that can figurate as a geodesic normal form for the elements in G or in any other group of intermediate growth?

Another language that naturally arises when studying a group by geometric means is the language $\mathcal{L}_{\text{geod}}$ consisting of words in $(S \cup S^{-1})^*$ determining a geodesic path beginning at the identity element in a Cayley graph. For instance, for hyperbolic groups $\mathcal{L}_{\text{geod}}$ is a regular language, hence has exponential growth (in the non-elementary case) and the set of all words in $\mathcal{L}_{\text{geod}}$ of length $\leq n$ can be listed by an algorithm in exponential time.

The growth of $\mathcal{L}_{\text{geod}}$ can be very different from growth of the language of geodesic normal form (i.e. of growth of the group). For instance, for $\mathbb{Z}^d$, $d \geq 2$ the growth is polynomial while the language of geodesics grows exponentially.

Problem 4.4. *a) What is the rate of exponential growth of $\mathcal{L}_{geod}^G$?*

b) To which class of functions belongs growth series of $\mathcal{L}_{geod}^G$?

c) Is there a group with intermediate (between polynomial and exponential) growth of the language of geodesics?

Problem 4.5. *a) To which class of languages does $\mathcal{L}_{geod}^G$ belong?*

b) What is the minimal complexity of an algorithm which for given n produces all words of length $\leq n$ in $\mathcal{L}_{geod}^G$?

One more natural language that can be associated to any finitely generated group is the word problem language $\mathcal{L}_{WP}$ consisting of all words from $\{S \cup S^{-1}\}^*$ representing the identity element. $\mathcal{L}_{WP}$ is a recursive set if and only if the word problem is solvable. The "nicer" the language $\mathcal{L}_{WP}$ the "better" the word problem.

Problem 4.6. *Is $\mathcal{L}_{WP}^G$ an indexed language?*

This is a longstanding open question which has a chance to have the positive solution because of the result of D. Holt and C. Röver [HR03] stating that the co-word problem language (i.e. the complement of $\mathcal{L}_{WP}$ in $(s \cup s^{-1})^*$) is indeed the indexed language. Therefore the study of growth series of indexed languages initiated in [GM99] looks quite relevant.

Problem 4.7. *a) To which class of functions does the growth series of $\mathcal{L}_{WP}^G$ belong?*

b) What is the second term of the asymptotic of the growth of $\mathcal{L}_{WP}^G$?

The latter question is equivalent to the question about the asymptotic of decay of probabilities $P_{11}^{(n)}$ of return for simple random walk on G. The first term is 7^n, which follows from the amenability of G and Kesten' criterion [Kes59].

We do not require in the last problems that the system of generators for G is the canonical one. The same type of questions can be attributed to any other self-similar group of intermediate growth or of branch type.

There are different ways to measure the complexity of algorithmic problems [LV97]. The one of them is based on use of Kolmogorov complexity was applied in [Gri85] to groups G_ω. The idea is as follows. Let M be an arbitrary group with system of generators $\{a_1, \dots, a_m\}$ and let $A = \{a_1, \dots, a_m, a_1^{-1}, \dots a_m^{-1}\}$ be the set of generators and its inverses considered as an alphabet. Using the order $a_1 < a_2 < \dots < a_m < a_1^{-1} < \dots < a_m^{-1}$ extend it lexicographically to the order on the set A^* of all words over A. Let $\{w_n\}_{n=0}^{\infty}$ be enumeration of elements of A^* in this ordering with $w_0 = \emptyset$. Let $\mathcal{L} \subset A^*$ be a subset consisting of words representing the identity element (i.e. $\mathcal{L} = \mathcal{L}_{WP}^M$) and let

$$\xi = \xi_{\mathcal{L}} = \xi_0 \xi_1 \dots \xi_{n-1} \dots$$

be a characteristic sequence of $\mathcal{L}$ i.e.

$$\xi_n = \begin{cases} 1 & \text{if } w_n \in \mathcal{L} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\xi^{(n)} = \xi_0 \xi_1 \dots \xi_{n-1}$ be the prefix of ξ of length n. Then the Kolmogorov complexity $K(\xi^{(n)})$ as a function of n is a quantitative measure of complexity of word problems. The growth of $K(n) = K(\xi^{(n)})$ when $n \to \infty$ is not faster than linear. For a typical sequence η (with respect to uniform Bernoulli measure on $\{0,1\}^{\mathbb{N}}$)

$$\lim_{n\to\infty} \frac{1}{n} K(\eta^{(n)}) = 1.$$

but for the sequences $\xi^{(n)}$ constructed above one has

$$\lim_{n\to\infty} \frac{1}{n} (\xi^{(n)}) = 0$$

if the group M is infinite [Gri85].

In the study of algorithmic problems it is more natural to use a version of Kolmogorov complexity, denoted KR, and called the complexity of resolution (or length conditional Kolmogorov complexity [LV97]).

The function

$$R(n) = KR(\xi^{(n)})$$

is the quantitative measure of undecidability of word problems and the WP is decidable if and only if $R(n)$ is bounded.

It is more natural to consider values of $R(n)$ at points of the form

$$N_n = \sum_{i=0}^{n} (2m)^i = \frac{(2m)^{n+1} - 1}{2m - 1}$$

which counts the number of all words in A^* of length $\leq n$. The value

$$r(n) = r_M^A(n) = R(N_n)$$

represents the amount of information needed to solve the word problem for elements of length $\leq n$.

The definition of Kolmogorov complexity or of the complexity of resolution relies on the notion of a universal Turing machine. The difference between complexities defined by different universal (or optimal) machines is uniformly bounded. Therefore it is natural to study functions $K(n)$ and $R(n)$ up to equivalence which ignores bounded functions. For study of growth of functions $r_G^A(n)$ (which we call WP growth function) more natural is to use Milnor equivalence [Mil68]:

$$\begin{gathered} f_1(n) \sim f_2(n) \Leftrightarrow \exists c \quad \text{s.t.} \quad f_1(n) \leq c f_2(cn), \\ f_2(n) \leq c f_1(cn), \quad n = 1, 2, \dots \, . \end{gathered}$$

Then the class of equivalence $[r_M^A(n)]$ does not depend on the choice of generating system and is bounded by the class $[2^n]$ of exponential function. Moreover, in case M is recursively presented (for instance, finitely presented) $[r_M(n)] \prec [n]$ (linear bound). There are finitely generated groups with $r(n) \sim 2^n$ and there are finitely presented groups with $r(n) \sim n$.

For the groups G_ω

$$r_{G_\omega}(n) \sim R(\omega^{[\log_2 n]}) \tag{4.5}$$

so the complexity of word problems depend on a complexity of the oracle ω determining the group [Gri85].

Relation (4.5) shows that there is infinitely many degrees of growth of complexity of word problem in finitely generated groups and that there are groups with incomparable growth of complexity.

A number of open questions can be formulated about $K(n)$, $R(n)$ and $r(n)$ and further investigations in this direction are desirable. Let us recall an open question from [Gri85]. Let

$$\beta = \overline{\lim_{n\to\infty}} \sqrt[n]{|\mathcal{L}_n|}$$

where $\mathcal{L}_n$ is the set of words of length n in $\mathcal{L}$. Then

$$R(n) \leq n^\delta \log n$$

where $\delta = \frac{\log\beta}{\log 2m}$. From Kesten's criterion of amenability [Kes59] follows that M is amenable if and only if $\delta < 1$. Thus for nonamenable groups the function $R(n)$ grows slower than n^ρ for some $\rho < 1$.

Problem 4.8. *Is there a finitely generated group for which the function $R(n)$ can not be estimated from above by a function of the form n^ρ where $\rho < 1$?*

If such a group exists it must be amenable. These and other considerations show that Kolmogorov complexity has interesting links to the subject of amenability and growth. It looks like the groups with most difficult word problems (from the point of view of Kolmogorov complexity) are amenable groups of exponential growth.

The idea to use a version of Kolmogorov complexity to the word problem has appeared in a few recent articles [KS05, Nab96].

Also the group G started to be popular among specialist in cryptography (the first time it was observed that G could be useful in cryptography was in [GZ91]). This is based on new developments in commercial cryptography which requires a groups with easy word problems but difficult conjugacy problems. The conjugacy problem for G will be considered in detail in the next section and this problem is certainly much harder than the word problem in G.

The group G is well-behaved not only with respect to the WP, but also to other algorithmic problems. One of them is the membership problem (MP) (sometimes called the generalized word problem). The membership problems asks for an algorithm which, given elements $g, h_1, \dots, h_m$, decides if g belongs to the subgroup generated by the elements $h_1, \dots, h_m$. This problem is also decidable in G. This follows from the subgroup separability property (every finitely generated subgroup is closed in the profinite topology i.e. can be presented as an intersection of subgroups of finite index) [GW03a] and the solvability of WP.

Problem 4.9. *Is it correct that every finitely generated branch group is subgroup separable?*

Problem 4.10. *Find a condition that insure that a self-similar group is subgroup separable (and hence has decidable membership problem).*

An example of self-similar group with undecidable MP is a direct product $F_7 \times F_7$ of two copies of a free group of rank 7 which follows from possibility represent $F_7 \times F_7$ as a group generated by states of finite automaton. Namely, first do this for F_7 [GM03], then for the direct product use a standard construction in the theory of automaton groups and use the Mikhailova construction [LS01], which is discussed below. It would be interesting to find a reasonable decision problem which is undecidable for G.

An interesting circle of problems initiated by study of the group G and other self-similar groups is the question about presentability of G and other self-similar groups of branch type by finite L-presentations.

By a finite L-presentation we mean a presentation of the form

$$M = \langle a_1, \dots, a_m \mid R, \tau^n(Q), n \geq 0 \rangle \tag{4.6}$$

where R and Q are finite sets of elements of a free group F_m with basis $a_1, \dots, a_m$ and τ an endomorphism of this group (i.e. a substitution $\tau\colon\ a_i \to w_i(a_\mu)$, $i = 1, \dots, m$ determined by some set of elements $w_i \in F_m$).

An L-presentation (4.6) is called pure if R is empty set and is called ascending if it is pure and τ induces an injective endomorphism $\tau\colon\ M \to M$.

The first example of a self-similar group with L-presentation was found by I. Lysionok (Lysenok) [Lys85] who showed that G has the following presentation

$$G \simeq \langle a, b, c, d \mid 1 = a^2 = b^2 = c^2 = d^2 = bcd = \sigma^n((ad)^4) = \sigma^n((adacac)^4), n \geq 0\rangle \tag{4.7}$$

where the substitution σ is defined as

$$\sigma\colon\ \begin{cases} a \to aca \\ b \to d \\ c \to b \\ d \to c \end{cases} \tag{4.8}$$

The presentation (4.7) is not pure but can be modified to the pure presentation

$$G \simeq \langle a, b, c, d \mid 1 = \sigma^n(r), n \geq 0, r \in \{a^2, b^2, c^2, d^2, bcd, (ad)^4, (adacac)^4\}\rangle.$$

This presentation is also ascending as σ induces the injective endomorphism of G.

The group G is not finitely presented and moreover the presentation (4.7) is minimal (the deletion of any relator changes the group). This fact was used in [Gri99] to show that the Schur multiplicator (i.e. the second homology group of G with coefficient in trivial G-module $\mathbb{Z}$) is infinite dimensional and is isomorphic to the direct sum of infinitely many copies of $\mathbb{Z}/2\mathbb{Z}$.

Infinitely presented groups given by finite L-presentation can be viewed as "finitely presented in generalized sense". Ascending L-presentations are useful because allow to embed a group "in a nice way" in a finitely presented group. Namely the ascending HNN-extension

$$\widetilde{M} = \langle M, t \mid t^{-1}mt = \tau(m), m \in M\rangle$$

of a group M given by ascending L-presentation

$$M = \langle a_1, \ldots, a_m \mid \tau^n(Q), n \geq 0\rangle$$

is a finitely presented group with the presentation

$$\widetilde{M} = \langle a_1, \ldots, a_m, t \mid Q, t^{-1}a_it = \tau(a_i), i = 1, \ldots, m\rangle.$$

The embedding $M \hookrightarrow \widetilde{M}$ preserves some of the properties of M (for instance to be amenable). In case of G

$$\begin{aligned}\widetilde{G} &= \langle a, b, c, d, t \mid 1 = a^2 = b^2 = c^2 = d^2 = bcd, t^{-1}at = aca, t^{-1}bt = d, t^{-1}ct = b, t^{-1}dt = c\rangle \\ &\simeq \langle a, t \mid a^2 = TaTatataTatataTataT = (Tata)^8 = (T^2ataTat^2aTata)^4 = 1\rangle\end{aligned}$$

where T stands for t^{-1} (the last presentation was found by L. Bartholdi [dlAGCS99]). $\widetilde{G}$ shares with G the property to be amenable but not elementary amenable but is a group of exponential growth in contrast to G which has intermediate growth between polynomial and exponential (Section 8). A more general point of view on L-presentations is presented in [Bar03].

Being amenable the group $\widetilde{G}$ does not contain a free subgroup on two generators. The theorem of R. Bieri and R. Strebel [Bau93] states that a finitely presented indicable group either is an ascending HNN extension with a finitely generated base group or contains a free subgroup of rank two (a group is indicable if it can be mapped onto $\mathbb{Z}$).

Problem 4.11. *Is it correct that a finitely presented indicable group not containing a free subgroup of rank 2 is an ascending HNN extension of a group with finite L-presentation?*

Problem 4.12. *a) Which self-similar groups have finite L-presentations?*

b) Which self-similar groups have ascending finite L-presentations?

We are now going to briefly describe the main steps in the existing method for finding L-presentations developed in [Lys85, Sid87b, Gri98, Gri99, Bar03]. Let M be a self-similar group.

Step 1. Find a suitable branch algorithm for solving the WP in M. This requires finding a suitable finitely presented group with solvable WP that covers M. In case of G one such group is the group Γ from (3.3). However, for our purposes it is better to replace Γ by another groups, by adding one more relator $(ad)^4$. The important property that should be achieved by the choice of analogue of covering group is the contraction of the length after the projections.

Step 2. Let Γ be the chosen covering group and let $M \simeq \Gamma/\Omega$. The second step consists of presenting Ω as a union $\bigcup\limits_{n=1}^{\infty} \Omega_n$ where $\Omega_n \triangleleft \Gamma$ consists of elements for which the branch algorithm stops its work (and "accepts" the element) on or before level n of the tree, and then analyzing the structure of the groups Ω_n, $n = 1, 2, \dots$. Assume that the cardinality of the alphabet (used to determine the self-similar structure on M) is d, that

$$\psi\colon \ \begin{cases} M \longrightarrow M \wr_{\times d} Sd \\ st_M(1) \longrightarrow \underbrace{M \times \cdots \times M}_{d} \end{cases}$$

is the embedding of type (3.10). Let $\nu\colon\ \Gamma \to M$ be the covering used in the branch algorithm, denote $\Xi = \gamma^{-1}(st_M(t))$ and let

$$\theta\colon\ \Xi \longrightarrow \underbrace{\Gamma \times \cdots \times \Gamma}_{d}$$

be the lift of ψ to Ξ. Denote further $\Omega_1 = \mathrm{Ker}\,\theta = \theta^{-1}(1)$ and $\Omega_n = \theta^{-n}(1)$ where 1 represent here the identity element in the direct product of copies of Γ.

One way of finding a generating set for Ω_1 (as normal subgroup) is through finding a presentations for Ξ and $\theta(\Xi)$ and comparing them. In this approach a question arises about presentations of finitely generated subgroups in direct products (which we will discuss a bit later). If $\theta(\Xi)$ has finite index in Γ^d then it is a finitely presented group whose presentation can be found by the standard Reidemeister–Schreier method. For instance, in case of G and covering group Γ given by the presentation

$$\langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = bcd = (ad)^4 = 1\rangle$$

$\theta(\Xi)$ has finite index in $\Gamma \times \Gamma$ (this is precisely why the relation $(ad)^4 = 1$ was added) and hence is a finitely presented group. In this case $\Omega_1 = \langle (ac)^8, (adacac)^4\rangle^{\Gamma}$.

Step 3. Finding a generating set for Ω_n requires finding a substitution σ which well "cooperates" with the embedding ψ and projections. In case of G the substitution σ given by (4.8) has the properties that $p_1\psi\sigma = id$ and $p_0\psi\sigma(r) = 1$ for any relator r in G (p_0, p_1 are the projections). Hence

$$\sigma^n((ad)^4) = (1, \sigma^{n-1}((ad)^4))$$
$$\sigma^n((adacac)^4) = (1, \sigma^{n-1}((adacac)^4))$$

for $n \geq 1$ and this leads to the relations

$$\Omega_n = \langle \sigma^i((ad)^4), 1 \leq i \leq n, \sigma^j((adacac)^4), 0 \leq j \leq n-1\rangle,$$
$$\theta(\Omega_n) = \Omega_{n-1} \times \Omega_{n-1}$$

and eventually to the presentation (4.7).

While for the group G the choice of the substitution σ is rather easy, for most self-similar groups it is quite difficult to find suitable Γ, σ and other tools which can be used in finding a L-presentation.

The above discussion have been touched the question about finite presentability of subgroups in direct products. As the groups Γ used in all known examples of computations of L-presentations are not far from virtually free groups we naturally come to the following question.

Problem 4.13. *Let Γ be a (nonelementary) hyperbolic group and $d \geq 2$. Which finitely generated subgroups of Γ^d have finite presentation?*

Recall that hyperbolic groups were defined by M.Gromov [Gro87]. In the above question we can distinguish as a special case the case when a subgroup is a subdirect product (i.e. its projection on each factor is Γ).

For the case when Γ is a free group and $d = 2$ we have the complete answer given by the theorem of G. Baumslag and J. Roseblade [BR84], [BW99]. Generalizations of this result to product of copies of surface groups are done in [BW99].

An interesting class of groups can be defined while studying the word problem in self-similar groups. Let me call them companions of self-similar groups. Namely, given a self-similar group M and a finitely presented group Γ which covers M let the companion group be $\mathcal{C}(M,\Gamma) = \Gamma/\Omega$ where $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$, $\Omega_n = \theta^{-n}(1)$ and

$$\theta\colon\ \Xi \longrightarrow \underbrace{\Gamma \times \cdots \times \Gamma}_{d}$$

is the homomorphism determined by the automaton. In a contracting situation $\mathcal{C}(M,\Gamma) = M$ but in general $\mathcal{C}(M,\Gamma)$ and M may be different. Companions of self-similar groups do not have the (T)-property of Kazhdan (private communication of A. Erschler).

Computation of presentations of companion groups starts with computation of $\Omega_1 = \mathrm{Ker}\,\theta$ and, as mentioned above, is related to finding a presentation of a subgroup in direct products. In case of $\Gamma = F_m$ and $d = 2$ in many examples the arising subgroups in $F_m \times F_m$ are of Mikhailova type [LS01] (i.e. subgroups generated by diagonal elements $(a_1,a_1),\dots,(a_m,a_m)$, where $a_1,\dots,a_m$ form a basis of F_m and elements of the form $(1,u_i)$, $u_i \in F_m$, $i = 1,\dots,k$).

For instance let us consider the automaton

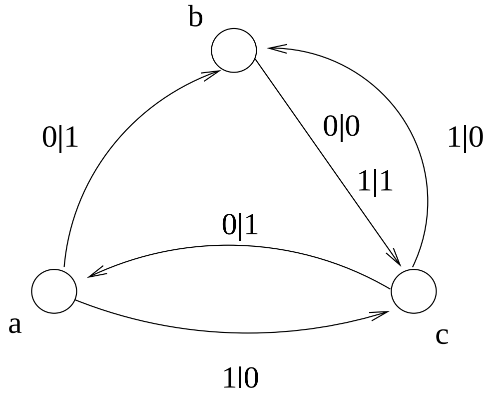


Figure 4.2:

that is $a = (b,c)\varepsilon$, $b = (c,c)$, $c = (a,b)\varepsilon$ where ε is nonidentity element of S_2. Let $M = \langle a,b,c\rangle$. Then

$$st_M(1) = \langle a^2, b, ca^{-1}, ac, aba^{-1}\rangle$$

and for the canonical embedding $\psi\colon\ M \to M \wr S_2$

$$\psi\colon\ \begin{cases} s_1 = a^2 \longrightarrow (bc, cb) \\ s_2 = b \longrightarrow (c,c) \\ s_3 = ca^{-1} \longrightarrow (ab^{-1}, bc^{-1}) \\ s_4 = ac \longrightarrow (b^2, ca) \\ s_5 = aba^{-1} \longrightarrow (bcb^{-1}, c). \end{cases}$$

Performing a Nielsen transformation on the set s_1, s_2, s_3, s_4, s_5 one can get a new set of generators u_1, u_2, u_3, u_4, u_5 of $st_M(1)$ for which

$$\psi\colon\ \begin{cases} u_1 \longrightarrow (a,a) \\ u_2 \longrightarrow (b,b) \\ u_3 \longrightarrow (c,c) \\ u_4 \longrightarrow (1, acb^{-2}) \\ u_5 \longrightarrow (1, cbc^{-1}b^{-1}) \end{cases} \tag{4.9}$$

so the image of θ in $F_3 \times F_3$ is a subgroup generated by pairs (4.9).

Problem 4.14. *What kind of presentations can be obtained for the Mikhailova subgroups in $F_m \times F_m$ determined by finite automata?*

Mikhailova subgroup $M < F_m \times F_m$ has the form of the semidirect product $(N \times N) \rtimes D$, where $N = \langle u_1, \dots, u_k \rangle^{F_m}$ and $D = \langle (a_1, a_1), \dots, (a_m, a_m) \rangle$ is the diagonal subgroup. Therefore its presentation can be described by finding a generating set for the free subgroup $N < F_m$, then for the product $N \times N$ and calculation of the action of the generators of D by conjugation on the generators of $N \times N$.

The generators for N can be calculated in terms of a spanning subtree of the Cayley graph of the group

$$\langle a_1, \dots, a_m \mid u_1 = \dots = u_k = 1 \rangle. \tag{4.10}$$

It may happen however that this group has unsolvable WP (which is equivalent to the fact that corresponding subgroup in $F_m \times F_m$ is not a recursive subset, which was the original idea of Mikhailova to show that the membership problem is undecidable in $F_m \times F_m$ [LS01]).

Problem 4.15. *Is there a finite invertible automaton over* $\{0, 1\}$ *for which the corresponding group* (4.10) *has undecidable WP?*

5 The conjugacy problem and the isomorphism problem

While the word problem was solved for G immediately after the group was discovered [Gri80] the conjugacy problem (CP) was solved only in the end of the 90-ies [Leo98, Roz98]. For p-groups G_ω with p odd and ω a periodic sequence, and for Gupta-Sidki p-groups the problem was solved in [WZ97] where it was shown that these groups are conjugacy separable.

The solution given in [Leo98, Roz98] is based on a different idea and in certain sense is more direct. As a byproduct it also leads to the conclusion that the group G has the conjugacy separability property. The result of [Leo98] is more general than the one given in [Roz98] as it deals with the whole class of groups G_ω, $\omega \in \Omega_2$.

Theorem 5.1. *The conjugacy problem is solvable in the group G_ω if and only if the sequence ω is recursive.*

We will describe the algorithm for the case of the group G modifying the exposition from [BGŠ03].

Let $x, g, h \in G$ and

$$g^x = h \tag{5.1}$$

In this case either both g, h are in $\mathrm{St}_G(1)$ or both g, h are not in $\mathrm{St}_G(1)$. We will view (5.1) as an equation in G with variable x, and g and h are considered to be the coefficients of the equation. The next lemma shows how (5.1) can be rewritten in terms of projections.

Lemma 5.1. *1. Let $g, h \in \mathrm{St}_G(1), g = (g_0, g_1), h = (h_0, h_1)$.*

(α) *If $x \in \mathrm{St}_G(1), x = (x_0, x_1)$ then the equation (5.1) is equivalent to the system equations*

$$\begin{cases} {g_0}^{x_0} = h_0 \\ {g_1}^{x_1} = h_1. \end{cases} \tag{5.2}$$

(β) *If $x \notin st_G(1), x = (x_0, x_1)a$ then* (5.1) *is equivalent to*

$$\begin{cases} {g_0}^{x_0} = h_1 \\ {g_1}^{x_1} = h_0. \end{cases} \tag{5.3}$$

2. Let $g, h \notin \mathrm{St}_G(1), g = (g_0, g_1)a, h = (h_0, h_1)a$.

(α) *if $x \in \mathrm{St}_G(1), x = (x_0, x_1)$ then* (5.1) *is equivalent to*

$$\begin{cases} (g_0 g_1)^{x_0} = h_0 h_1 \\ x_1 = g_1 x_0 (h_1)^{-1}. \end{cases} \tag{5.4}$$

(β) *$x \notin st_G(1), x = (x_0, x_1)a$ then* (5.1) *is equivalent to*

$$\begin{cases} (g_0 g_1)^{x_0} = h_1 h_0 \\ x_1 = (g_0)^{-1} x_0 h_1. \end{cases} \tag{5.5}$$

The solutions of the systems (5.2), (5.3), (5.4), (5.5) are considered under the condition that (x_0, x_1) is an element of G.

The cases listed in the lemma are demonstrated by the diagrams in Figure 5.1. .

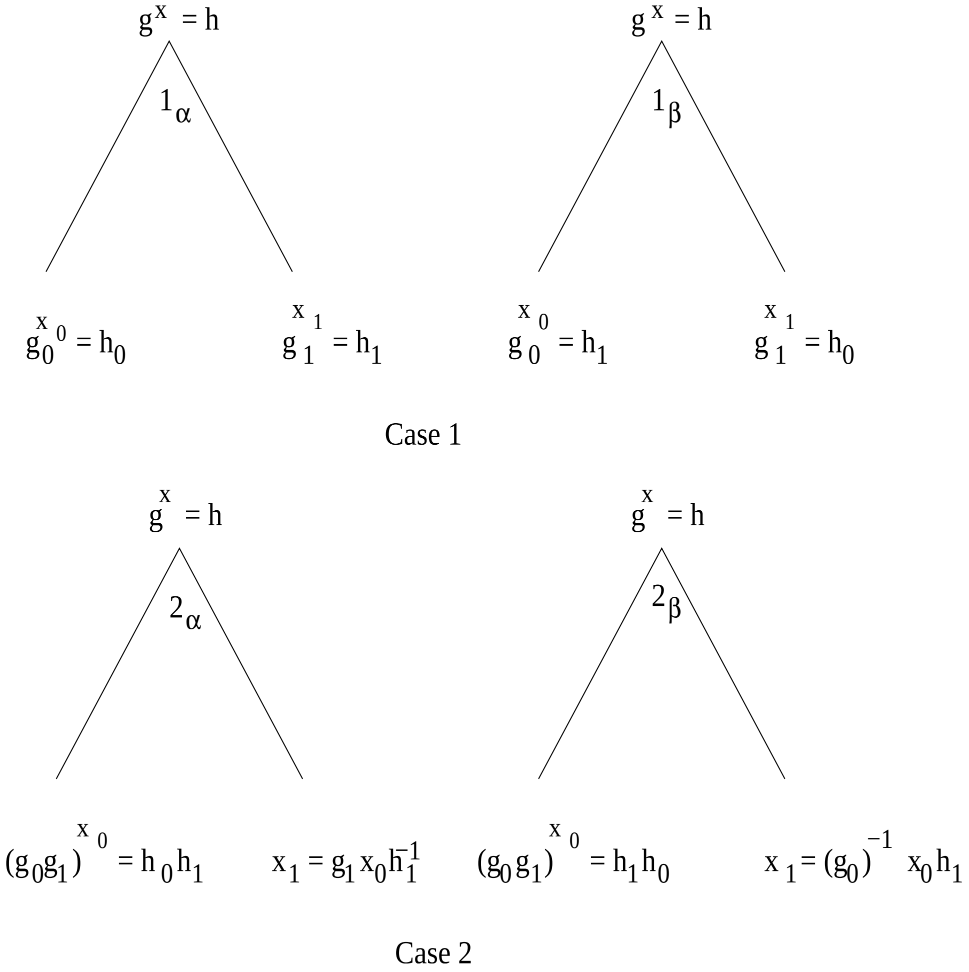


Figure 5.1:

We call the diagrams representing Case 1 independent, while the diagrams representing Case 2 are called dependent (the right vertex on the bottom is dependent on the left vertex). Also we call vertices labelled by a conjugacy equation "c-type" and other vertices ''d-type". This diagram will be used later to draw solution trees for CP.

Recall that $K = \langle (ab)^2 \rangle^G$, $K > K_1$, $K_1 = K \times K$ where each factor acts on the corresponding subtree, and that K_1 has finite index in G. Thus there are only finitely many cosets Kx.

Let

$$Q(g, h) = \{Kx : g^x = h\}. \tag{5.6}$$

The next lemma contains the main idea of the algorithm: finding the set $Q(g, h)$ from the equations given in the bottom of the diagrams in Figure 5.1.

Lemma 5.2. *1. Let $g = (g_0, g_1), h = (h_0, h_1) \in G$ and let for some $u_i, v_j, w_s \in G, i \in I, j \in J, s \in S$, where I, J and S are some indexing sets,*

$$Q(g, h) = \{Ku_i : i \in I\},$$
$$Q(g_0, h_0) = \{Kv_j : j \in J\},$$
$$Q(g_1, h_1) = \{Kw_s : s \in S\}.$$

Then for every $i \in I$ there is $j \in J$ and $s \in S$ such that the element $z = (v_j, w_s)$ is in G and $Ku_i = Kz$.

2. Let $g = (g_0, g_1)a, h = (h_0, h_1)a \in G$ and let

$$Q(g, h) = \{Ku_i : i \in I\}$$
$$Q(g_0 g_1, h_0 h_1) = \{Kv_j : j \in J\}.$$

Then for every $i \in I$ there exist $j, s \in J$ such that the element $z = (v_j, g_1 v_s {h_0}^{-1})$ is in G and $Ku_i = Kz$.

This lemma corresponds to the case (α) of the Lemma 5.1. The statement corresponding to the case (β) is similar and we omit it. Later when we quote Lemma 5.2 we will assume that it covers all the cases and hope that this will not lead to any confusion. Let us call the sets defined by (5.6) the Q-sets.

While the diagrams drawn in Figure 5.1 show how to move from the equation on the top to the equations on the bottom, the last lemma provides the way of moving in the opposite direction (by which we mean the computing of the Q-set on the top of the diagram using the Q-sets on the bottom).

In case the equations on the bottom are simpler than the equation on the top this suggests the following construction. Given the equation (5.1) construct for each $t \geq 1$ the set of trees (we call them solution trees) of depth $\leq t$ by application to the root vertex any of the rules described by Figure 5.1 and then iterating this process independently at every new arising c-type vertex in such a way that the tree will not exceed the t-th level. This procedure leads to a set of decorated trees whose vertices are labelled by the equations of conjugacy type or by the relations corresponding to vertices of d-type. If for some t there is a possibility to compute for any tree of described type the Q-sets for c-type vertices which are leaves of the tree, then the Q-sets of the other vertices can be computed as well. This leads to the computation of Q-sets of all vertices including the root vertex.

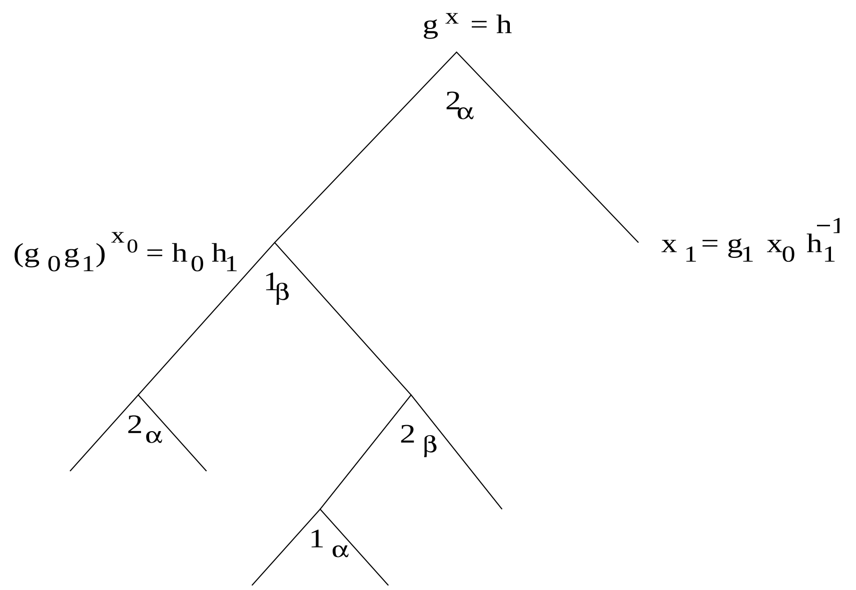


Figure 5.2:

Observe that g and h are conjugate if and only if $Q(g, h)$ is nonempty. This leads to a solution of CP in our situation when we have the reduction of the total length $\tau =\mid g \mid + \mid h \mid$ of coefficients of (5.1) (in case $\tau > 2$).

Indeed, in the light of Lemma 3.1, the length reduction is obvious for Case 1 of Lemma 5.1. For Case 2 we have only the estimate (3.7) but if we assume that g and h have reduced form $a * a * \cdots a*$ (and we can always assume this, if not we just conjugate one or both coefficients by a suitable element from the set $\{a, b, c, d\}$), then $|g_0 g_1| + |h_0 h_1| \leq |g| + |h|$ (which is again follows from Lemma 3.1). If d occurs at least in one of coefficients, then we again get a reduction of the length when moving in the tree one level down, as projecting d produces the empty symbol in one of two directions. If this is not the case then moving a level down may not decrease τ but at least the parameter τ will not increase. Further, projecting c produces d and projecting b produces c, so after at most three moves down we get a reduction of τ in all equations on the third level, for any decorated tree.

Thus after at most $t = 3\tau$ steps down in the tree we will get equations with $\tau \leq 2$. If $\tau = 0$ then the equation is just the relation $1 = 1$. In the other cases the coefficients belong to the set $\{a, b, c, d\}$. As there is only finitely many such equations, we may assume when proving the decidability of CP that we know Q-sets for all such equations. Indeed, we can compute these sets explicitly. Namely, it is known that a, b, c, d belong to different conjugacy classes. Simple arguments show that

$$\begin{aligned} Q(a,a) &= \{1, a, (ac)^4\} \\ Q(b,b) &= \{1, b, c, d\} \\ Q(c,c) &= \{1, b, c, d\} \\ Q(d,d) &= \{1, b, c, d, ada, (ad)^2, bada, badad\}. \end{aligned}$$

Also we have the following table of lifts of pairs of cosets (we only show the representatives for each coset; the symbol $-$ means that the corresponding pair does not belong to G))

	1	a	b	c	d	ada	$(ad)^2$	$bada$	$badad$	$(ac)^4$
1	1	$-$	d	$-$	$-$	$-$	$-$	$-$	$-$	$-$
a	$-$	$-$	$-$	b	c	$-$	$-$	$-$	$-$	$-$
b	ada	$-$	$(ad)^2$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
c	$-$	b	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
d	$-$	$bada$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
ada	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$(ad)^2$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$(ac)^4$	$-$	$-$
$bada$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$badad$	$-$	$-$	$-$	$-$	$-$	$-$	d	$-$	$(ad)^2$	$-$
$(ac)^4$	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	1

So to solve CP in G for the pairs of elements g, h we do as follows. Compute τ; construct all solution trees of depth not greater 3τ which have leafs of c-type labelled by equations with coefficients belonging to the set $1, a, b, c, d$ (if one of coefficients is 1 then the second is also 1 and the equation becomes $1 = 1$, in which case the Q-set consists of all cosets Kx); for each such a tree find Q-sets of the leafs and the Q-sets of all other vertices finishing the computation in the root vertex. This completes the description of the algorithm. The algorithm also allows to prove the following

Theorem 5.2. *The group G is conjugacy separable.*

Recall that being conjugacy separable means that two elements are conjugate if and only if they are conjugate in each finite quotient. In other words, to show that if two elements g, h are not conjugate in G then they are not conjugate in some finite quotient one has to consider the images of any g, h in $G/\operatorname{St}_G(3\tau{+}1)$. They will not be conjugate either, as for the quotient group a similar algorithm of verification of conjugacy works.

Although the algorithm is quite nice, it looks as if it has large complexity both in space and in time. Namely one has to construct all trees of the sort we consider of depth $\leq 3\tau$ and a rough estimate for the number of such trees is two iterations of the exponential function. Then for each tree one has to solve the problem of lifting of Q-sets (but this is not hard). So a rough upper bound for complexity both in space and in time is $e^{e^{\tau}}$.

This makes the group G a good candidate for use in cryptography where groups with easy WP but difficult CP play a special role . But maybe the CP is indeed not so difficult in G? This is our next question.

Problem 5.1. *What is the complexity of CP for G? Is it subexponential, exponential or superexponential?*

Problem 5.2. *(a) Is conjugacy problem solvable for a group generated by a finite set of finite invertible automata?*

(b) In particular is it solvable for a self-similar group?

Problem 5.3. *Are all self-similar groups with solvable CP conjugacy separable?*

It is well known that the classical Dehn isomorphism problem is undecidable and there is no algorithm which, given a finite presentation, determines if a group is trivial or not. Nevertheless the isomorphosm problem can be considered for special classes of groups and special type of presentations and then it may happen to be decidable.

Problem 5.4. *Is the isomorphism problem decidable for the class of self-similar groups?*

Problem 5.5. *Is the isomorphism problem decidable for recursively presented branch groups?*

I would expect that the answer to both questions is negative. But there are some results showing that for smaller subclasses an algorithm exists.

One can consider the isomorphism problem for a class of groups containing non recursively presented groups for instance for the class G_ω. In 1984 I observed that for each $\omega \in \Omega$ there are at most countably many sequences $\eta \in \Omega$ such that $G_\omega \simeq G_\eta$. The proof of this fact was based on the statement that the bijection between canonical set of generators of G_ω and G_η extends to a homomorphism $G_\omega \to G_\eta$ if and only if $\omega \simeq \eta$. The problem of isomorphism for the class G_ω, $\omega \in \Xi$, was recently solved by V. Nekrashevych [Nek05] who showed that G_ω is isomorphic to G_η if and only if ω can be obtained from η by some permutation of symbols 0,1,2 applied to all entries. The proof is based on the technique of [LN02] or on the result from [GW03b], stating that under certain conditions an action of a branch group on a rooted tree is essentially unique (unique up to the procedure of deletion of levels) .

A new idea to the solution of isomorphism problem is advanced by V. Nekrashevych in [Nek05]. Namely, to certain self-similar groups of branch type he associates a topological dynamical system and proves that the groups are isomorphic if and only if the corresponding systems are conjugate. In some cases he is able to show that the groups are not isomorphic by showing that the corresponding dynamical systems are not conjugate.

This is the first case when non-isomorphism of groups is established by means of dynamical systems.

One can also consider a classification problem for the self-similar groups. For instance one can try to classify them when fixing a complexity.

By a complexity we mean the parameter (m, n) where m is the cardinality of the states of the automaton generating the corresponding group while n is the cardinality of the alphabet. There are six groups of complexity (2,2). There is a hope that soon we will have the classification of self-similar groups of complexity (3,2). The next steps would be the consideration of groups of complexity (2,3) and (3,3).

6 Subgroup structure and branch groups

The group G has very nice and interesting subgroup structure. The main property is that its lattice of subnormal subgroups has branching structure, following the structure of the tree on which the group acts. Such groups are called branch groups (see relevant definitions below). Moreover, the group G is regular branch over the subgroup $K = \langle (ab)^2 \rangle^G$. The latter means that K has finite index in G and, if we denote by K_n the subgroup of $\mathcal{G} = \mathrm{Aut}\ T$ equal to the direct product of 2^n copies of K (each factor acts on a subtree of T with root at the n-th level), then K_n is a subgroup of K of finite index.

This automatically implies that all K_n, $n = 1, 2, \ldots$ are subgroups of G and that $\{K_n\}_{n=1}^{\infty}$ is a descending sequence of normal subgroups of finite index in G with trivial intersection. The existence of such a sequence plays a crucial role in many considerations, in particular in proving that G is a just infinite group, i.e., is infinite but every proper quotient is finite [Gri00b]. Thus all normal subgroups in G have finite index.

The question of existence of a residually finite just infinite group different from classical examples like $SL(n, \mathbb{Z})/\text{centre}$, $n \geq 3$ [Men65], [BMS67] was raised in [CM82] and answered in [GS83, Gri84a] by producing examples of just infinite groups of branch type. Indeed branch groups constitute one of three classes into

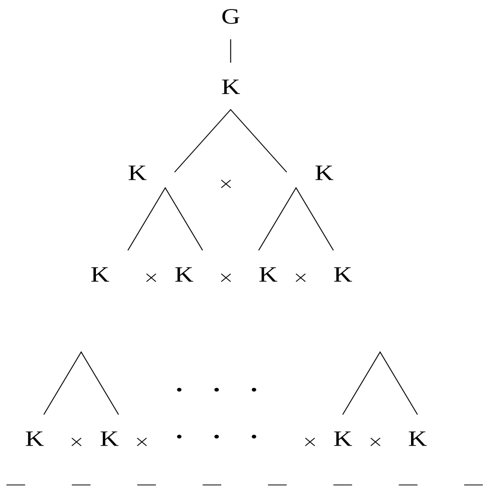


Figure 6.1:

which the class of just infinite groups naturally splits [Gri00b]. The other two classes are related to hereditary just infinite groups and to simple groups [Gri00b].

Let L be a group acting on a rooted tree T. Consider the following subgroups called respectively stabilizer of level n, rigid stabilizer of a vertex u and rigid stabilizer of level n:

$$st_L(n) = \{g \in L\colon\ u^g = u, \forall u \in V_n\},$$
$$\mathrm{rist}_L(u) = \{g \in L\colon\ u^g = u \quad \forall u \in T \backslash T_u\},$$
$$\mathrm{rist}_L(n) = \langle \mathrm{rist}_L(u)\colon\ u \in V_n \rangle = \prod_{u \in V_n} \mathrm{rist}_L(u),$$

where V_n denotes the set of vertices on level n, and T_u denotes the subtree with root at vertex u)

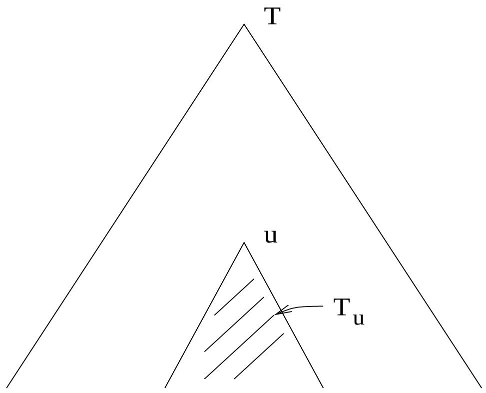


Figure 6.2: $\mathrm{rist}_G(u)$ acts nontrivially only on a subtree T_u

Although a stabilizer always has finite index, a rigid stabilizer can have infinite index and can even be trivial. We get the following classification of actions:

1) $\mathrm{rist}_L(n)$ has finite index in L for all $n = 1, 2, \ldots$

2) $\mathrm{rist}_L(n)$ has infinite index starting from some n, but is an infinite subgroup for all $n = 1, 2, \ldots$

3) $\mathrm{rist}_L(n)$ is finite group starting at some level n (and hence is trivial starting at some level $m \geq n$).

A group L is called a branch group if for some sequence $\overline{m} = \{m_n\}_{n=1}^{\infty}$ of integers $m_n \geq 2$ (called branch index) it has a faithful, spherically transitive (i.e., transitive on levels) action of type 1) on spherically homogeneous rooted tree $T_{\overline{m}}$ defined by the sequence $\overline{m}$ [Gri00a]. An action of type 1) is called a branch action, an action of type 2) is called weakly branch and action of type 3) is called diagonal.

A group L acting faithfully on a d-regular rooted tree $T = T_d$ (that is, with $m_n = d$, $n = 1, 2, \ldots$) is called regular branch over a subgroup $M < L$ if $[L : M] < \infty$ and M contains as a subgroup of finite index the group

$$M_1 = \underbrace{M \times \cdots \times M}_{d} \tag{6.1}$$

i-th factor in (6.1) acts on the corresponding subtree T_i of T with root u_i at first level and we use the natural identification of T_i and T to induce the action of M from T to T_i.

A regular branch group L is also a branch group because $\mathrm{rist}_L(n) \geq M_n$ where

$$M_n = \underbrace{M \times \cdot \times M}_{d^n}$$

(here each factor acts on the corresponding subtree with root at n-th level), and clearly $[G : M_n] < \infty$, $n = 1, 2, \ldots$.

A group L is called weakly regular branch over a subgroup M if $M \neq \{1\}$ and M contains M_1 as a subgroup (but not necessary of finite index).

We do not introduce the notion of a diagonal group since every countable residually finite group acts faithfully on some spherically homogeneous tree. Just take a descending sequence of normal subgroups of finite index intersecting trivially, form a tree in which the vertices are the left cosets and the incidence is induced by inclusion (the root is the whole group), and let the group act on the coset tree by left multiplication.

The group G is regular branch group for its action on the binary rooted tree as defined in Section 1. Indeed, $\mathrm{rist}_G(n) \geq K_n$ for any $n = 1, 2, \ldots$ where $K = \langle (ab)^2 \rangle^G$.

Other examples of regular branch groups include Gupta–Sidki p-groups and $IMG(z^2 + i)$ [BGN03] to cite a few [Gri00a, BGŠ03]. The Basilica group (which is $IMG(z^2 - 1)))$ is an example of a weakly branch group.

Let us now give more information about the structure of stabilizers and rigid stabilizers. The abelization G_{ab} is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and the group G has 7 subgroups of index 2:

$$\begin{gathered} \langle b, ca \rangle, \quad \langle c, ad \rangle, \quad \langle d, ab \rangle \\ \langle b, a, a^c \rangle, \quad \langle c, a, a^d \rangle, \quad \langle d, a, a^b \rangle \\ st_G(1) = H = \langle b, d, b^a, d^a \rangle, \end{gathered}$$

listed in accordance with the number of generators 2, 3 or 4. Normal closures of generators and factors by them are

$$\begin{aligned} A = \langle a \rangle^G &= \langle a, a^b, a^c, a^d \rangle, \quad G/A \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \\ B = \langle b \rangle^G &= \langle b, b^a, b^{ad}, b^{ada} \rangle, \quad G/B \simeq D_8 \\ C = \langle c \rangle^G &= \langle c, c^a, c^{ad}, c^{ada} \rangle, \quad G/C \simeq D_8 \\ D = \langle d \rangle^G &= \langle d, d^a, d^{ac}, d^{aca} \rangle, \quad G/D \simeq D_{16} \end{aligned}$$

(here D_{2n} is the dihedral group of order $2n$). To formulate the next statement and give more information on the top part of the lattice of normal subgroups consider the following subgroups of G

$$\begin{aligned}
K &= \langle (ab)^2 \rangle^G, \quad L = \langle (ac)^2 \rangle^G, \quad M = \langle (ad)^2 \rangle^G \\
\overline{B} &= \langle B, L \rangle, \quad \overline{C} = \langle C, K \rangle, \quad \overline{D} = \langle D, K \rangle \\
T &= K^2 = \langle (ab)^4 \rangle^G \\
T_m &= \underbrace{T \times \cdots \times T}_{2^n} \\
K_m &= \underbrace{K \times \cdots \times K}_{2^m} \\
M_m &= T_{m-1} K_m.
\end{aligned}$$

In T_m and K_m each factor acts on a corresponding subtree with roots at the m-th level.

Theorem 6.1 ([Roz93], [BG02a]). *The following hold in G.*

(i)

$$st_G(n) = \begin{cases} H & \text{if } n = 1 \\ \langle D, T \rangle & \text{if } n = 2 \\ \langle M_2, (ab)^4 (adacac)^2 \rangle & \text{if } n = 3 \\ \underbrace{st_G(3) \times \cdots \times st_G(3)}_{2^{n-3}} & \text{if } n \geq 4 \end{cases}.$$

(ii)

$$rist_G(n) = \begin{cases} D & \text{if } n = 1 \\ K_n & \text{if } n \geq 2 \end{cases}.$$

(iii) in the lower central series $\{\gamma_n(G)\}_{n=1}^{\infty}$

$$\gamma_{2^m+1}(G) = M_m, \qquad m = 1, 2, \dots,$$

(iv) the derived series $\{G^{(n)}\}_{n=1}^{\infty}, \{K^{(n)}\}_{n=1}^{\infty}$ of groups G and K respectively

$$\begin{aligned}
G^{(n)} &= rist_G(2n-3), \qquad n \geq 3 \\
K^{(2)} &= rist_G(2n), \qquad n \geq 2.
\end{aligned}$$

It is important for the study of various properties of the group G to get as full information as possible about the lattice of subgroups of finite index, normal subgroups, subgroups closed in the profinite topology. For the moment the lattice of normal subgroups is tolerably well understood, and its upper part below the subgroup $H = st_G(1)$ is given in Figure 6 (see [BG02a]).

Detailed study of normal subgroups of index $\leq 2^{11}$ is carried out in [CSST01] where it is shown in particular that for every normal subgroup $N \triangleleft G$, $N \neq \{1\}$, there exists n such that $M_{n+1} < N < st_G(n)$. The problem is solved in [Bar05] via using Lie algebras constructed from the lower central series of G. It follows from his work that the normal subgroup growth of G is equal to $n^{\log_2 3}$, and that every normal subgroup of G is generated as a normal subgroup by one or two elements.

The following problem remains open:

Problem 6.1. *What is the subgroup growth of G?*

Maximal subgroups play a special role in group theory. The question whether all maximal subgroup of G have finite index (let us call such a property (P)) was open for a long time before it was answered positively by E. Pervova [Per00]. She proved that only subgroups of index 2 (listed above) are maximal subgroups in G and hence the Frattini subgroup $\phi(G)$ is equal to the commutator subgroup G'. This gives a very simple and efficient criterion to determine when a set of elements $S \subset G$ generates G, namely if and only if its

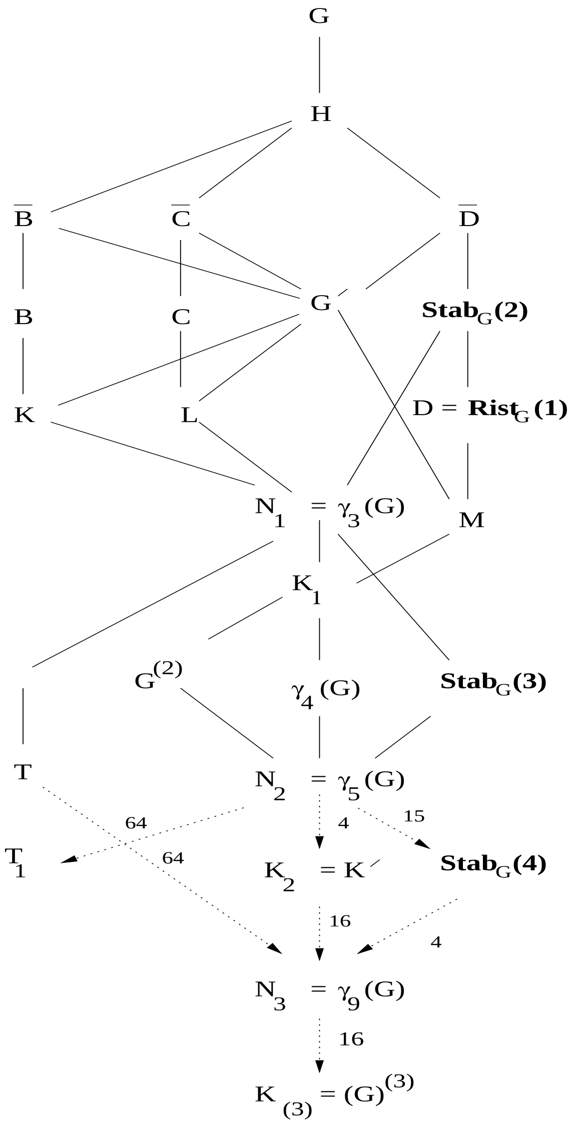


Figure 6.3: The top of the lattice of normal subgroups of G below H.

image $\overline{S}$ in $G/\Phi(G) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ generates the quotient. It is obvious that this is a necessary condition. To show sufficiency let us assume that $\overline{S}$ generates $G/\Phi(G)$ but $\langle S\rangle$ is a proper subgroup L in G. Then L is a subgroup of some maximal subgroup $\tilde{L}$ in G which has index 2 so the image of L in $G/\Phi(G)$ will be of index 2. Contradiction.

Pervova extended her result to Gupta–Sidki p-groups and some other groups [Per02a]. Her proof is quite tricky and doesn't seem to work for all branch groups.

Problem 6.2. *Is there a finitely generated branch group with a maximal subgroup of infinite index?*

In [GW03a] Pervova's result is improved by showing that not only G has property (P) but also any group L abstractly commensurable with G has property (P) (recall that two groups are abstractly commensurable if they have isomorphic subgroups of finite index). In particular, every subgroup of G finite index possesses (P). By the same arguments as in [GW03b] one can show that if L is a branch group and has (P) then any group abstractly commensurable with L has (P).

Weakly maximal subgroups constitute another important class of subgroups. (These are subgroups of infinite index maximal with respect to this property, i.e., adding to the group a single element from its complement extends it to a subgroup of finite index). The group G has many such subgroups. For instance, denote by ∂T the boundary of the tree, that is the set of all geodesic rays connecting the root with infinity. The action of G on T extends naturally to an action on ∂T. Then for any $\xi \in \partial T$, the stabilizer

$$P_\xi = st_G(\xi) = \{g \in G\colon\ \xi^g = \xi\}$$

is a weakly maximal subgroup [BG02a]. We call P_ξ a parabolic subgroup.

It is also easy to produce an example of a weakly maximal subgroup in G not of parabolic type. For instance, let $L = (Q \cdot U) \rtimes \langle a \rangle$ where Q is the diagonal subgroup of G consisting of elements (g, g), $g \in B$ (g has to belong to the subgroup $B \cdot \langle (ad)^2 \rangle$ in order for the pair (g, g) belong to G) and $U = \langle c, c^a \rangle$. Then L is a finitely generated weakly maximal subgroup of G. Indeed let us show that if $x \notin L$ then $\langle L, x \rangle$ has finite index in G. As $a \in L$ we can assume $x \in st_G(1)$, $x = (x_0, x_1)$. Let $x_0 = yz$ where $y \in B$ and $z \in \langle a, d \rangle$. The element (y, y) belongs to Q so we can replace x by element $(z, y^{-1}x_1)$. Taking an element in U of the form (z, w) we can replace x by $(1, w^{-1}y^{-1}x_1)$, with $w^{-1}y^{-1}x_1 \neq 1$. Let $N = \langle w^{-1}y^{-1}x_1 \rangle^G$. Then N is a normal subgroup of finite index and the group $\langle L, x \rangle$ contains the group $\psi^{-1}(N \times N)$ and therefore has finite index in G.

Problem 6.3. *Describe all weakly maximal subgroups in G.*

The following description of the parabolic subgroup $P = P_G(1^\infty)$ for the rightmost path $\xi = 1 \cdots 1 \cdots = 1^\infty$ is given in [BG02a]. Denote $B = \langle b \rangle^G$, $Q = B \cap P$, $R = K \cap P$ where $K = \langle (ab)^2 \rangle^G$. Then the following decompositions hold

$$\begin{aligned} P &= (B \times Q) \rtimes \langle c, (ac)^4 \rangle \\ Q &= (K \times R) \rtimes \langle b, (ac)^4 \rangle \\ R &= (K \times R) \rtimes \langle (ac)^4 \rangle \end{aligned} \tag{6.2}$$

where the factors $B \times Q$, $K \times R$ act on subtrees with roots at the first level. (6.2) should be viewed as recursive formulas describing P, Q, R in terms of action of the same groups on corresponding subtrees. Here we use again self-similarity of involved groups. By iterating formulae (6.2) we get the following decomposition for P

$$P = (B \times ((K \times ((K \times \cdots) \rtimes \langle (ac)^4 \rangle)) \rtimes \langle b, (ac)^4 \rangle)) \rtimes \langle c, (ac)^4 \rangle$$

where each factor from the direct or semi-direct product acts on the corresponding subtree as indicated in Figure 6.4. The product $B \times K \times K \times \cdots \times K \times \cdots$, where each factor acts on the corresponding subtree according to 6.4, is a subgroup of the group P on which act the active factors $\langle c, (ac)^4 \rangle$, $\langle b, (ac)^4 \rangle$, $\langle (ac)^4 \rangle \ldots$ from the decomposition above. These factors are not, formally speaking, elements of G (because they act not on the whole tree but only on certain subtrees). In order to get a corresponding element of G we need to lift the projection by using the i-th power of the endomorphism σ (see (4.8)).

Thus active factors $\langle b, (ac)^4 \rangle$,$\langle (ac)^4 \rangle$, $\langle ac \rangle^4 \ldots$ which correspond to the rightmost ray on the diagram 6.4 will be represented by subgroups $\sigma(\langle b, (ac)^4 \rangle)$,$\sigma^2(\langle (ac)^4 \rangle)$, $\sigma^3(\langle ac \rangle^4) \ldots$. This is explained by the fact that for an arbitrary element $g \in \langle b, (ac)^4 \rangle$ the following holds: $\sigma(g) = (1, g)$.

A parabolic subgroup P_ξ can be presented as an intersection of subgroups of finite index

$$P_\xi = \bigcap_{n=1}^{\infty} P_{n,\xi}$$

where $P_{n,\xi}$ is the stabilizer of the vertex of level n belonging to the path ξ. This implies that P_ξ is closed in profinite topology. More generally, any weakly maximal subgroup $L < G$ can also be presented as an intersection of subgroups of finite index. Indeed, there is a maximal subgroup Q_1 of index 2 which contains L, and, as it has property (P), there is a subgroup Q_2 of index 2 in Q_1 containing L, and so on. We get a descending sequence $\{Q_n\}_{n=1}^{\infty}$ of subgroups of finite index which contain L.

$$\bar{L} = \bigcap_{n=1}^{\infty} Q_n$$

If we suppose that their intersection is different from L we get a contradiction with the weak maximality of L. Hence $L = \bigcap_{n=1}^{\infty} Q_n$ and L is closed in the profinite topology.

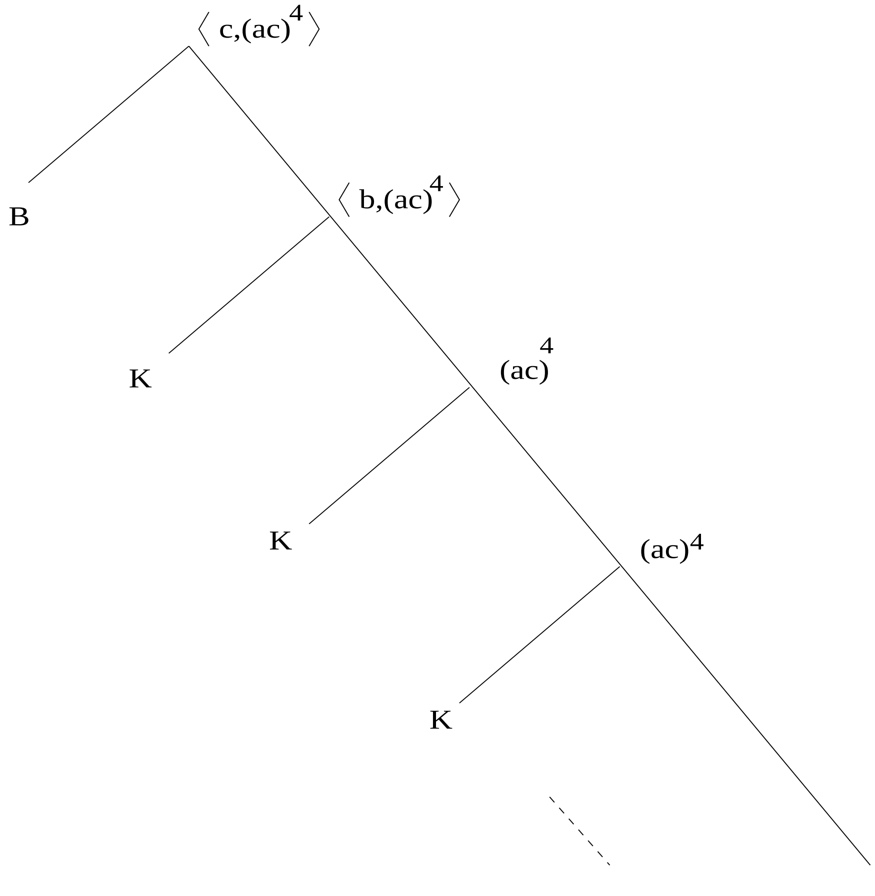


Figure 6.4: Decomposition of parabolic subgroup.

Problem 6.4. *Describe in algebraic terms the subgroups of G which are closed in profinite topology.*

As was already mentioned in Section 4, finitely generated subgroups of G are closed in profinite topology (see [GW03a]). Such groups are called subgroup separable (SS). Free groups, polycyclic groups, surface groups, some 3-manifold groups and free products of groups with property (SS) are known to be (SS). One of the consequences of the (SS) property is the solvability of the membership problem.

The following problem is inspired by an analogous result of L. Ribes and P. Zaleski [RZ93] in the case of a free group.

Problem 6.5. *Is the product $A_1A_2\dots A_n$ of finitely generated subgroups of G closed in the profinite topology?*

The next property that we are going to discuss is the congruence subgroup property (CSP). We say that a group L acting on rooted tree $T_{\overline{m}}$ has the CSP if every subgroup $M < L$ of finite index contains $st_L(n)$ for some $n \geq 1$. Thus we consider $\{st_L(n)\}_{n=1}^{\infty}$ as the principal congruence subgroups sequence. The choice of the descending sequence of stabilizers looks natural but is not the only possible. For instance, for branch groups, one can also study the congruence subgroup problem with respect to the sequence of rigid stabilizers.

The group G has the CSP. Indeed every subgroup L of finite index in G contains a normal subgroup of finite index and hence by Theorem 4 [Gri00b] contains the commutator subgroup $(\mathrm{rist}_G(n))'$ of a rigid stabilizer for sufficiently large n. Therefore it contains the group $K'_n \cong \underbrace{K' \times \dots \times K'}_{2^n}$. But K' contains $st_G(5)$, hence $L > \underbrace{st_G(5) \times \dots \times st_G(5)} \geq st_G(n+5)_{2^n}$ (each factor in the above products acts on the corresponding subtree with root at the n-th level).

An example of a branch p-group acting on the p-regular rooted tree (p odd prime) and without CSP is constructed in [Per02b].

The CSP for groups acting on trees is important first of all because it allows to identify the closure of the group in Aut T with its profinite completion. For groups without CSP the problem of description of profinite completion is harder. In some cases it is solved in [Per04].

Let L be a residually finite group and $\{M_n\}_{n=1}^{\infty}$ be a descending sequence of subgroups of finite index with intersection that has a trivial core (i.e., $\cap_{g\in G} P^g = \{1\}$, where P denotes $\cap_{n=1}^{\infty} M_n$) . Call such a sequence a T-sequence. Then a tree of type $T_{\overline{m}}$ and a faithful action of L on T can be constructed in canonical way. Namely as the set of vertices of T one takes the set of cosets gM_n, $g \in L$, $n = 0, 1, 2, \dots$ ($M_0 = L$) (the root vertex is represented by the coset $1 \cdot L$), and two vertices gM_n and hM_k, $n \leq k$ are joined by an edge if and only if $k = n + 1$ and $hM_k \subset gM_n$. The action of L on the set of vertices is by left multiplication.

Conversely, given a tree $T = T_{\overline{m}}$ and faithful action of L on T one gets a sequence of subgroups of finite index of the form $\{st_L(u_n)\}_{n=1}^{\infty}$ where $\{u_n\}_1^{\infty}$ is the sequence of vertices of a geodesic ray $\xi \in \partial T$ joining the root vertex with infinity.

If $\{M_n\}_{n=1}^{\infty}$ is a sequence of normal subgroups, then $M_n = st_L(n)$, $n = 1, 2, \dots$ Thus the study of actions on rooted trees is closely related to the study of descending sequences of subgroups of finite index whose intersection has trivial core. Call two sequences $\overline{M} = \{M_n\}_{n=1}^{\infty}$, $\overline{N} = \{N_n\}_{n=1}^{\infty}$ equivalent and write $\{M_n\}_{n=1}^{\infty} \sim \{N_n\}_{n=1}^{\infty}$ if for any n there is k s.t. $M_n < N_k$ and vice versa, for any k there is n s.t. $M_n > N_k$.

Problem 6.6. *(i) Is there a branch group with infinitely many equivalence classes of normal T-sequences?*

(ii) If the answer to the above question is positive how complicated can be the lattice of classes?

Let us say that two actions of L on trees $T_{\overline{M}}$ and $T_{\overline{N}}$ given by a normal T-sequences $\overline{M} = \{M_n\}_{n=1}^{\infty}$ and $\overline{N} = \{N_n\}_{n=1}^{\infty}$ are equivalent if $\overline{M} \sim \overline{N}$. The study of actions of a group on a rooted tree up to this equivalence is an interesting topic. For a group of branch type there should not be too many different actions. One result in this direction is obtained in [GW03b] where it is showm that under certain conditions a branch group has only one action up to deletion of levels.

An important role in the study of lattice of subgroups of a group L belongs to the group of automorphisms of L. For many groups L of branch type Aut L coincides with the normalizer $N_{\mathrm{Aut}\ T}(L)$ of L in the group of automorphisms of the tree. There are at least three approaches to proving this. One was given by Sidki [Sid87a] in the case of the 3-group of Gupta and Sidki, another by Lavreniuk and Nekrashevich [LN02] for branch groups which have a so-called saturated sequence of characteristic subgroups, and a third one [GW03b] via the property of uniqueness of actions mentioned above.

Problem 6.7. *Is it correct that for any branch group L acting on a tree T*

$$Aut\ L = N_{Aut\ T}(L)?$$

If true, this helps calculating Aut L, as was used in [Sid87a] for Gupta–Sidki 3-groups and in [GS04] for G, where it was shown that Out $G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \ldots$ is an elementary 2-group of infinite rank (the generators of this group are explicitly calculated). As a consequence of the result in [GS04] we get that every normal subgroup in G is characteristic.

It is shown in [LN02] if L is a weakly branch group then

$$\text{Aut } L = N_{\text{Homeo } \partial T}(L),$$

where Homeo $\partial T(L)$ is the group of homeomorphisms of the boundary ∂T.

An interesting approach to the study of normal and characteristic subgroups of groups acting on rooted trees is due to Kaloujnin [Kal45, Kal48] and is based on the use of so-called tables and their characteristic called height. The modern exposition of Kaloujnin method leading to some new results is done in [CSLST04] and [LNS]. It would be interesting to apply Kaloujnin method for studying characteristic and normal subgroups of finitely generated self-similar groups of branch type.

7 Finite quotients and groups of finite type

Any group L acting faithfully on a rooted tree T is residually finite: the approximating sequence of finite quotient groups is $L_n = L/st_L(n)$, $n = 1, 2, \ldots$. The group L_n acts on the subtree $T_{[n]}$ where the action is induced by "forgetting" the action of L below level n. The study of the sequence $\{L_n\}$ can be useful for the study of L and its profinite completion.

Let us consider the sequence $G_n = G/st_G(n)$ approximating the group G. The first three groups are $G_1 = C_2$, $G_2 = C_2 \wr C_2$, $G_3 = C_2 \wr C_2 \wr C_2$ (so they are the full groups of automorphisms of the trees $T_{[1]}$, $T_{[2]}$, $T_{[3]}$). Starting from the fourth level the character of the sequence $\{G_n\}$ completely changes, in particular the number of generators remains to be 3. If we keep the notation a, b, c, d for the generating elements of G_n then the following statement holds [Gri00b, dlH00].

Proposition 7.1. *When $n \geq 3$ the map* (3.4) *induces an embedding*

$$\psi^{(n)}\colon\ G_{n+1} \longrightarrow G_n \wr C_2 = (G_n \times G_n) \rtimes \langle a \rangle \tag{7.1}$$

the image of which has index 8 and has the form

$$((B_n \times B_n) \rtimes \langle (a, c), (c, a) \rangle) \rtimes \langle a \rangle$$

where B_n is the image of B in G_n.

Corollary 7.1.

$$|G_n| = 2^{5 \cdot 2^{n-3}+2}.$$

Indeed $|G_3| = 2^7$ and (7.1) imply

$$|G_{n+1}| = \frac{1}{4}|G_n|^2,$$

for $n \geq 3$.

The above proposition and corollary can be used in two different directions. First, it can be used for computation of the Hausdorff dimension of the closure $\overline{G}$ of G in Aut T (recall that $\overline{G}$ is isomorphic to the profinite completion $\widehat{G}$ because of the congruence subgroup property).

By definition the Hausdorff dimension of a closed subgroup L in a profinite group M with a descending chain $\{M_n\}$ of open normal subgroups is a number from [0,1] defined as

$$\text{Dim } L = \liminf_{n\to\infty} \frac{\log |LM_n/M_n|}{\log |M/M_n|}.$$

In our case $M = \text{Aut } T, M_n = st(n)$,

$$|M/M_n| = 2^{1+2+\cdots+2^{n-1}} = 2^{2^n - 1}$$

and hence

$$\text{Dim } \overline{G} = \lim_{n\to\infty} \frac{5 \cdot 2^{n-3} + 2}{2^n - 1} = \frac{5}{8}.$$

It was mentioned in the introduction that profinite groups of the form $\bar{L}$ where L is a self-similar group play an important role in Galois Theory and in the theory of profinite groups.

Problem 7.1. *(i) What is the set of values of Hausdorff dimensions of groups of the form $\bar{L}$ where L is a self-similar group?*

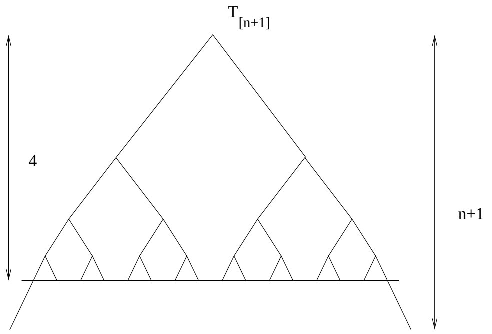
(ii) In particular, are the numbers Dim $\bar{L}$ *always rational?*

(iii) Is there an algorithm that, given an automaton A, computes Dim $\bar{L}$*?*

We note that M. Abert and B. Virag have shown that, with probability 1, the Hausdorff dimension of the closure of three randomly chosen tree automorphisms is 1 [AV05]. Z. Sunik has constructed concrete examples of regular branch self-similar groups of Hausdorff dimension arbitrary close to 1 [Sun].

The second application of Proposition 7.1 leads to the idea of groups of finite type which we are going to explore now. But before doing this let us consider a finitary version.

If $g_0, g_1 \in G_n$ and $x \in S_2$ then the element $(g_0, g_1)x \in \text{Aut } T_{[n+1]}$ belongs to G_{n+1} if and only if it belongs to the image of the map (7.1). In the language of portraits it means that the portrait (the labelling of vertices of $T_{[n+1]}$ up to level n by elements of S_2) of the element $(g_0, g_1)x$, has label x at the root vertex, has the portrait of the element g_0 in left subtree and the portrait of the element g_1 in right subtree (the left and the right subtrees have their roots on the first level of $T_{[n+1]}$).



Moreover the labelling inside the subtree $T_{[4]}$ must coincide with a portrait of some element of the group G_4. Vice versa, if this condition is satisfied then $(g_0, g_1)x$ belongs to G_{n+1}. Indeed, since the index of the image $\overline{B}$ of B in the group G_3 coincides with the index $|G : B| = 8$ it follows that $B \geq st_G(3)$. If $(h_0, h_1)y \in G_{n+1}$ has the same portrait as $(g_0, g_1)x$ on the subtree $T_{[4]}$ then $x = y$ and $(g_0, g_1)x((h_0, h_1)x)^{-1} = (g_0h_0^{-1}, g_1h_1^{-1}) \in st_{G_{n+1}}(4)$. Therefore $g_0h_0^{-1}$, $g_1h_1^{-1} \in st_{G_n}(3)$ and so $g_0h_0^{-1}$, $g_1h_1^{-1} \in B_n$. But this implies that $(g_0h_0^{-1}, g_1h_1^{-1})$ belongs to the image of $\psi^{(n+1)}$, and therefore $(g_0, g_1)x((h_0, h_1)x)^{-1} \in G_{n+1}$ and $(g_0, g_1)x \in G_{n+1}$.

¿From these considerations we get

Proposition 7.2. *A labelling of $T_{[n+1]}$ is a portrait of some element in G_{n+1} if and only if for each subtree $T' \subset T_{[n+1]}$ which is isomorphic to $T_{[4]}$ the pattern of the labelling inside T' determines an element of the group G_4 (after identification of T' with $T_{[4]}$).*

This statement is illustrated in Figure 7.

Let us call a configuration on $T_{[4]}$ which correspond to elements in Aut $T_{[4]} \backslash G_4$ forbidden patterns. Then what was written in the previous paragraphs can be reformulated as follows.

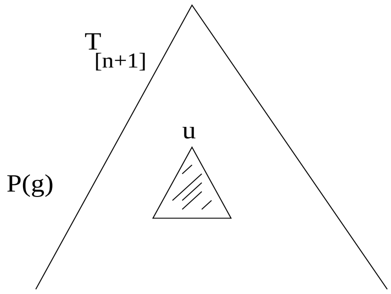


Figure 7.1: $\forall T' \subseteq T_{[n+1]}$ the corresponding pattern is not forbidden.

Proposition 7.3. *A labelling of $T_{[n+1]}$ is a portrait of some element of G_{n+1} if and only if it doesn't contain a forbidden pattern.*

In a way analogous to the case of the group G one can define a core portrait of an element of G_n and the depth $d(g)$, $g \in G_n$. Again as in 3.16 the inequality $d_n(g) \leq \log|g| + 1$ holds. The next problem is analogous to the Problem 3.4 (ii).

Problem 7.2. *(i) Find a description of the portraits of elements of fixed length in the groups G_n, $n = 1, 2, \ldots$?*

(ii) What is the maximal length of the elements in G_n and how do the the portraits of the elements of maximal length look like?

In all cases we consider the length of the elements in G_n with respect to the systems of generators that is the image of the canonical one.

Similar arguments which were used to prove the Proposition 7.1 lead to the following

Proposition 7.4. *A core portrait $P_{core}(g)$, $g \in \mathrm{Aut}\, T$ determines an element of G if and only if the corresponding portrait $P(g)$ of g does not contain a forbidden pattern.*

Corollary 7.2. *A portrait $P(g)$, $g \in \mathrm{Aut}\, T$ determines an element of the closure $\overline{G}$ of G in $\mathrm{Aut}\, T$ if and only if it does not contain a forbidden pattern.*

Remark. The verification if $P_{\mathrm{core}}(g)$ satisfies the condition of Proposition 7.2 requires only verification of the condition on the subtree $T_{[d+4]}$ where d is the depth of $P_{\mathrm{core}}(g)$.

In Ergodic Theory there is a notion of a subshift of finite type [Kit98]. Given an alphabet X, and a finite set $F \subset X^*$ of words (called forbidden words) one considers the subspace $\Omega_F \subset \Omega$ of the space $\Omega = X^{\mathbb{N}}$ of infinite sequences consisting of those sequences $x = x_0 x_1 \ldots x_n \ldots$ that do not contain forbidden subwords. This set is closed with respect to the shift τ, where

$$(\tau x)_n = x_{n+1}.$$

The dynamical system (τ, Ω_F) is called a subshift of finite type.

In our situation the "ray" $\mathbb{N}$ is replaced by a tree and the set F is replaced by a set of forbidden patterns.

Remark. In the definition of a subshift of finite type the finite set F can be replaced by a finite set of forbidden words of the same length. This corresponds to our case because all forbidden patterns have the same size, namely the size of the tree $T_{[4]}$.

The example of G suggests the following definition. Let $T = T_m$ be an m-regular tree, A be a subgroup of $\mathrm{Aut}\, T_{[d]}$ for some $d \geq 1$, $B = \mathrm{Aut}\, T_{[d]} \backslash A$ and let F be the set of portraits of the elements in B. The elements of the set F will be called forbidden patterns and d will be called the depth of the forbidden patterns.

Definition 7.1. *A closed subgroup $M \subset \mathrm{Aut}\, T_m$ is called a subgroup of finite type if there is a set F of the type described in the previous paragraphs such that an element $g \in \mathrm{Aut}\, T_m$ belongs to M if and only if its portrait $P(g)$ does not contain a forbidden pattern.*

The examples of groups of finite type are the full group of automorphisms of the tree and the closure $\overline{G}$ of G in Aut T_2. The group $\overline{IMG(z^2+i)}$ is of finite type, as are the completions of Gupta–Sidki p-groups.

Proposition 7.5. *Let $M \subset$ Aut T_m be a closed subgroup of finite type. Then M is a branch weakly self-similar group.*

Proof. It is obvious that M is a self-similar group. Let $L = st_M(d)$ where d is the depth of the forbidden patterns determining the group and let $L_1 = \underbrace{L \times \cdots \times L}_{m}$ where each factor acts on the corresponding subtree with root at the first level.

We claim that L_1 is a subgroup of L. Indeed if $k \in L_1, k = (k_1, \ldots, k_m), k \in L$ and $P(k)$ is a portrait of k then $P(k)$ has no forbidden patterns with the root at level ≥ 1. But all the labels in a pattern with root at the first level are equal to the identity element. As the identity element belongs to A it does not belong to B so such a pattern is not forbidden. Therefore L_1 is a subgroup of M and even a subgroup of L. L is an open (normal) subgroup in M (and so has finite index). As $L = st_M(d)$, $L_1 \geq st_M(d+1)$ so L_1 also is an open subgroup in Aut T therefore $[M : L_1] < \infty$ and M is a regular branch over L. □

Problem 7.3. *Let M be a profinite group of finite type.*

(i) What conditions on the set of forbidden patterns F imply that M is finitely generated as a profinite group?

(ii) What conditions on F imply that M contains an abstract dense branch subgroup?

Remark. A necessary condition for M to be finitely generated is that the abelianization M_{ab} is finitely generated. Let $C_2 = \{1, -1\}$ be the multiplicative group of order 2. There is a canonical homomorphism $M \xrightarrow{\xi} \prod_{n=0}^{\infty} C_2$ (Cartesian product) which in component n sends an element $g \in M$ to the element of C_2 equal to the product of the signatures of the labels of the vertices on level n. Of course, to have M finitely generated the image has to be a finitely generated.

If M is given by a set of forbidden patterns then the image of ξ is not necessarily determined by a finite set of forbidden patterns as shown in the example below.

In the case of G, after changing $\{1, -1\}$ to $\{0, 1\}$, we have

$$\begin{aligned}
\alpha &= \xi(a) = 100\ldots \\
\beta &= \xi(b) = 0\ 110\ 110\ldots \\
\gamma &= \xi(c) = 0\ 101\ 101\ldots \\
\delta &= \xi(d) = 0\ 011\ 011\ldots\ .
\end{aligned}$$

These elements generate a subgroup $C_2 \times C_2 \times C_2$ in $\bigoplus_{\theta}^{\infty} \mathbb{Z}/2\mathbb{Z}$.

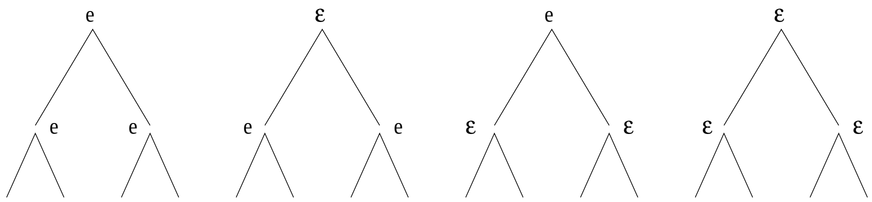
The description of the elements of a group of finite type by a set of forbidden patterns allows us to "visualize" them and hence to get a better understanding of the structure of the group. Subshifts of finite type are particular case of sofic systems [LM95] for which the set of forbidden words constitutes a regular language (i.e. can be described by a finite automaton).

Problem 7.4. *(i) What is the analogue of sofic systems for self-similar profinite groups or for regular branch profinite groups?*

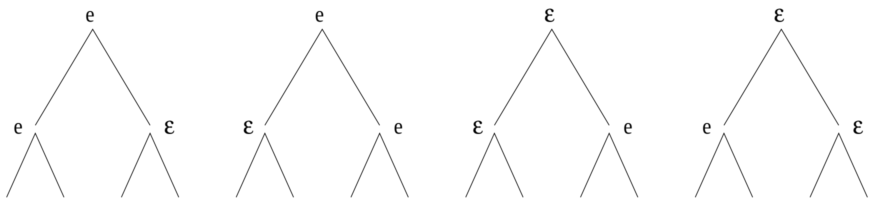
(ii) What is the class of "languages" of forbidden patterns that serves to describe any self-similar profinite group or any regular branch profinite group?

For instance this question is interesting even for the Basilica group given by the automaton in Figure 9.

One of the simplest examples of a group of a finite type which is different from Aut T and acts spherically transitively is the following. Take a binary tree, $d = 2$ and consider the subgroup $A \subset$ Aut $T_{[2]}$ which acts transitively on $T_{[2]}$, has index 2 in Aut $T_{[2]}$ and whose elements are given by the configurations



Then the forbidden patterns are



For such a set of forbidden patterns the image of ξ in $\bigoplus_{n=0}^{\infty} \mathbb{Z}/2\mathbb{Z}$ has only two elements, namely $0000\dots$ and $1000\dots$.

8 Around Milnor's question

Let L be a finitely generated group with a (finite) system of generators S and let $|g|$ be the length of the element $g \in L$ with respect to S i.e. the smallest n such that

$$g = s_{i_1} \dots s_{i_n},$$

for some $s_{i_j} \in S \cup S^{-1}, j = 1, \dots, n$.

The growth function $\gamma(n) = \gamma_L^S(n)$ is the function

$$\gamma(n) = \#\{g \in L\colon\ |g| \leq n\}$$

defined already in Section 3 Equation (3.15)in a more general situation and let

$$\gamma^{sph}(n) = \#\{g \in L\colon\ |g| = n\}$$

be the spherical growth function.

The group L has polynomial growth if there are constants c and d such that $\gamma(n) \leq cn^d$. A theorem of Gromov states that the class of finitely generated virtually nilpotent groups is exactly the class of groups of polynomial growth.

The free group $\mathbb{F}_2$ of rank 2 has exponential growth, as does any group containing F_2 as a subgroup (or even a free submonoid on 2 generators). For instance, finitely generated solvable groups which are not virtually nilpotent contain a free submonoid on 2 generators and hence have exponential growth. For more on growth see [Gri90, GdlH97, dlH00]. The growth cannot be superexponential because of the trivial estimate $\gamma(n) \leq (2|S|)^{n+1}$.

Two functions $\gamma_1(n)$ and $\gamma_2(n)$ are called equivalent if there is a constant c such that $\gamma_1(n) \leq c\gamma_2(cn)$, $\gamma_2(n) \leq c\gamma_1(cn)$, $n = 1, 2, \dots$. This equivalence is denoted by $\sim$ while the corresponding preorder is denoted by $\precsim$. The following problem was raised by Milnor.

Problem 8.1. *[Mil68] Is it correct that the growth function of a finitely generated group is either equivalent to a polynomial function n^d or to an exponential function 2^n?*

This problem was solved in [Gri83, Gri84a] where the first examples of groups of intermediate between polynomial and exponential growth where constructed. Indeed the group G happened to be the first known example of a group of intermediate growth.

The idea of the proof of the fact that G has intermediate growth with estimates

$$e^{n^\alpha} \leq \gamma_G(n) \leq e^{n^\beta} \tag{8.1}$$

for some constants $0 < \alpha, \beta < 1$ is the following.

That G has a super polynomial growth follows from the fact that G is an infinite finitely generated torsion group and from Gromov's description of groups of polynomial growth (indeed finitely generated nilpotent torsion group is finite; this is a classical result of group theory). But such an argument doesn't gives a lower bound in (8.1). To get such a bound one can observe that G is (abstractly) commensurable with $G \times G$ (i.e. G and $G \times G$ contain subgroups of finite index which are isomorphic). To see this let us use the embedding (3.4). As the image Im ψ contains $B \times B$ where $B = \langle b \rangle^G$ and as $\psi^{-1}(B \times B) = D = \langle d \rangle^G$ and both B and D have finite index in G the commensurability of G and $G \times G$ follows.

As $\gamma_{G \times G}(n) \sim \gamma_G^2(n)$ and a group and a subgroup of finite index have equivalent growth functions we get the equivalence $\gamma_G(n) \sim \gamma_G^2(n)$ from which the lower bound in (8.1) easily follows.

More precise but still easy arguments show that α can be taken to be 0.5. This was done in [Gri84a] by showing that $\psi(B_1(4n)) \supset B_1(n) \times B_1(n)$ where $B_1(n)$ is the ball of radius n centered at the identity element in the Cayley graph of G i.e. for any pair of elements $g_0, g_1 \in G$, $|g_i| \leq n$, $i = 0, 1$ such that $(g_0, g_1) \in G$ there is an element $g \in G$ of length $|g| \leq 4n$ such that $\psi(g) = (g_0, g_1)$. Indeed this is a consequence of Lemma 3.2. Much more delicate arguments show that α can be taken as 0.504 [Leo01], [Bar01]. For the upper bound we again use the embedding ψ but now iterating it three times.

Namely let $L = st_G(3)$ be the stabilizer of the third level in the binary tree $T = T_2$. The group L is a subgroup of finite index in G and let

$$\psi_3 \colon\ L \to \underbrace{G \times \cdots \times G}_{8}$$

be a monomorphism corresponding to the threefold iteration of ψ. The crucial lemma is the

Lemma 8.1. *[Gri84a] If $g \in L$ and*

$$\psi(g) = (g_1, \ldots, g_8).$$

Then

$$\sum_{i=1}^{\infty} |g_i| \leq \lambda |g| + 10 \tag{8.2}$$

where $\lambda = 3/4$.

Actually ψ_3 can be extended to a monomorphism

$$\psi_3 \colon\ G \to G \wr_Y (C_2 \wr C_2 \wr C_2)$$

where $Q = C_2 \wr C_2 \wr C_2$ is identified with the group Aut $T_{[3]}$, $T_{[3]}$ is the part of T up to the third level, and Y is the set of leaves of $T_{[3]}$ (i.e. the set of vertices of the third level) on which Q acts.

If

$$\psi(g) = (g_1, \ldots, g_8) \cdot \sigma \tag{8.3}$$

where $\sigma \in Q$ then the inequality (8.2) still holds.

A hint for getting the upper bound of type (8.1) will be given below. Let us now show how to deduce from (8.2) that G has subexponential growth.

Because $\gamma(n)$ is a semimultiplicative function i.e. $(\gamma(m+n) \leq \gamma(m) \cdot \gamma(n))$ the limit

$$\delta = \lim_{n \to \infty} \sqrt[n]{\gamma(n)}$$

exists and $\delta = 1$ if and only if the group has subexponential growth.

The inequality (8.2) implies the existence of a constant C such that

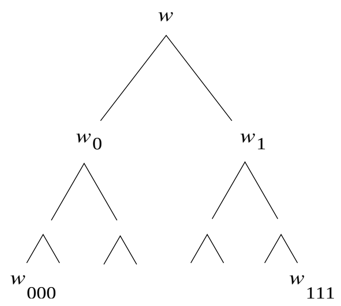
$$\gamma(n) \leq C \sum \gamma(n_1) \cdots \gamma(n_8) \tag{8.4}$$

where the sum is taken over 8-tuples $(n_1, \ldots, n_8)$ with $n_1 + \cdots + n_8 \leq \frac{3}{4}n + 10$. As the number of such tuples can be estimated by a polynomial $\mathcal{P}(n)$ (of degree 8) and as $\gamma(n_i) \sim \delta^{n_i}$ when n_i is large the inequality (8.4) leads to the inequality $\delta \leq \delta^{3/4}$ which implies $\delta = 1$, so G has subexponential growth. Additional considerations show that (8.4) is enough to conclude the existence of $\beta < 1$ for which the upper bound in

(8.1) holds. For details see [Gri84a, Gri98]) where the bound β is given by $\beta = \log_{32} 31$. A better technique was used in [Bar98], [MP01], [BŠ01] to improve the value of the bound β to $\beta = 0.767\ldots$.

The smaller λ and the smaller the level ℓ of the tree for which an inequality of type (8.2) holds, the better estimate for β can be obtained (in our case $\lambda = \frac{3}{4}$, $\ell = 3$). As was pointed out to me by Z. Sunik the constant $3/4$ can be replaced by $2/3$. Let us consider briefly Sunik's arguments.

If g is an element stabilizing the third level and g is represented by a reduced word w of length equal to $|g$, then we can draw a tree of projections for w:



(8.5)

where $w_i = r(\varphi_i(w))$, $i = 0, 1$, φ_i, $i = 0, 1$ are the results of the rewriting processes obtained through the reduction described in Section 4. The words w_{ij} are obtained in a similar way from w_i, and w_{ijk} are obtained from w_{ij}. Thus w_{ijk} are reduced words (perhaps not geodesic in G) representing the elements $g_1, \ldots, g_8$ from (8.3). We are going to show that

$$\sum_{ijk} |w_{ijk}| \leq \frac{2}{3}|w| + C$$

for some constant C (the value of constant C is not important for the estimate of the growth). Let us split all the occurrences of letters from the set $\{b, c, d\}$ in $w_{000}, \ldots, w_{111}$ into two sets E and F. In the first set we include those occurrences of letters that came from w (under the rewriting process) as a result of a collision (reduction) of the form (ii) i.e. $xy = z$. Such a letter is called a "killer".

The letters (i.e. the corresponding occurrences) that are not "killers" are called "neutral" elements. They constitute the set F. Let $\xi = |E|$ and $\eta = |F|$. Then

$$\sum_{i,j,k} |w_{ijk}|_* \approx (|E| + |E|) = \xi + \eta,$$

where $|w|_*$ denotes the *-length (that is a number of occurrences of the symbols b, c, d and therefore the *-length is approximately half of the actual length) and $\approx$ means "equality mod some additive constant". Denote $t = \sum_{i,j,k} |w_{ijk}|_*$ and assume that $|w| = 2n$, so that $|w|_* \approx n$.

As each "killing" reduces one letter we have $t \leq n - \xi$. Let δ_0 be the number of occurrences of the letter d in w, δ_1 the total number of occurrences of the letter d in w_0 and w_1, and δ_2 the total number of occurrences of the letter d in w_{00}, w_{01}, w_{10} and w_{11} (so δ_i is the total number of d-symbols in the words at level i in the tree in (8.5), $i = 0, 1, 2$).

As each neutral element in the decomposition from the root vertex (alma mater) to the second level has once to be equal to d (this follows from the rewriting rules) $\delta_0 + \delta_1 + \delta_2 \geq \eta$. As d produces a symbol 1 for one of its two projections this gives a reduction of the number of a's in the next level which also implies reduction of the $*$-length (this is correct up to ± 1 for each word in the decomposition. Therefore

$$t \precsim n - (\delta_0 + \delta_1 + \delta_2) \leq n - \eta.$$

We have a system

$$\begin{cases} t \precsim n - \xi \\ t \precsim n - \eta \\ t \approx \xi + \eta \end{cases}$$

Adding these relations we get

$$2t \precsim \frac{2}{3}(2n).$$

But $2t$ represents the total length of the eight projections and $2n$ was the length of w, hence we are done.

Improvements of the growth estimate for the group G and for the groups $G\omega$ are provided in [Bar98], [MP01] and [BŠ01]. Let us focus on the estimate given in [Bar98]. The idea is to replace the word length $|g|$ by the weight length $|g|_\omega$ defined on the generating elements $s \in S \cup S^{-1}$ by positive numbers $\omega(s)$ with extension to arbitrary elements $g \in G$ by

$$|g|_\omega = \min[\omega(s_1) + \ldots + \omega(s_n)]$$

where the minimum is taken over all the presentations of g as a product $g = s_1 \ldots s_n$ of generators $s_i \in S \cup S^{-1}$ and then consider the corresponding growth function. It is easy to see the rate of growth of a group does not depend on the choice of the weight so one can try to get a better estimate using an appropriate weight. In order to have a good contracting property for projections, the weight has to satisfy special properties. In case of G this is realized as follows.

Let $\eta \approx 0.811$ be the real root of the polynomial $x^3 + x^2 + x - 2$ and let ω be a weight on G defined by

$$\begin{aligned} \omega(a) &= 1 - \eta^3 = \eta^2 + \eta - 1 \\ \omega(b) &= \eta^3 = 2 - \eta - \eta^2 \\ \omega(c) &= 1 - \eta^2 \\ \omega(d) &= 1 - \eta. \end{aligned}$$

Then the following statement holds [Bar98].

Lemma 8.2. *Let $g \in st_G(1)$ with $\psi(g) = (g_0, g_1)$. Then*

$$\eta(|g|_\omega + |a|_\omega) \geq |g_0|_\omega + |g_1|_\omega. \tag{8.6}$$

To see this observe that

$$\begin{aligned} |g|_\omega + |c|_\omega &> |d|_\omega \\ |b|_\omega + |d|_\omega &> |c|_\omega \\ |c|_\omega + |d|_\omega &= |b|_\omega \end{aligned}$$

and therefore the minimal (with respect to ω) presentation of g as a product of generators has the form (3.2). Let the word u be a ω-minimal presentation of g. Construct the words u_0, u_1 using ψ just as in the case of the standard word length. Note that

$$\begin{aligned} \eta(|a|_\omega + |b|_\omega) &= |a|_\omega + |c|_\omega \\ \eta(|a|_\omega + |c|_\omega) &= |a|_\omega + |d|_\omega \\ \eta(|a|_\omega + |d|_\omega) &= |b|_\omega. \end{aligned} \tag{8.7}$$

As $\psi(b) = (a, c)$ and $\psi(aba) = (c, a)$, each b in u contributes $|a|_\omega + |c|_\omega$ to the total weight of u_0 and u_1; similar arguments apply to c and d.

Grouping together pairs of generators in (3.2) such as ba, ca, da we see that $\eta|g|_\omega$ is a sum of left-hand terms (with a possible difference in a single term $\eta|a|_\omega$). At the same time $|g_0|_w + |g_1|_w$ is bounded by the total weight of the letters in u_0 and u_1, which is the sum of the corresponding right-hand terms in (8.7) and this leads to (8.2).

The difference between Lemma 8.1 and Lemma 8.2 is that in (8.2) we get essential reduction of the word length only after we project on the third level, while for the length $|g|_\omega$ the reduction is present already at the first level. This leads to the better upper estimate $\beta \leq 0.767$. (See Prop. 4.3 in [Bar98], Th. 5.9 in [MP01] and Th. 6.1 in [BŠ01].)

We formulate now some open questions on the growth of G.

Problem 8.2. *(i) Does the limit*

$$\lim_{n\to\infty} \frac{\gamma_G(n+1)}{\gamma_G(n)} \tag{8.8}$$

exist, where $\gamma_G(n)$ is the growth function with respect to the canonical generating set $\{a, b, c, d\}$ of G?

(ii) Is it correct that the limit (8.8) *exists for every finite system of generators of* G*?*

(iii) Answer the corresponding questions for the spherical growth function $\gamma_G^{sph}(n)$.

Remark. Of course, the subexponential growth of G implies that if the limit (8.8) exists, it is equal to 1.

Problem 8.3. *(i) Is there an* α *such that*

$$\gamma_G(n) \sim e^{n^\alpha}?$$

(ii) If the answer to (i) is NO what are the limits

$$\overline{\lim_{n\to\infty}} \frac{\log\log\gamma_G(n)}{\log n}$$
$$\underline{\lim_{n\to\infty}} \frac{\log\log\gamma_G(n)}{\log n}?$$

Problem 8.4. *(i) Is there a constant* $\alpha > 0$ *such that*

$$\gamma_G^{sph}(n) \succeq e^{n^\alpha}?$$

(ii) Is it correct that $\gamma_G^{sph}(n) \sim \gamma_G(n)$*?*

Problem 8.5. *(i) Is it correct that the generating series* $\Gamma_G(z) = \sum\limits_{n=0}^{\infty} \gamma_G(n) z^n$ *satisfies a some differential—functional equation?*

(ii) What can be said about the singular point 1 of $\Gamma_G(z)$*? For instance, is it correct that* $\Gamma(z)$ *behaves as* $e^{c/(z-1)^\alpha}$ *when* $z \to 1$ *in the interval* $(1-\varepsilon, 1)$, $\varepsilon > 0$, *where* α *and* c *are some constants?*

(iii) What is the set of singular points of $\Gamma_G(z)$ *on the unit circle? Is the unit disc* $\{z : |z| < 1\}$ *the natural domain of convergence for* $\Gamma_G(z)$*?*

Remark. (i) In case

$$\lim_{z\nearrow 1} \frac{\Gamma(z)}{e^{c(z-1)^\alpha}} = B$$

for some constant B the asymptotic of $\gamma(n)$ has the form

$$\gamma(n) \approx e^{Dn^\delta}$$

with some constants D and δ so we get a positive answer to some of the above problems.

(ii) As the coefficients of a McLauren series of an algebraic function grow either polynomially or exponentially $\Gamma_G(z)$ is a transcendental function.

The next discussion based on [Gri88] relates the problem of calculation of the growth of G to a similar problem for the automaton given by Figure 1.4. Let A be a finite automaton of the type defined in Section 1. Let $A^{(n)} = A \circ \cdots \circ A$ be the n-fold composition of A and $\tilde{A}^{(n)} = \min(A^{(n)})$ be the minimization of A^n (see [Eil74] for a minimization algorithm). Let $\gamma_A(n)$ be the number of states of $\tilde{A}^{(n)}$. The function $\gamma_A(n)$ is called the growth function of the automaton A.

Remark. Let $S = S(A) = \langle A_q \colon\ q \in Q\rangle$ be the semigroup generated by states of A and let

$$\gamma_S^{\mathrm{middle}}(n) = \#\{s \in S \mid s = A_{q_1} \dots A_{q_n}, q_i \in Q\}$$

be the function counting the number of elements which can be presented as a product of exactly n generators. Clearly

$$\gamma_S^{\mathrm{sph}}(n) \le \gamma_S^{\mathrm{middle}}(n) \le \gamma_S(n).$$

The following proposition is obvious.

Proposition 8.1.

$$\gamma^{middle}_{S(A)}(n) = \gamma_A(n).$$

Indeed, the states of the automaton $A^{(n)}$ are in bijection with products $\{A_{q_1} \dots A_{q_n}\}$ of length n, while the states of the automaton $\min(A^{(n)})$ are in bijection with elements of the semigroup $S(A)$ which can be represented by products of length n.

In case A is the automaton given by Figure 1.4 the semigroup $S(G)$ coincides with the group $G(A)$ generated by the automaton A and the middle growth function coincides with the ordinary growth function (because the identity element is represented by one of the states). Therefore the study of the growth of the automaton reduces to the study of the growth function of G and we get the example of an invertible automaton of intermediate growth as observed in [Gri88]. It would be interesting to develop methods to study the growth of automata and apply them to study the growth of groups. Interesting examples of noninvertible automata of intermediate growth are constructed in [RS02a, RS02b].

Let us conclude this section by an example which on one side is very similar to G but at the same time has different asymptotics of growth as shown by A. Erschler [Ers04a].

Let $\mathcal{E} = G(E)$ be a group generated by the automaton E given by the Figure 8.

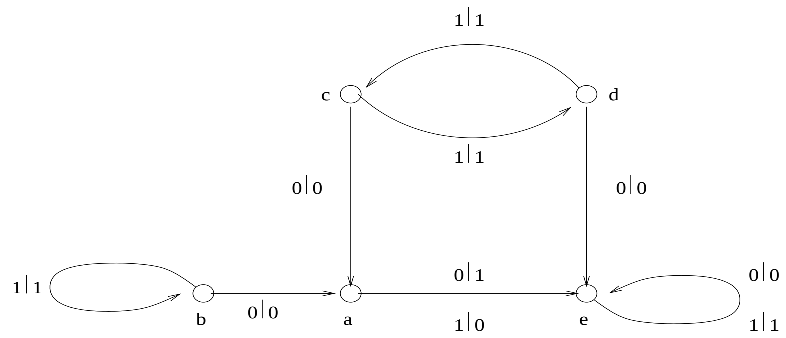


Figure 8.1: Automaton E

Then the generators E_a, E_b, E_c, E_d have order two, $E_bE_cE_d = 1$ and the group $\mathcal{E} = G(E)$ is a group of intermediate growth with growth estimates

$$e^{\frac{n}{\log^{2+\varepsilon} n}} \preceq \gamma_{\mathcal{E}}(n) \preceq e^{\frac{n}{\log^{1-\varepsilon} n}} \tag{8.9}$$

for any $\varepsilon > 0$. Therefore the automaton E also has intermediate growth with the same estimates as in (8.9). The group $\mathcal{E}$ is isomorphic to the group G_ω, where $\omega = (01)^\infty$. As it was already mentioned, the groups G_ω, $\omega \in \xi$) have intermediate growth.

The methods used in [Ers04b] are based on the study of asymptotic properties of random walks on groups and the corresponding Poisson boundary.

An important role in the study of formal languages is played by formal power series [SS78]. In the context of theory of groups analogous role is played by the complete growth functions studied in [GN97], where their rationality is shown in the case of hyperbolic groups, and also a relation to the growth function of the language of all geodesics and the problem of the spectrum of the discrete Laplace operator (Section 13) is indicated. Formal power series and their connections to regular and context-free languages are well studied [SS78]. On the other hand, recent results of K. Roever, D. Holt, and M. Elder, M. Gutierrez, Z. Sunik indicate that indexed languages are useful in the study of various properties of G. It would be good to develop formal power series methods for such languages as well.

9 Amenability and elementary classes

Amenable groups were introduced by J. von Neumann in 1929 [vN29] in his study of the algebraic roots of the phenomenon known as Banach–Tarskii Paradox [Gre69, Wag93]. Recall that a group M is called amenable if there is a measure μ defined on all subsets of M, satisfying the properties

1. $0 \leq \mu(E) \leq 1, \mu(M) = 1$ (positivity and normalization),

2. $\mu(gE) = \mu(E)$ (left invariance),

3. $\mu(E \cup F) = \mu(E) + \mu(F)$ if $E \cap F = \emptyset$ (additivity).

Here $E, F \subset M$ are arbitrary subsets and $g \in M$ is na arbitrary element.

Finite groups and commutative groups are amenable (the first fact is obvious, the second is nontrivial and based on the axiom of choice [Gre69]). We emphasize that the measure μ is assumed to be only finitely additive, so it should not be confused with Haar measure, which for discrete groups is a measure that has positive mass at each point and hence cannot satisfy the conditions imposed on μ in the case of an infinite group. Von Neumann also observed that the free group F_k of rank $k \geq 2$ is not amenable and that the class of amenable groups is closed with respect to the operations

(i) taking a subgroup,

(ii) taking a factor group,

(iii) extension

(iv) direct limit.

Let AG be the class of amenable groups, NF be the class of groups without free subgroups of rank ≥ 2 and EG be the class of elementary amenable groups, i.e. the smallest class of groups containing finite groups, commutative groups and closed with respect to the operations (i)–(iv). Then the inclusions

$$EG \subseteq AG \subseteq NF$$

hold. The questions whether the equalities $EG = AG$ or $AG = NF$ hold were raised in [Day57]. The second question in the form of the conjecture "the group is non-amenable if and only if it contains a free subgroup with two generators" is sometimes called von Neumann conjecture [Ver82] (although there is no written confirmation that von Neumann indeed asked this question). This question was answered negatively in [Ol′80], [Adi82] (see more on this problem and the history of it solution in [GK93, GS02]).

Surprisingly even through the class EG was introduced (not explicitly) already by von Neumann in 1929 and the theory of invariant means and their applications was intensively developed [Gre69, Pat88] the first examples of amenable groups not in the class EG appeared only in 1983 when the first examples of groups of intermediate growth were discovered. For instance, G belongs to $AG \backslash EG$. Indeed it was already known in 1957 [AVŠ57] and later rediscovered a few times that groups of subexponential growth are amenable. At the same time it was shown in [Cho80] that groups from EG cannot have intermediate growth. They also cannot be finitely generated infinite torsion groups (i.e. Burnside groups). So the fact that $G \notin EG$ follows from several properties of G.

One more property that prevents the group G from being elementary amenable is its branch property, discussed in Section 6.

Theorem 9.1. *Let M be a finitely generated branch group. Then $M \notin EG$.*

The attempts initiated by J. von Neumann and M. Day to give an algebraic description of the class of amenable groups failed but a new idea may come and a new investigations in this direction are needed. An important step in understanding of the algebraic features of amenable groups is to understand amenable just infinite groups, since any finitely generated amenable group has a just infinite quotient, which also must be amenable. As the class of just infinite groups splits into three subclasses: branch, almost hereditarily just infinite and almost simple [Gri00b], a natural problem is to try describe in algebraic terms the amenable groups inside each subclass.

Problem 9.1. *a) Is it correct that a branch group is non-amenable if and only if it contains a free subgroup with two generators?*

b) Is there a hereditarily just infinite amenable group which does not belong to the class EG?

c) Is there an infinite finitely generated amenable simple group?

d) Is there a finitely generated infinite torsion amenable group of bounded exponent?

It seems unlikely to have a positive answer on questions b) and c). Most known examples of finitely generated branch groups are amenable but S. Sidki and J.W. Wilson recently constructed an examples of finitely generated branch groups with free subgroups on two generators [SW03].

A rich source of amenable but not elementary amenable groups is the class of self-similar groups. As was already mentioned the self-similar groups of intermediate growth (such as G) are of this type. Remarkably, there are also amenable but not elementary amenable automaton groups that "have nothing to do" with the class of groups of intermediate growth, as shown by L. Bartholdi and B. Virag in [BV03]. The group they have been studying is the so-called Basilica group B, defined by the automaton in Figure 9, introduced for the first time in [GŻ02a, GŻ02b] as the group generated by the automaton The group B is a torsion

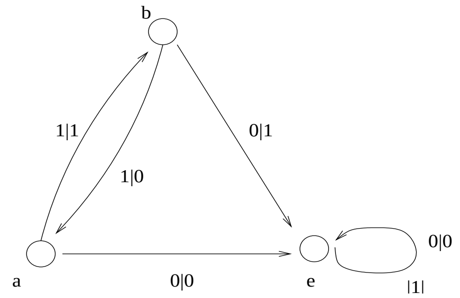


Figure 9.1: The automaton generating Basilica group

free group of exponential growth in which a and b generate a free monoid. It was shown in [GŻ02a] that B does not contain a free subgroup on two generators and does not belong to the class SG of subexponentially amenable groups. The latter is the class of groups containing all groups of subexponential growth and closed with respect to operations (i)–(iv). As the groups of subexponential growth are amenable the inclusion $SG \subseteq AG$ holds and in [Gri98] (see also [dlAGCS99, GZ03]) the question about the equality $SG = AG$ was raised.

Negative answer to this question was provided in [BV03] by showing that B is amenable. The proof has probabilistic flavor and uses Kesten's criterion of amenability [Kes59]. In a paper by V. Kaimanovich the class of amenable self-similar groups is extended by new examples and the methods of entropy for random walks are applied in the proof od amenability [Kai].

A large class of self-similar groups without a free subgroup of rank 2 was found by S. Sidki in [Sid04]. This is the class of groups generated by automata determining a language of polynomial growth. In [BKNV] the amenability is shown for the subclass consisting of groups generated by bounded automata.

Problem 9.2. *a) Is there an algorithm which allows to determine which self-similar groups are amenable?*

b) Is there a non-amenable self-similar group without free subgroups on two generators?

We see that branch groups and groups generated by finite automata give fundamental contributions to the study of the phenomenon of amenability.

One of the important topics is the study of growth of Følner sequences in amenable groups. Recall that a sequence $\{E_n\}_{n=1}^{\infty}$ of finite subsets of a group M is Følner sequence if for any $g \in M$

$$\frac{|gE_n \triangle E_n|}{|E_n|} \underset{n\to\infty}{\longrightarrow} 0$$

($\triangle$ denotes symmetric difference). A group M is amenable if and only if there is such a sequence [Føl57]. The next notion was introduced by A. Vershik [Ver82].

Let M be a finitely generated amenable group with a system of generators S and let $f_M(n)$, $n = 1, 2, \dots$ be the infimum of the cardinalities of the finite subsets $E \subset M$ with the property

$$\frac{|sE \triangle E|}{|E|} < \frac{1}{n}$$

for $s \in S$. The function $f_M(n)$ is called the Følner function for G. The rate of growth of $f_M(n)$ when $n \to \infty$ does not depend on the choice of the finite generating set and is an important asymptotic invariant and, until recently, very little was known about it. Recent results in this direction were obtained in [Ers03] where the invariant is computed for wreath products in terms of its values for the factors. New progress is achieved in [Ers] were, based on a generalization of automaton groups (the author call her groups "piecewise automatic groups"), A. Erschler showed that given any function $f\colon \mathbb{N} \to \mathbb{N}$ there exists a finitely generated group M of intermediate growth for which the Følner function satisfies $f_M(n) \geq f(n)$ for all sufficiently large n. Erschler's construction uses some features of the construction of groups G_ω and, in particular, it uses the topology in the space of finitely generated groups discussed in Section 2.

The book in progress by C. Pittet and L. Saloff-Coste [PSC01] also contributes to the subject.

Problem 9.3. *a) What are the possible types of growth of Følner functions in the case od finitely generated amenable branch groups?*

b) The same question for self-similar groups.

Although we know that all groups G_ω, $\omega \in \Omega$ are amenable (they have intermediate growth if $\omega \in \Xi$ and are virtually metabilian if $\omega \in \Omega \backslash \Xi$ there is a number of interesting open questions about these groups related to the amenability phenomenon.

Problem 9.4. *a) Find the growth of the Følner function $f_\omega(n)$ for the group G_ω?*

b) What is the shape of optimal Følner sets in G_ω?

c) Answer the above questions for the group G.

We even do not know if the sequence of balls $\{B_1^G(n)\}_{n=1}^\infty$ in the group G is a Følner sequence (although we know that some subsequence $B_1^G(n_i)$ must be a Følner sequence). The corresponding question is equivalent to the question stated in Problem 8.2(i).

To formulate the next question, let us present the groups G_ω in the form of quotients F_4/N_ω where F_4 is a free group of rank four with generators a, b, c, d and N_ω consisting of all words $w(a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, c^{\pm 1})$ representing the identity in G_ω. Let

$$N = \bigcap_{\omega \in \Omega} N_\omega.$$

Problem 9.5. *(a) Is the group $\mathcal{N} = F_4/N$ finitely presented?*
(b) Is $\mathcal{N} = F_4/N$ amenable?

The next observation is due to V. Nekrashevich

Theorem 9.1. *The group $\mathcal{N}$ is a contracting self-similar group generated by the automaton given in Figure 9.*

The automaton described by Figure 9 has five states representing the generators a, b, c, d of N and the identity element e and it is an automaton over the alphabet $\mathcal{A} = \{0,1\} \times \{0,1,2\}$ consisting of six elements. The star $*$ represents any element of $\{0,1,2\}$. The proof of Theorem 9.1 is straightforward if one recalls the definition of the groups G_ω and the rule of action of initial automata on sequences described in Section 1.

The action of $\mathcal{N}$ on the 6-regular rooted tree T_6 given by the above automaton presentation of G is not level transitive. For each $\omega \in \Omega$ there is a binary $\mathcal{N}$-invariant subtree T_ω in T_6 consisting of vertices $(u, v) \in \mathcal{A}^*$ whose second coordinate follows the path ω and the restriction of the action of $\mathcal{N}$ on T_ω factorized by the kernel of the action is isomorphic to the action of G_ω on the binary tree. Thus the above presentation of $\mathcal{N}$ models a simultaneous action of the groups G_ω for all $\omega \in \Omega$. The group $\mathcal{N}$ has exponential growth, it is residually finite and has uncountably many homomorphic images. In [Gri84b] a residually finite group of intermediate growth with uncountably many homomorphic images was constructed.

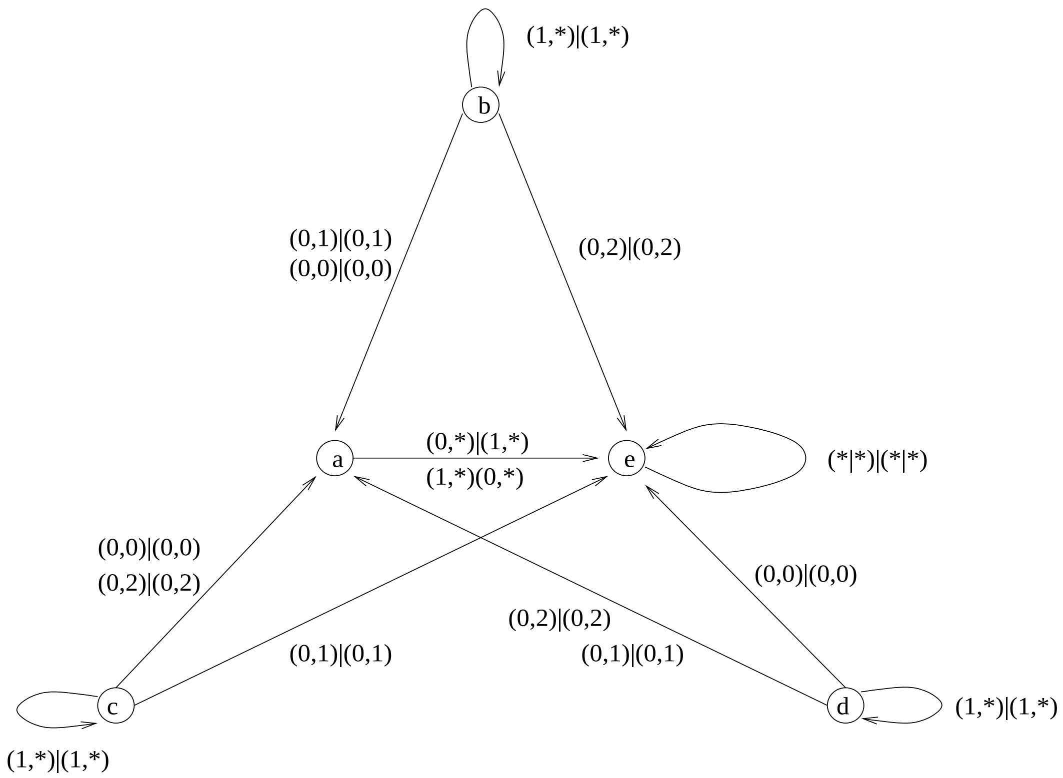


Figure 9.2:

10 Representations

In the case a group is not virtually abelian the description of all unitary representations is a "wild" problem. But even if there are no examples of classification of all irreducible unitary representations of a group that is not of type (I) [Tho68] a lot of work has been done in the study and use of particular representations or classes of representations.

One of the most important representations is the left regular representation λ_G and many problems in mathematics can be solved by answering a particular question about such representations. An important role belongs to the finite dimensional representations, especially in the case when the group in question is a MAP-group ("maximally almost periodic"), that is, the finite dimensional representations separate the elements of the group. For a finitely generated group this is the case if and only if the group is residually finite.

Groups acting faithfully on rooted trees and, in particular, self-similar groups belong to this class. A natural sequence of finite dimensional representations that can be associated with a group M acting on a tree $T = T_{\overline{m}}$ is the sequence of permutational representations $\{\pi_n\}_{n=1}^{\infty}$ given by the action of the group on the levels of the tree.

Another important class of representations are the quasiregular representations $\lambda_{M/H}$ in the space $\ell^2(M/H)$ given by the left action of M on the set M/H of left cosets gH. For a group M acting on a rooted tree T a natural first choice for a subgroup is the parabolic subgroup $P = P_\xi$ defined as the stabilizer $st_M(\xi)$ of some point ξ on the boundary ∂T of the tree. In case M acts spherically transitively the particular choice of ξ usually doesn't play an important role and the representations λ_{M/P_ξ} share many common properties. On the other hand, when M is a branch group λ_{M/P_ξ} are irreducible representations and no two of them are isomorphic [BG00a]. The family of representations $\{\lambda_{M/P_\xi},\ \xi \in \partial T\}$ separates the elements of M since

$$\bigcap_{\xi \in \partial T} P_\xi = \{1\}.$$

The groups P_ξ already appeared in Section 6 as examples of weakly maximal subgroups. These subgroups are closed in the profinite topology as

$$P_\xi = \bigcap_{n=1}^{\infty} P_{n,\xi}$$

where $P_{n,\xi} = st_M(u_n)$ and u_n is the unique vertex on level n that belongs to the path ξ. Thus the coefficients of the representation λ_{M/P_ξ} are approximated by the coefficients of the finite dimensional representations $\lambda_{M/P_{n,\xi}}$ (which are isomorphic to the permutational representations π_n). The weak maximality of P_ξ and the fact the P_ξ is closed in the profinite topology make the representations λ_{M/P_ξ} especially important from many points of view.

An interesting new development is the appearance of Gelfand pairs associated to self-similar groups. Gelfand pairs play important role in many topics, in particular in the study of random walks [Dia88]. It is shown in [BG02a] that (M_n, P_n), $n = 2, 3, \dots$ are Gelfand pairs where $M_n = M/st_M(n)$, $P_n = st_{M_n}(u_n)$ (u_n any vertex on level n) and the group M is either G or Gupta–Sidki p-group. In [BG02a] a direct proof is provided showing that the corresponding Hecke algebra $\mathcal{L}(M, P)$ is commutative, while in [BdlH03] it is observed that Gelfand pairs can be constructed for those branch groups whose action on the tree is 2-point transitive on levels (transitive on the sets of pairs of vertices of the same level with fixed distance between them.) The action of G is 2-point transitive while the action of Gupta–Sidki p-groups is not. Perhaps at the moment there is no example of a branch group M such that (M, P_n) are not Gelfand pairs.

Problem 10.1. *Is there a branch group M such that for some n the pair (M_n, P_n) is not a Gelfand pair?*

Branch groups allow not only to produce examples of Gelfand pairs consisting of finite groups but also to produce examples of Gelfand pairs consisting of profinite groups. Namely for G the pair $(\overline{G}, \overline{P}_n)$ where the bar denotes the closure in Aut T is such an example.

The idea to use group actions on rooted trees to study Gelfand pairs is very fruitful. It allows not only construction of new examples of such pairs, but also clarifies the classical results by presenting them in a new angle, as it was done in [CSSFa, CSSFb].

A part of the process of studying the quasiregular representations (finite dimensional or infinite dimensional), looking for Gelfand pairs and understanding the dynamics of a group acting on rooted trees is the investigation of the action on level n of the stabilizer $st_M(u_n)$, where u_n is a vertex at level n. For instance, in case of G the action of $st_G(u_n)$ has $n+1$ orbits consisting of $1, 1, 2, 2^2, \dots, 2^{n-1}$ points and these orbits are depicted in Figure 10.

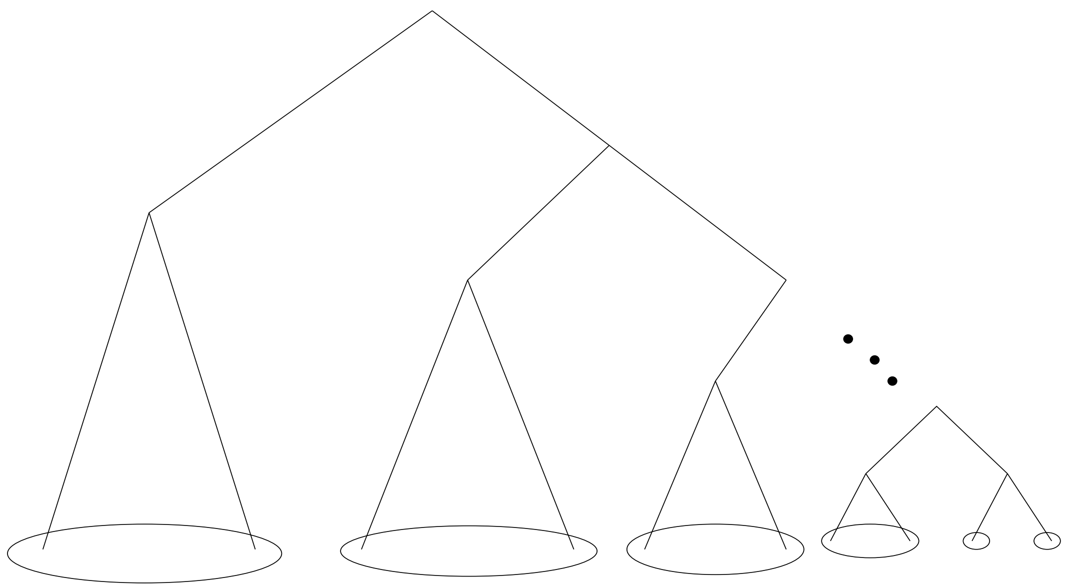


Figure 10.1: Orbits of the action of $st_G(1^n)$ on n-th level.

An important class of representations arises from actions of groups by measure preserving transformations. For any group M acting on a rooted tree T one can consider the dynamical system $\mathcal{D} = (M, \partial T, \nu)$ where ν is uniform measure on the boundary ∂T of the tree (i.e. ν is the product $\bigoplus_{n=1}^{\infty} \nu_n$ where ν_n is the uniform measure $\{\frac{1}{m_n}, \dots, \frac{1}{m_n}\}$ supported on a set of cardinality m_n).

The corresponding unitary representation ρ in the Hilbert space $\mathcal{H} = L^2(\partial T, \nu)$ given by

$$(\rho(g)f)(x) = f(g^{-1}x)$$

is sum of finite dimensional representations. Indeed, let $\mathcal{H}_n$ be the subspace spanned by the characteristic functions of cylinder sets of rank n (each such set consists of those $\xi \in \partial T$ that pass through some fixed vertex at level n). Then $\mathcal{H}_n$, $n = 0, 1, 2, \dots$ are $\rho(M)$ invariant subspaces and $\mathcal{H}_n$ naturally embeds into $\mathcal{H}_{n+1}$.

Let $\mathcal{H}_n^{\perp}$ be the corresponding orthogonal complement to H_n in H_{n+1}, $n = 0, 1, \dots$. Then

$$\mathcal{H} = \mathbb{C} \oplus \bigoplus_{n=0}^{\infty} \mathcal{H}_n^{\perp}$$

and

$$\rho = 1_{\mathbb{C}} \oplus \bigoplus_{n=0}^{\infty} \rho_n^{\perp} \tag{10.1}$$

where $\rho_n^{\perp}$ is the restriction of ρ on $\mathcal{H}_n^{\perp}$. The finite dimensional representation $\rho_n = \rho|_{\mathcal{H}_n}$ allows a decomposition

$$\rho_n = 1_C \oplus \bigoplus_{i=0}^{n-1} \rho_i^{\perp}$$

and is isomorphic to the permutational representation π_n, which in turn is isomorphic to the quasiregular representation λ_{M/P_n}. There is a big difference between the representations λ_{M/P_ξ}, $\xi \in \partial T$ and ρ, but they are both approximated by the same sequence of finite dimensional representations ρ_n.

Problem 10.2. *Is there an algorithm which, given a finite automaton, finds a decomposition of the representation ρ_n into irreducible components?*

The following question may be useful as steps in solving the above problem.

Problem 10.3. a) *Is there an algorithm which, given an automaton A, describes the orbits of the action of $G(A)$ on the levels of the corresponding tree?*

b) *Is there an algorithm which, given an automaton A, describes the orbits of the action of the stabilizer $P_n = st_{G(A)}(u_n)$ (where u_n the rightmost vertex at level n) on the n-th level, $n = 1, 2, \dots$?*

Problem 10.4. a) *Is there an algorithm which, given a finite automaton A, finds all irreducible representations of the finite quotients $M_n = M/st_M(n)$, $n = 1, 2, \dots$, where $M = G(A)$?*

b) *In particular, find all irreducible representations of the finite quotients of G.*

The solution of Problem 10.4 b) would give a complete description of the irreducible representations of the profinite completion $\widehat{G}$ since all such representations are in natural bijection with the irreducible representations of the groups G_n, $n \geq 1$. This follows from the following three facts about G and $\widehat{G}$:

(i) congruence subgroup property for G,

(ii) just infiniteness of $\widehat{G}$,

(iii) absence of faithful finite dimensional representations for G.

The last fact follows from the following observation obtained jointly with T. Delzant.

Theorem 10.1. *Let M be a branch group. Then M is not linear (i.e. it does not have a faithful finite dimensional representation over any field).*

The study of representations over fields different from $\mathbb{C}$ (in particular modular representations) of the group G (and other self-similar groups and groups acting on trees) is important for many topics in algebra. The first results in this direction are obtained in [Sid97] where an infinite dimensional irreducible representation of the Gupta–Sidki 3-group over a field of characteristic 3 is constructed. It is not known if such a representation exists for G (replace the characteristic 3 by 2).

Problem 10.5. *Does the group G have an irreducible representation over a field of characteristic 2?*

This question as well as the next one is related to Kaplansky Conjecture on Jacobson radical [Kap70] (see also [Pas98]). Recall that the Jacobson radical is the intersection of all maximal left ideals of the group algebra over the give field.

Problem 10.6. *Is it correct that the fundamental ideal $\omega(\mathbb{F}_2[G])$ of the group algebra $\mathbb{F}_2[G]$ of G over the field $\mathbb{F}_2$ of two elements coincides with the Jacobson radical $\mathcal{J}(\mathbb{F}_2[G])$ of this algebra?*

Positive answer to this question would give a counterexample to Kaplansky Conjecture. A theorem of Lichtman [Lih63] states:

Let M be a finitely generated group and let $\mathbb{K}$ be a filed of positive characteristic p. If $\mathcal{T}(\mathbb{K}[G]) = \omega(\mathbb{K}[G])$ then

(i) M is a p-group.

(ii) If $M \neq 1$, then $M \neq M'$.

(iii) If M is an infinite group, then M has an infinite, residually finite homomorphic image.

(iv) Any maximal subgroup of M is normal of index p.

(v) If H is a subgroup of M of infinite index, then there exists a chain of subgroups $M = M_0 > M_1 > M_2 > \cdots > H$ with $M_{i+1} \triangleright M_i$ and $|M_{i+1}/M_i| = p$.

The group G satisfies these condition with $p = 2$. The properties listed in (i), (ii), (iii) hold because G is an infinite residually finite 2-group. The property (iv) is established by E. Pervova [Per00] and (v) follows from a generalization of the result of Pervova on subgroups of finite index given in [GW03a]. Indeed, let H be a subgroup of infinite index. Then H is contained in a maximal subgroup G_1 of G which has index 2 by (iv). Now apply the same argument to the pair (G_1, H) etc. The above discussion shows that the group G is a candidate for a contrexamples to Kaplansky Conjecture.

11 C^*-algebras

One can naturally associate at least three C^*-algebras $C^*_\rho(M)$, $C^*_{M/P}(M)$, $C^*_r(M)$ to a group M acting on a rooted tree. These algebras are defined as (norm) closure of the corresponding unitary representations ρ, $\lambda_{M/P}$ or λ_M of the group algebra $\mathbb{C}[M]$. The algebra $C^*_r(M)$ is called reduced C^*-algebra of the group M.

It is easy to check that, for the corresponding norms $\| \ \|_\rho$, $\| \ \|_{M/P}$, the inequality

$$\|x\|_\rho \geq \|x\|_{M/P}$$

holds for any element $x \in \mathbb{C}[M]$. Therefore there is a canonical $*$-homomorphism $\varphi\colon\ C^*_\rho \to C^*_{M/P}$. In case the pair $(M, M/P)$ is amenable φ is an isomorphism [BG00a, Nek05]. In case M is amenable group $C^*_\rho(M)$ and $C^*_{M/P}(M)$ are quotients of $C^*_r(M)$ (because any unitary representation of amenable group is weakly contained in the regular representation [Gre69]).

Reduced C^*-algebras play a fundamental role in the theory of C^*-algebras and in the study of unitary representations of groups. Unfortunately very little is known about them. One of rare results is the simplicity of C^*_r in case of a free group of finite rank and in case of groups having the so-called Powers property [Pow75], [BCdlH94].

The study of the algebras $C^*_{M/P}(M)$ is interesting not only because of the relations to quasiregular representations but also because of their use in the study of spectral properties of regular graphs which will be discussed in the next section.

We concentrate our attention here on $C^*_\rho(M)$.

Proposition 11.1. *$C^*_\rho(M)$ is residually finite dimensional algebra (i.e. embeds into $\bigoplus_{i=0}^{\infty} M_{n_i}(\mathbb{C})$ for some sequence n_i of integers [BL00]).*

This follows from the decomposition (10.1).

Just infinite groups play important role in group theory [Gri00b, BGŠ03]. Perhaps a similar role should belong to just infinite-dimensional C^*-algebras.

Problem 11.1. *Let M be a self-similar just-infinite group.*

(a) *Is it correct that $C^*_\rho(M)$ has only finite-dimensional proper images?*

(b) *Answer the above question for G.*

In case of the group G the representations $\rho_n^\perp$ (of dimension 2^n) are irreducible [BG00a] and the embedding

$$C^*_\rho(G) \underset{\theta}{\hookrightarrow} \mathbb{C} \oplus \bigoplus_{n=0}^{\infty} M_{2^i}(\mathbb{C}) \tag{11.1}$$

holds with the image being a subdirect sum (its projection on each summand is onto). We are going to describe the embedding more precisely and to show that the algebra $C^*_\rho(G)$ is self-similar in the sense that the embedding

$$\psi\colon\ G \hookrightarrow G \wr S_2$$

from (3.4) has a natural extension to the embedding

$$\widetilde{\psi}\colon\ C^*_\rho(G) \hookrightarrow M_2(C^*_\rho(G)).$$

To see this let us mention that the partition $[0,1] = [0,\frac{1}{2}] \cup [\frac{1}{2},1]$ (or $\partial T = \partial T_0 \sqcup \partial T_1$ where T_0, T_1 are subtrees with a root at first level) leads to the decomposition $H = H_0 \oplus H_1$ where

$$\begin{aligned} H &= L^2([0,1),m) \\ H_0 &= L^2\left(\left[0,\frac{1}{2}\right],m_0\right) \\ H_1 &= L^2\left(\left[\frac{1}{2},1\right],m_1\right), \end{aligned}$$

and m, m_0, m_1 are restrictions of the Lebesgue measure on corresponding intervals. The natural isomorphisms $H \simeq H_i$, $i=0,1$ allow to represent an operator in H by a 2×2 operator matrix. This leads to the following recursive relations for the generators of G:

$$\begin{aligned} \rho(a) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} \rho(a) & 0 \\ 0 & \rho(c) \end{pmatrix}, \\ \rho(c) &= \begin{pmatrix} \rho(a) & 0 \\ 0 & \rho(d) \end{pmatrix}, \quad \rho(d) = \begin{pmatrix} 1 & 0 \\ 0 & \rho(b) \end{pmatrix}. \end{aligned} \tag{11.2}$$

Similar recursions hold for the matrices $\rho_n(a)$, $\rho_n(b)$, $\rho_n(c)$, $\rho_n(d)$, $n = 1,2,\dots$. For $n=0$

$$\rho_0(a) = \rho_0(b) = \rho(c) = \rho_0(d) = 1$$

where 1 is the scalar unity and for $n \geq 1$

$$\begin{aligned} \rho_n(a) &= \begin{pmatrix} 0_{n-1} & 1_{n-1} \\ 1_{n-1} & 0_{n-1} \end{pmatrix}, \quad \rho_n(b) = \begin{pmatrix} \rho_{n-1}(a) & 0_{n-1} \\ 0_{n-1} & \rho_{n-1}(c) \end{pmatrix} \\ \rho_n(c) &= \begin{pmatrix} \rho_{n-1}(a) & 0_{n-1} \\ 0_{n-1} & \rho_{n-1}(d) \end{pmatrix}, \quad \rho_n(d) = \begin{pmatrix} 1_{n-1} & 0_{n-1} \\ 0_{n-1} & \rho_{n-1}(b) \end{pmatrix} \end{aligned} \tag{11.3}$$

where 0_n and 1_n are respectively a zero matrix and identity matrix of order 2^n.

The representation ρ_n is a subrepresentation of ρ_{n+1} because H_n naturally embeds in H_{n+1} as it was mentioned above. To a function f on level n one can associate a function $\tilde{f}$ on level $(n+1)$ given by

$$\tilde{f}(u0) = \tilde{f}(u1) = f(u),$$

where u is any vertex of level n. The complement of H_n in H_{n+1} consists of a function f which satisfies

$$f(u0) = -f(u1),$$

$|u| = n$. This shows that the restrictions of ρ_{n+1} on $\mathcal{H}_n^\perp$ (which we have denoted $\rho_n^\perp$) satisfy, in an appropriate basis, relations similar to those in (11.3)

$$\rho_0(a) = -1, \quad \rho_0(b) = \rho_0(c) = \rho_0(d) = 1$$

and for $n \geq 1$

$$\begin{aligned} \rho_n^\perp(a) &= \begin{pmatrix} 0_{n-1} & 1_{n-1} \\ 1_{n-1} & 0_{n-1} \end{pmatrix}, & \rho_n^\perp(b) &= \begin{pmatrix} \rho_{n-1}(a) & 0_{n-1} \\ 0_{n-1} & \rho_{n-1}(c) \end{pmatrix} \\ \rho^\perp(c) &= \begin{pmatrix} \rho_{n-1}(a) & 0_{n-1} \\ 0_{n-1} & \rho_{n-1}(d) \end{pmatrix}, & \rho_n^\perp(d) &= \begin{pmatrix} 1_{n-1} & 0_{n-1} \\ 0_{n-1} & \rho_{n-1}(b) \end{pmatrix}. \end{aligned} \tag{11.4}$$

The relations (11.2) can be interpreted as ψ-images of operators $\rho(a)$, $\rho(b)$, $\rho(c)$, $\rho(d)$. They extend to any operator $\rho(x)$, $x \in \mathbb{C}[G]$ and hence to any element of C_ρ.

The relations (11.4) can be used for construction of matrices corresponding to the generators a, b, c, d in the representations $\rho_n^\perp$, $n = 1, 2, \ldots$. Namely for appropriate choices of bases in $H_n^\perp$, $n \geq 1$, the elements $\theta(\rho(s))$, $s \in \{a, b, c, d\}$, have the form:

$$\begin{aligned} \theta(\rho(a)) &= \bigoplus_{n=-1}^{\infty} a_n, \quad a_{-1} = 1, a_1 = -1 \\ a_n &= \begin{pmatrix} 0 & 1_{n-1} \\ 1_{n-1} & 0 \end{pmatrix}, \qquad 1_{n-1} - \text{identity matrix of size } 2^{n-1} \\ \theta(\rho(b)) &= \bigoplus_{n=1}^{\infty} b_n, \qquad b_{-1} = b_0 = 1, \\ b_n &= \begin{pmatrix} a_{n-1} & 0 \\ 0 & c_{n-1} \end{pmatrix}, \\ \theta(\rho(c)) &= \bigoplus_{n=-1}^{\infty} c_n, \qquad c_{-1} = c_0 = 1, \\ c_n &= \begin{pmatrix} a_{n-1} & 0 \\ 0 & d_{n-1} \end{pmatrix}, \\ \psi(\rho(d)) &= \bigoplus_{n=-1}^{\infty} d_n, \qquad d_{-1} = d_0 = 1, \\ d_n &= \begin{pmatrix} 1_{n-1} & 0 \\ 0 & b_{n-1} \end{pmatrix}. \end{aligned}$$

For instance

$$C_4 = \left(\begin{array}{c:c}
\begin{array}{cccc:cccc}
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1\\
\hdashline
1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0
\end{array} & \bigcirc \\
\hdashline
\bigcirc & \begin{array}{c:c}
\begin{array}{c:c}
\begin{matrix}1&0\\0&1\end{matrix} & \bigcirc \\
\hdashline
\bigcirc & \begin{matrix}1&0\\0&1\end{matrix}
\end{array} & \bigcirc \\
\hdashline
\bigcirc & \begin{array}{c:c}
\begin{matrix}0&1\\1&0\end{matrix} & \bigcirc \\
\hdashline
\bigcirc & \begin{matrix}-1&0\\0&1\end{matrix}
\end{array}
\end{array}
\end{array}\right).$$

The elements $\theta(\rho(s))$, $s \in \{a, b, c, d\}$ generate $C^*_\rho(G)$. Even though the generators of $C^*_\rho(G)$ are described explicitly, it is not easy to study the properties of the algebra and at the moment very little is known about $C^*_\rho(G)$.

In a similar way an embedding

$$\psi\colon\ L \hookrightarrow L \wr S_m$$

of self-similar group L given by relations of the type (3.10) can be extended to an embedding

$$\widetilde{\psi}\colon\ C^*_\rho(L) \hookrightarrow M_m(C^*_\rho(L))$$

and a matrix and operator recursions of the type similar to (11.2), (11.3) hold. The above construction leads to a large family of self-similar algebras.

Any residually finite dimensional algebra embedded in $\bigoplus\limits_{i=1}^{\infty} M_{n_i}(\mathbb{C})$ has a faithful trace of the form

$$\operatorname{tr} x = \sum_{i=1}^{\infty} \frac{1}{2^{n_i}} \operatorname{tr} x_i.$$

Unfortunately, in our situation this trace does not agree with the embedding ψ. Having a self-similar algebra C^* and an embedding

$$\xi\colon\ C^* \to M_d(C^*)\xi(c) = (c_{ij})_{i,j=1}^d.$$

it is natural to work with a trace τ such that

$$\tau(\xi(c)) = \frac{1}{d}\sum_{i=1}^{d} \tau(c_{ii})$$

for any $c \in C^*$.

If M is a group generated by a finite automaton over an alphabet on d letters then the space $\mathcal{H}_n$ has dimension d^n and the limit

$$\lim_{n\to\infty} \frac{1}{d^n} Tr_n \rho_n(g) \tag{11.5}$$

exists for any element $g \in M$, where Tr_n is the standard (not normalized) trace on matrix algebra $M_{d^n}(\mathbb{C})$. The value $\tau(g)$ of this limit has the properties of a trace and can be extended to $\mathbb{C}[M]$ and to $C^*_\rho(M)$. The obtained trace agrees with the embedding ψ.

The trace τ is group-like if $\tau(g) = 1$ for $g = 1$ and $\tau(g) = 0$, for $g \neq 1$.

Problem 11.2. *Let τ be a trace on $C_\rho(M)$ defined by (11.5) where M is a group generated by finite automaton.*

(a) *Is τ unique normalized trace that agrees with the embedding $\psi\colon\ C_\rho \to M_d(C_\rho)$ given by the automaton structure.*

(b) *Under which conditions τ is faithful?*

(c) *Under which conditions τ is group-like trace?*

The values $\tau(g)$, $g \in M$ satisfy the system of equations

$$\tau(g) = \sum_{i=1}^{d} \tau(g_{ii}) \tag{11.6}$$

where

$$\psi(g) = (g_{ij})_{i,j=1}^d$$

and the matrix elements g_{ij} belong to the set $0 \cup M$. If g_{ij} is nonzero, then it is an element of the group M of length not greater then the length of g. This shows that the system S of equations (11.6), where g runs over M, splits as union $\bigcup_{n=1}^{\infty} S_n$ of finite systems of equations where S_n consist only of the equations with $|g| \leq n$ (the length is considered with respect to the generating set of M given by the set of the states of the automaton). The question 11.2 (a) becomes a question on uniqueness of the solution of a system of equation. In case M is a contracting group everything reduces to the system (11.6) with g running through the core. For instance for the group G we get the system

$$\begin{cases} \tau(a) = 0 \\ \tau(b) = \dfrac{1}{2}(\tau(a) + \tau(c)) \\ \tau(c) = \dfrac{1}{2}(\tau(a) + \tau(d)) \\ \tau(d) = \dfrac{1}{2}(\tau(1) + \tau(b)) \end{cases}$$

with unique solution $\tau(b) = \frac{6}{7}$, $\tau(c) = \frac{5}{7}$, $\tau(d) = \frac{3}{7}$ and $\tau(g)$ can be effectively computed for any element $g \in G$ (by use of core portraits).

Orthogonal to the contracting case is the class of automatons determining group-like trace. A condition which implies such a situation is a local nontriviality of the action of M on rooted tree T_d by which we mean the triviality of the kernel of any homomorphism

$$st_M(u) \longrightarrow st_M(u)|_{T_u}$$

given by restriction of $st_M(u)$ on the subtree with a root at u. In other words if an element fixes a vertex u and acts trivially on the subtree T_u then it acts trivially on the whole tree.

Example of such actions are given by automata from Figures 1.5 and 1.6 [GŻ01].

In case τ is group-like trace on C_ρ the factor algebra $C_\rho/\text{ker } \tau$, where

$$\text{ker } \tau = \{x\colon\ \tau(xx^*) = 0\}$$

is isomorphic to $C^*_r(G)$.

Problem 11.3. (a) *Is there an algorithm which, given an automaton, checks if the action of corresponding group is locally nontrivial?*

(b) *Give an example of automaton for which the trace τ is group-like and faithful.*

Perhaps the lamplighter group given by automaton from Figure 1.6 answers positively the question 11.3(b). At the same time the trace determined by the automaton given by Figure 1.5 is not faithful. This follows from the fact that reduced $C_r^*(F_m)$ algebra of a free group of rank $m \geq 2$ is simple.

12 Schreier graphs

Schreier graphs associated to a group acting on a rooted tree are very interesting combinatorial objects.

Given a group L, a subgroup $H < L$ and a finite system of generators S of M the Schreier graph $\Gamma(M, H, S)$ consists of vertices represented by cosets gH, $g \in M$ and of edges of the form (gH, sgH) where $s \in S \cup S^{-1}$. To complete the definition one should consider them as oriented graphs with edge labelling by the elements of $S \cup S^{-1}$, but this is often ignored. Schreier graphs are k-regular graphs (viewed as non-oriented graphs) where the degree k of each vertex is equal to $2|S|$ (loops contribute 2 to the degree). Every k-regular graph with even degree $k = 2m$ can be realized as a Schreier graph of the form $\Gamma(F_m, H, S)$ where F_m is a free group of rank m and S is a basis of F_m [dlH00].

Let L be a group generated by a finite automaton over an alphabet on d letters. Then M naturally acts on d-regular rooted tree $T = T_d$ as was described in Section 1 and for any point of the boundary $\xi \in \partial T$ (represented by geodesic ray joining the root with infinity) one can consider the Schreier graphs Γ, and Γ_n, $n = 1, \dots,$ where

$$\Gamma = \Gamma(L, P, S),$$
$$\Gamma_n = \Gamma(L, P_n, S),$$

$P = st_L(\xi)$, $P_n = st_L(u_n)$ and u_n is the unique vertex at level n belonging to the path ξ_n. The graphs Γ_n are finite because P_n has finite index in M and Γ is infinite graph if the M-orbit of ξ is infinite (this holds for instance when the action is level transitive). As $P = \bigcap_{n=1}^{\infty} P_n$ and as $\{P_n\}$ is descending sequence, the graph Γ is the limit of the sequence $\{\Gamma_n\}$ in the sense of topology on the space of graphs considered in [GŻ99] (which is analogous to the topology discussed in Section 2). More precisely

$$(\Gamma, v) = \lim_{n\to\infty} (\Gamma_n, v_n) \tag{12.1}$$

where $v = P$, $v_n = P_n$ are reference points and convergence in the space of marked graphs (i.e. graphs with reference point) means that the neighborhoods of v_n of radius R in Γ_n isomorphically stabilize to the corresponding neighborhood in Γ when $n \to \infty$.

The graph Γ and the approximating sequence $\{\Gamma_n\}$ keep important information about the group. Recent investigations show that the study of the asymptotic properties of Γ and $\{\Gamma_n\}$ is related to many topics in computer science, combinatorics, geometry, dynamics and other areas of mathematics.

To a k-regular graph Γ one can associate a Markov operator M acting on the Hilbert space $\ell^2(\Gamma)$ of square summable functions defined on the vertices of Γ by

$$(Mf)(x) = \frac{1}{k} \sum_{y \sim x} f(y)$$

where $y \sim x$ is the adjacency relation. M is a self-adjoint contraction and $sp\ M \subseteq [-1, 1]$. If Γ is finite connected graph then 1 is simple eigenvalue of M. If Γ is infinite connected graph then $1 \in sp\ M$ if and only if Γ is amenable graph [dlAGCS99].

Using the spectral decomposition

$$M = \int_{-1}^{1} \lambda dE(\lambda)$$

(where $E(\lambda)$ is a family of orthoprojectors) one can define for each vertex u the spectral function $\sigma_u(\lambda) = \langle E(\lambda)\delta_u, \delta_u\rangle$ where δ_u is delta function and $\langle\ ,\ \rangle$ denotes the standard scalar product in the Hilbert space $\ell^2(\Gamma)$. Let μ_n be the corresponding measure $(\mu([a,b]) = \sigma_n(a+) - \sigma_u(b))$. The observation made in [GŻ99] shows that μ_u^Γ is a (vector-valued) continuous function on the space of marked k-regular graphs.

In particular, the relation

$$\mu_v^\Gamma = \lim_{n\to\infty} \mu_{v_n}^{\Gamma_n}$$

is a consequence of (12.1). The next three questions are the most important in spectral theory of graphs:

(a) What is the norm (i.e. the spectral radius) of operator M?

(b) What is the spectrum *sp* M (as a subset of $[-1,1]$)?

(c) What can be said about the spectral measures μ_v? In particular, how μ_v decomposes in absolutely continuous, singular and discrete parts?

There are very few examples of computation of spectra of infinite Schreier graphs given in [BG00a], [GŻ01], [Sun].

In case Γ is transitive graph (i.e. Aut Γ acts transitively on the set of vertices) the measures μ does not depend on v. This holds, for instance, when P is trivial subgroup and hence Γ is the Cayley graph in this case.

The spectral measure μ_v has moments

$$P_{v,v}^{(n)} = \int_{-1}^{1} \lambda^n \, d\mu_v$$

equal to probabilities of return to v for a simple random walk on Γ with beginning in v. It is clear that in a non-transitive case this probabilities depends on v, therefore the measure μ_v depends on v in general.

In our situation one can attach to Γ a measure μ_* which is defined as

$$\mu_* = \lim_{n\to\infty} \mu_n \tag{12.2}$$

where μ_n is a counting (or cumulative) measure on Γ_n which counts the ratio of the number of eigenvalues (including multiplicities) of the Markov operator on M_n which belong to a given interval and the total number of eigenvalues (i.e. the size of Γ_n).

The limit (12.2) exist for any covering system of graphs (in our case Γ_{n+1} covers Γ_n because P_{n+1} is a subgroup of P_n). This follows from one theorem of Serre [Ser97] (see [BG00a, GŻ04]). The measure μ_* was named KNS-spectral measure (Kesten–von Neumann–Serre) in [GŻ04]. The same measure was named empiric measure in [BG00a] where the first calculations of μ_* appeared. An open question is the question on the relation between the supports of the measures μ_* and μ_v, $v \in V(\Gamma)$. The union of supp μ_v gives the spectrum of M as a set. Another question that can be asked is to find conditions under which the support of μ_* coincides with *sp* M? Some results in this direction are provided in [GŻ99].

The graphs Γ and $\{\Gamma_n\}$ associated with the group G, or with the Gupta–Sidki 3-group and in some other cases are substitutional graphs, i.e. there is a substitutional rule which allows us to construct Γ_{n+1} from Γ_n by application of substitutions. A formal definition can be found in [BG00a], along with other references. Without getting into the details let us demonstrate this on the example of G.

The graph Γ_1 in this case has the form

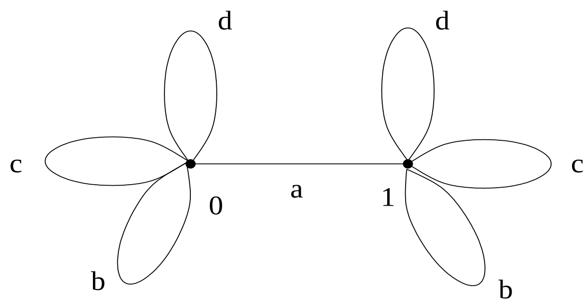


Figure 12.1:

and the substitutional rule is

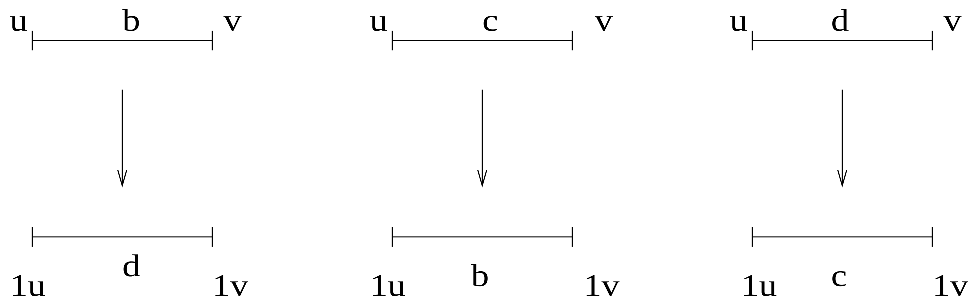


Figure 12.2:

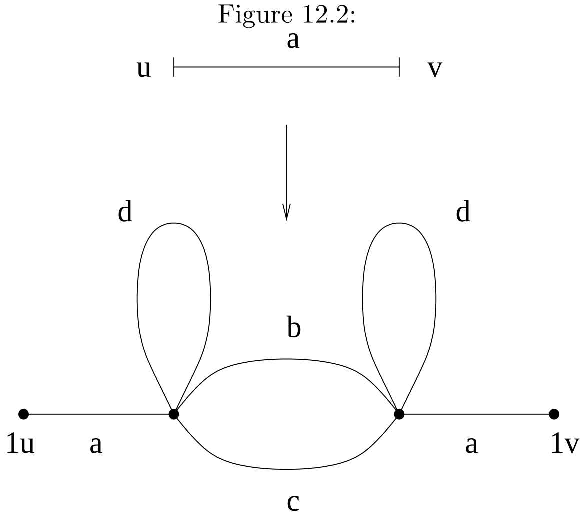


Figure 12.3:

For instance Γ_2 and Γ_3 look like

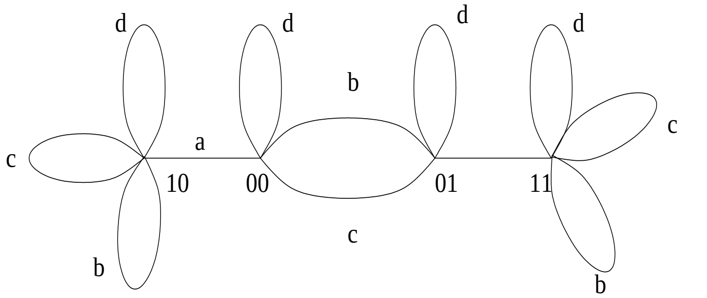


Figure 12.4:

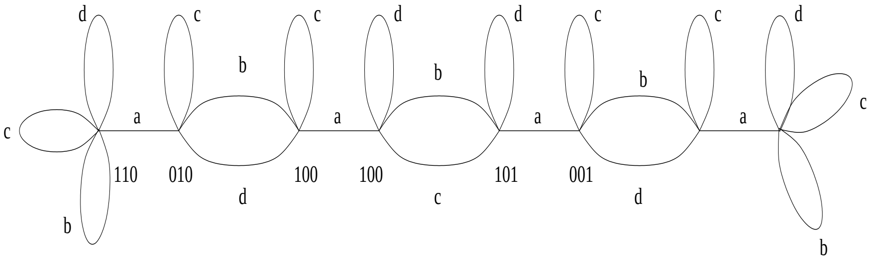


Figure 12.5:

The infinite graph Γ looks as

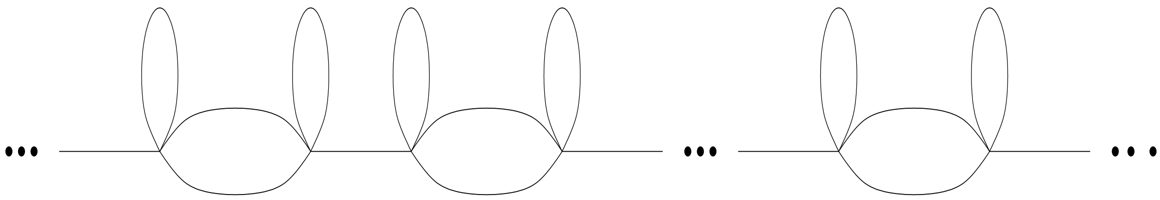


Figure 12.6:

if the sequence $\xi \in \partial T$ (determining the subgroup $P = st_G(\xi)$) is different from $1\ \ 1\cdots 1\cdots$ and looks as

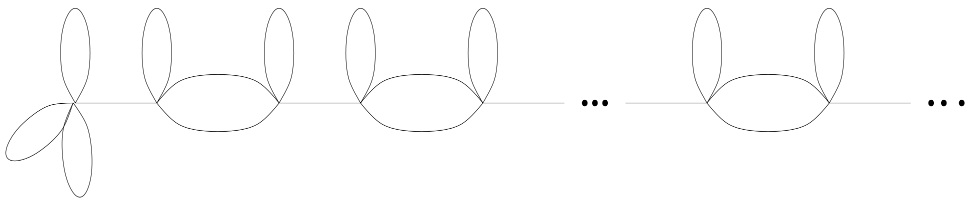


Figure 12.7:

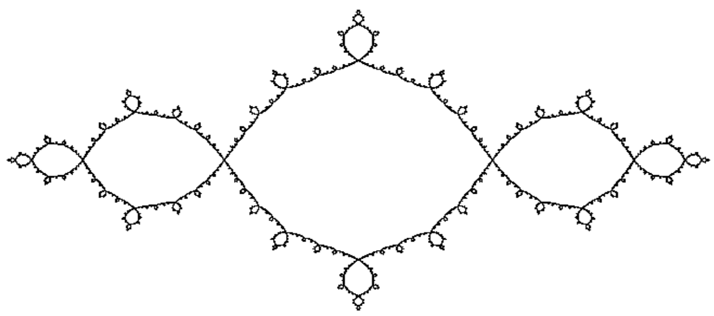


Figure 12.8: Graph Γ for Basilica group

in the opposite case (we draw Γ without labelling of edges).

Problem 12.1. *For which automata the sequence $\{\Gamma_n\}$ of associated Schreier graphs is substitutional (i.e. can be described by a substitutional rule)?*

Important characteristics of a graph are its diameter and the girth (i.e. the size of the shortest nontrivial cycle).

Problem 12.2. *Describe all possible asymptotic behaviors of diameters and girths in graphs of the form $\Gamma_n = \Gamma(L, P_n, S)$, $n = 1, 2, \dots$ where L is a self-similar group.*

Problem. *a) Describe all possible rates of growth of graphs $\Gamma = \Gamma(L, P, S)$ where L is a self-similar group.*
b) Describe the set of all degrees of polynomial growth of graphs of type $\Gamma = \Gamma(L, P, S)$, where L is a self-similar group.

In case of the group G the diameters of Γ_n grow exponentially while the growth of limit graph Γ is linear. In case of Gupta–Sidki 3-group or Fabrikowskii–Gupta group the sequence $\{\Gamma_n\}$ also is substitutional, the diameters grow exponentially but the degree of polynomial growth is $\log_2 3$ [BG00a]. Indeed the graph $\Gamma = \Gamma(L, P, S)$ always has polynomial growth in case the group L is contracting [BG02a], [BGŠ03]. An approach to calculation of the degree of polynomial growth of graphs γ of the above type and of diameters of the corresponding sequences $\{\Gamma_n\}$ is recently found by Bondarenko and Nekrashevych.

An interesting example of an automaton (and a group) determining a graph Γ of intermediate growth is constructed in [BCSN]. For the Basilica group the graph Γ is shown in Figure 12.8, which particularly explains the name of the group (more on this in [Kai].

There are many examples when the graph Γ grows exponentially. For instance, the two-state automaton given by Figure 1.6 determines the Cayley graph of the lamplighter group for almost every point $\xi \in \partial T$ (with respect to uniform measure) and hence has the exponential growth. Indeed Γ has exponential growth for any point $\xi \in \partial T$ since P is either trivial or cyclic [GŻ01].

Let X_n be a sequence of finite connected k-regular graphs and $|X_n| \to \infty$. Let λ_n be the second largest eigenvalue (after 1) of the Markov operator on X_n. Then the sequence $\{X_n\}_{n-1}^{\infty}$ is called a sequence of expanders if there exists $\delta > 0$ such that $\lambda_n \le 1 - \delta$, $n = 1, 2, \dots$. The bigger δ is the better expanding properties the family of graphs $\{X_n\}$ has . But δ can not be larger than $1 - \frac{2\sqrt{k-1}}{k}$ (this follows from Alon–Boppana Theorem [Lub94, GŻ99]).

When the second largest eigenvalue of the Markov operator on a connected k-regular graph is no greater than $\frac{2\sqrt{k-1}}{k}$ the graph is called Ramanujan graph. Expanders and Ramanujan play important roles in many topics including rather concrete applications (for more on this see in [Lub94, LPS88]). However, it is difficult to give explicit constructions and currently there are very few conceptually different constructions.

There are indications that sequences of the form $\{\Gamma_n\}$ constructed with the help of some particular finite automata are sequences of expanders (and perhaps in some cases even of Ramanujan graphs). For instance this holds for the automaton given by Figure 1.5 (computer experiments) with a rigorous proof of a weaker fact (namely that they are the so called asymptotic Ramanujan graphs).

By the Chung inequality [Ter99, Chu97] the diameter D of a finite connected graph X can be estimated using the second eigenvalue:

$$D \leq \frac{\log |X| - 1}{-\log \mu} + 1.$$

Thus if the group $L = G(A)$ generated by a finite automaton A acts transitively on levels and determines a sequence $\{\Gamma_n\}_{n=1}^{\infty}$ of expanders then the corresponding sequence of diameters grows linearly, since $|\Gamma_n| = d^n$ in this case (d is the cardinality of the alphabet).

Problem 12.3. *a) Find an automaton producing a sequence $\{\Gamma_n\}_{n=1}^{\infty}$ of expanders.*

b) Find an automaton producing a sequence $\{\Gamma_n\}_{n=1}^{\infty}$ of Ramanujan graphs.

Problem 12.4. *a) Is there an algorithm which, given an automaton, determines the type of growth (polynomial, intermediate or exponential) of the diameters of Γ_n?*

13 Spectra and rational maps

Given a finite automaton A one can associate with it at least three spectra: the spectrum of the Cayley graph $\Gamma(G(A), Q)$ of the group $G(A)$ generated by the automaton A with respect to the system of generators given by the set of states, the spectrum of the Schreier graph $\Gamma(G(A), P, Q)$ where P is a stabilizer of a point on the boundary of the tree T_d (d-cardinality of the alphabet) and finally the spectrum of the Hecke type operator $\mathcal{M}$ in $L^2(\partial T, \nu)$ defined by

$$(\mathcal{M}f)(x) = \frac{1}{2|Q|} \sum_{q \in Q} (\rho(q) + \rho(q^{-1}))f(x)$$

where ρ is the representation described in Section 10.

As ρ is a direct sum of finite dimensional representations $\mathcal{M}$ has pure point spectrum. At the same time for most known examples the spectrum of the Markov operator on an infinite regular graph is continuous or at least is very far from discrete. However, there are exceptions, for instance the lamplighter group represented by automaton given by Figure 1.6 for which the spectrum is discrete.

The three defined spectra correspond to representations λ_G, $\lambda_{G/P}$ and ρ. If $\Gamma = \Gamma(G(A), P, Q)$ is amenable then its spectrum coincides with the spectrum of $\mathcal{M}$ [BG00a]. In general only the inclusion $sp\ M_\Gamma \supset sp\ \mathcal{M}$ holds. This follows from the existence of a surjective $*$-homomorphism $C^*_\rho \to C^*_{\lambda_{G/P}}$ discussed in Section 11. The computation of the spectrum of $\mathcal{M}$ is based on the relation

$$sp\ \mathcal{M} = \bigcup_{n=1}^{\infty} sp\ \mathcal{M}_n \tag{13.1}$$

where $\mathcal{M}_n = \mathcal{M}|_{\mathcal{H}_n}$ and $\mathcal{H}_n$ is the space of dimension d^n defined in Section 10. The operator $\mathcal{M}_n$ is similar to the Markov operator on the finite Schreier graph $\Gamma_n = \Gamma(G(A), P_n, Q)$. The union (13.1) is increasing union since Γ_{n+1} covers Γ_n.

The computation of $sp\ \mathcal{M}$ reduces to computation of spectra of finite matrices defined by a recursion given by the diagram of A, as shown by example in Section 11. There is no general method which allows us to solve this problem efficiently. However, in some cases explicit computation is possible. Let us demonstrate this with a few examples.

For the group G we have the recursion (11.3) which can be rewritten in the form

$$\begin{aligned} a_0 &= b_0 = c_0 = d_0 = 1 \\ a_n &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b_n = \begin{pmatrix} a_{n-1} & 0 \\ 0 & c_{n-1} \end{pmatrix} \\ c_n &= \begin{pmatrix} a_{n-1} & 0 \\ 0 & d_{n-1} \end{pmatrix}, \quad d_n = \begin{pmatrix} 1 & 0 \\ 0 & b_{n-1} \end{pmatrix}, \end{aligned}$$

where we suppress ρ as well as the dimensions of the identity and zero matrices. Then

$$\mathcal{M}_n = a_n + b_n + c_n + d_n = \begin{pmatrix} 2a_{n-1} + 1 & 1 \\ 1 & \mathcal{M}_{n-1} - a_{n-1} \end{pmatrix}$$

and it is not clear how this relation could be used to get a recursion for the spectrum. But, surprisingly, the problem can be solved if instead of considering the standard spectral problem of inversion of the matrix $\mathcal{M}_n - \lambda I_n$ one considers the problem of inversion of the pencil of matrices

$$\begin{aligned} R_n^{(1)}(\lambda, \mu) &= -\lambda a_n + b_n + c_n + d_n - (\mu + 1) I_n \\ &= \mathcal{M}_n - (\lambda + 1) a_n - (\mu + 1) I_n \end{aligned}$$

(the coefficients $-(\lambda + 1)$ and $\mu + 1$ are chosen instead of λ and μ because of simplification in the resulting formulas). The next recurrence for the determinant $|R_n^{(1)}(\lambda, \mu)|$ is the first step in the calculation of the spectrum.

Lemma 13.1 ([BG00a]). *For $n \geq 2$ we have*

$$|R_n^{(1)}(\lambda, \mu)| = (4 - \mu^2)^{2^{n-2}} \left| R_{n-1}^{(1)} \left(\frac{2\lambda^2}{4 - \mu^2}, \mu + \frac{\mu \lambda^2}{4 - \mu^2} \right) \right|. \tag{13.2}$$

Proof. Observe that as $a_n^2 = 1$ (1 stands for the identity matrix)

$$(2a_{n-1} - \mu)(2a_{n-1} + \mu) = 4 - \mu^2$$

and

$$|2a_{n-1} - \mu| = \begin{vmatrix} -\mu & 2 \\ 2 & -\mu \end{vmatrix}_{2^{n-1}} = (\mu^2 - 4)^{2^{n-2}}.$$

Therefore

$$\begin{aligned} |R_n^{(1)}(\lambda, \mu)| &= \begin{vmatrix} 2a_{n-1} - \mu & -\lambda \\ -\lambda & \mathcal{M}_{n-1} - a_{n-1} - (\mu + 1) \end{vmatrix} \\ &= \begin{vmatrix} 2a_{n-1} - \mu & -\lambda \\ 0 & \mathcal{M}_{n-1} - a_{n-1} - (\mu + 1) - \dfrac{\lambda^2}{4 - mu^2}(2a_{n-1} + \mu) \end{vmatrix} \\ &= |2a_{n-1} - \mu| \cdot \left| \mathcal{M}_{n-1} - \left(1 + \frac{2\lambda^2}{4 - \mu^2}\right) a_{n-1} - \left(1 + \mu + \frac{\mu\lambda^2}{4 - \mu^2}\right) \right|, \end{aligned}$$

which proves the claim. $\square$

Using the relation (13.2) $(n - 1)$ times we come to the relation

$$|R_n^{(1)}(\lambda, \mu)| = (4 - \mu^2)^{2^{n-2} + 2^{n-3} + \cdots + 2 + 1^0} R_1^{(1)}(F_1^{(n-1)}(\lambda, \mu))$$

where $F_1\colon\ \mathbb{R}^2 \to \mathbb{R}^2$ is the rational map

$$F_1\colon \begin{cases} \lambda \longrightarrow \dfrac{2\lambda^2}{4-\mu^2} \\ \mu \longrightarrow \mu + \dfrac{\mu\lambda^2}{4-\mu^2} \end{cases}$$

and $F^{(n)} = \underbrace{F \circ \cdots \circ F}_{n}$ is the n-fold composition of F. Thus the spectrum Σ_n of the pencil $R_n^{(1)}(\lambda,\mu)$ (i.e. the set of (λ,μ) for which $R_n(\lambda,\mu)$ is not invertible) heavily depends on the dynamics of the map F_1.

If Σ is the closure of $\bigcup\limits_{n=1}^{\infty} \Sigma_n$ then the spectrum of $\mathcal{M}$ is the projection on the μ-axis of the intersection of Σ and the line $\lambda = -1$. On the other hand Lemma 13.1 shows that the spectrum Σ is F_1-invariant, i.e $F_1^{-1}(\Sigma) = \Sigma$. We see that at this point the spectral problem meets a problem from dynamics requiring a description of invariant subsets. In some examples arising from self-similar groups these invariant sets have quite complicated form, as we will see later.

Fortunately in case of G the problem of finding invariant sets and the study of the dynamics of F_1 has an easy solution because of the following observation.

Lemma 13.2. *The map F_1 is semi-conjugate to the map $\alpha\colon\ x \to 2x^2-1$ and the diagram*

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{F_1} & \mathbb{R}^2 \\ \psi\big\downarrow & & \big\downarrow\psi \\ \mathbb{R} & \xrightarrow[\alpha]{} & \mathbb{R} \end{array}$$

where $\psi(\lambda,\mu) = \frac{4-\mu^2+\lambda^2}{4\lambda}$ is commutative.

The map $x \to 2x^2-1$ is known as Chebyshev–von Neumann–Ulam map. It is conjugate to the map $x \to x^2-2$. Lemma 13.2 shows that the map is integrable in certain sense and has invariant family of hyperbolas

$$H_\theta(\lambda,\mu) = 4-\mu^2+\lambda^2+4\lambda\theta$$

showed in Figure 12.2 (right). The F_1-preimage of H_θ is the union of H_{θ_1}, and H_{θ_2} where θ_1,θ_2 are α-preimages of θ. In other words the following relation holds:

$$H_\theta(F_1(\lambda,\mu)) = H_{\theta_1}(\lambda,\mu)H_{\theta_2}(\lambda,\mu).$$

Moreover, induction allows us to show that the factorization

$$|R_n^{(1)}(\lambda,\mu)| = (2-\lambda-\mu)(2+\lambda-\mu)\prod_{\theta} H_\theta(\lambda,\mu)$$

holds, where the product is taken over

$$\theta \in \bigcup_{i=0}^{n-2} \alpha^{-i}(0).$$

This leads to computation of counting measures μ_n on the graphs Γ_n and KNS spectral measure $\mu = \lim\limits_{n\to\infty} \mu_n$ which in this case is supported on $[-\frac{1}{2},0] \cup [\frac{1}{2},1]$ and is absolutely continuous with respect to Lebesgue measure:

$$\begin{aligned} d\mu(x) &= \frac{|1-4x|\ dx}{2\pi\sqrt{x(2x-1)(2x+1)(1-x)}} \\ &= \{(\lambda,\mu)\colon\ 4-\lambda^2+\mu^2-\mu\xi = 0\}. \end{aligned}$$

Its density is shown in Figure 13

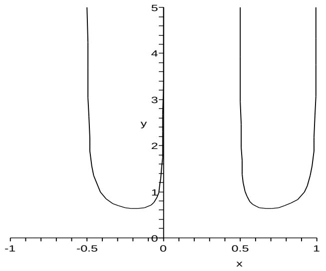


Figure 13.1: KNS spectral measure related to G.

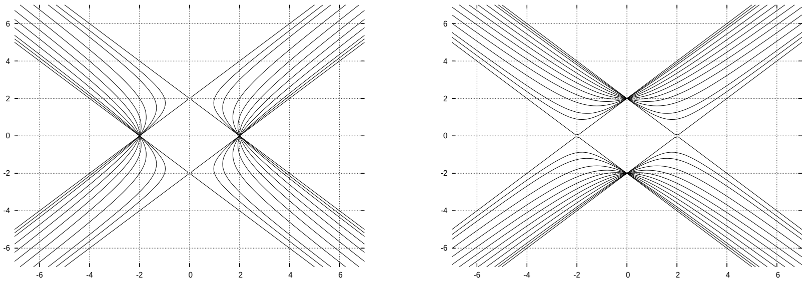


Figure 13.2:

There is another invariant family

$$U_\xi = \ 4 - \lambda^2 + \mu^2 - \mu\xi$$

of hyperbolas transversal to H_θ and shown in Figure13 (left). Indeed not only the family $\{U_\xi\}$ is F_1-invariant but each hyperbola U_ξ is F_1-invariant so in this case the diagram

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "F"] \arrow[d, "\varphi"'] & \mathbb{R}^2 \arrow[d, "\varphi"] \\ \mathbb{R} \arrow[r, "id"] & \mathbb{R} \end{tikzcd}$$

is commutative, where

$$\varphi(\lambda, \mu) = \frac{4 - \lambda^2 + \mu^2}{\mu}.$$

Another example is provided by the lamplighter group L realized as a group generated by the automaton from Figure 1.6. Let u, v be the generators of L given by the states of the automaton and $w = v^{-1}u$. Then w acts on sequences by changing the first symbol and hence w coincides with the transformation a from the previous example. The operator recursion in this case is

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad u = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}, \quad v = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$$

Let

$$R^{(2)}(\lambda, \mu) = u + u^{-1} + v + v^{-1} - \lambda - \mu w$$

be the pencil of operators (for the representation ρ in $L^2(\partial T, \nu)$) and let $R_n^{(2)}$ be a finite dimensional approximation given by a restriction of the action on level n. Then [GŻ01]

$$|R_n^{(2)}(\lambda, \mu)| = (\mu - \lambda)^{2^{n-1}} |R_{n-1}^{(2)}(F_2(\lambda, \mu))|$$

if $n \geq 2$, where

$$F_2\colon \begin{cases} \lambda \longrightarrow \dfrac{2-\lambda^2+\mu^2}{\mu-\lambda} \\ \mu \longrightarrow \dfrac{2}{\lambda-\mu}. \end{cases}$$

The map F_2 is also "integrable", namely it is semi-conjugate to the identity map via the map $\psi(\lambda,\mu) = \lambda+\mu$. The last fact allows to understand the dynamics of F_2 (more information can be found in [GŻ02b]) but the factorization of $|R_n(\lambda,\mu)|$ requires some extra work and relies on use of continuous fractions and (implicitly) Chebyshev polynomials.

The main result from [GŻ01] states that the Markov operator on the Cayley graph $\Gamma(L,\{u,v\})$ has a pure point spectrum concentrated on the points $\cos\frac{p}{q}\pi$, $q = 2,3,\ldots$, $p \in \mathbb{Z}$. The spectral measure μ is therefore discrete and is concentrated on the above points with values

$$\mu\left(\cos\frac{p}{q}\pi\right) = \frac{1}{2^q-1}, \quad (p,q)=1.$$

As a consequence of this result a question of M. Atiyah [Ati76] was answered in [GLSŻ00] by providing a counterexample to the strong Atiyah conjecture [Lüc02].

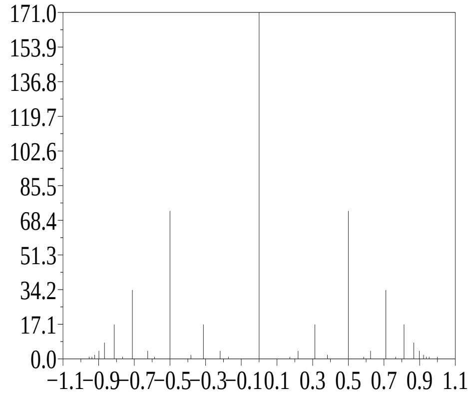


Figure 13.3:

More sophisticated examples of integrable rational maps of $\mathbb{R}^2$ leading to computation of spectra of Schreier graphs of Gupta–Sidki 3-group and Fabrikovsky–Gupta groups are given in [BG00a].

The Basilica group B given by the automaton from Figure 9.1 is an example of intermediate situation when the recursion for the determinant exists but the dynamics of the corresponding map is unknown and we do not know what the spectrum in this case is (although we know that it has 1 is an accumulating point).

Let a,b be the two generators of B given by the (nonidentity) states of the automaton and let

$$R^{(3)}(\lambda,\mu) = a+a^{-1}+\lambda(b+b^{-1})-\mu.$$

Then

$$|R_n^{(3)}(\lambda,\mu)| = \lambda^{2^n}|R_{n-1}^{(2)}(F_3(\lambda,\mu))|$$

where

$$F_3\colon \begin{cases} \lambda \longrightarrow -2+\dfrac{\lambda(\lambda-2)}{\mu^2} \\ \mu \longrightarrow \dfrac{\lambda-2}{\mu^2}. \end{cases}$$

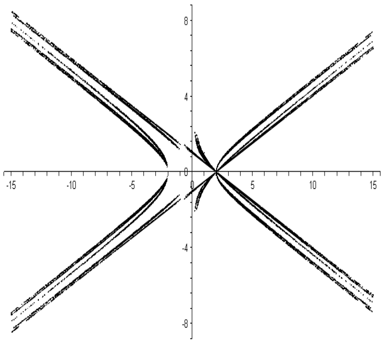


Figure 13.4:

The spectrum of $R^{(3)}(\lambda,\mu)$ is F_3-invariant and diagrams made on a computer give an impression that here we deal with a strange attractor (see Figure 13.4).

Unfortunately, for most of the examples we are unable even to get a recursion for the determinant of the type exhibited in the above examples.

Problem 13.1. *a) Is there an algorithm that, given an automaton A, finds a suitable pencil $R(\lambda_1,\dots,\lambda_k)$ of matrices representing the set of generators for $G(A)$ and such that there is a rational map $F\colon \mathbb{R}^k \to \mathbb{R}^k$ (k is the number of parameters involved in the pencil) for which the recursion of type*

$$|R_n(\lambda_1,\dots,\lambda_k)| = U\cdot V^{d^n}|R_{n-1}(F(\lambda_1,\dots,\lambda_k))| \tag{13.3}$$

holds, where U and V are some functions (and d is the cardinality of the alphabet)?

b) Give an example when such a recursion does not exist.

Problem 13.2. *For the examples when the recursion of type* (13.3) *exists and, in particular, for Basilica group describe the topological structure of the invariant subsets.*

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