On the asymptotic spectrum of random walks on infinite families of graphs

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Abstract

We observe that the spectral measure of the Markov operator depends continuously on the graph in the space of graphs with uniformly bounded degree. We investigate the behaviour of the largest eigenvalue and the density of eigenvalues for infinite families of finite graphs. The relations to the theorems of Alon-Boppana and Greenberg are indicated.

1 Introduction

The theory of the spectrum of the Markov (or Laplace) operator on graphs and groups is an important part of the theory of Markov processes. It is related to many topics in the theory of random walks on groups and graphs, abstract harmonic analysis, the theory of operator algebras, K-theory etc.

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The spectral radius, spectrum and the spectral measure of the Markov operator on the homogeneous tree T_k of degree k was computed by Kesten in [12]. The number $\frac{2\sqrt{k-1}}{k}$, the value of the spectral radius for this graph, plays an important role in the theory.

First of all there is a large class of infinite regular graphs of degree k with the same value of the spectral radius as was first observed in [6]. These are graphs with cogrowth beween 1 and $\sqrt{k-1}$. (In [12] and [6] the case when k is even is considered. But the same arguments work for odd k if we keep in mind the spherical homogeneity of the simple random walk on a group $\mathbb{Z}_2 * \ldots * \mathbb{Z}_2$).

The second place where these numbers appear is the theory of Ramanujan graphs (see [14], [16], [20] and [23]).

The third place is the theory of the largest eigenvalue λ_1, not equal to 1, of the Markov operator on finite graphs. In particular, the theorem of Alon-Boppana claims:

Theorem 1 (Alon-Boppana) *Let $\{X_{n,k}\}_{n=1}^{\infty}$ be a sequence of k-regular, finite, connected graphs where k is fixed and the number of vertices of $X_{n,k}$ tends to infinity. Then*

$$\liminf_{n\to\infty} \lambda_1(X_{n,k}) \geq \frac{2\sqrt{k-1}}{k}.$$

In this paper we will clarify and simplify some considerations related to this kind of problem. Another goal is to consider the problem of approximation of the spectrum of the Markov operator on infinite graphs by the spectra of operators on the sequence of finite graphs which approximate the given graph. For this purpose we introduce the compact topology on the space of graphs with uniformly bounded degree (which generalizes the topology introduced in [8]) and observe that the spectral measure (a measure on $[-1,1]$) depends continuously on the graph in this space. More precisely in Section 3 we will show:

Theorem 2 *The spectral measure is a continuous (measure valued) function on the space of marked graphs.*

The most regular case to which we can apply our results is the case of Cayley graphs; Let F_k be the free group on k generators. Let H_n be a sequence of normal subgroups in F_k such that $H_{n+1} \subset H_n$ for every $n \in \mathbb{N}$.

Let $H = \bigcap H_n$. Then the Cayley graphs of F_k/H_n converge to the Cayley graph F_k/H (for generators we choose the images of generators of F_k).

Some versions of convergence of graphs and corresponding limit theorems for spectral measures were considered in [4] and [18] (the latter paper is also a good survey on spectra of infinite graphs). The idea of approximation of an infinite graph by finite graphs is presented also in [14] and other books and articles. It comes from the group theory where the corresponding property is called residual finiteness. But in contrast to the group case where not every finitely generated group is residually finite every infinite regular graph can be approximated by finite regular graphs (see Proposition 1 below).

We hope that the approximation method may help to compute the spectrum (and even the spectral measure) of the Markov operator for some interesting infinite residually finite groups (for instance for the groups of intermediate growth constructed in [7] and [8]).

2 The space of graphs with uniformly bounded degree

In this paper we consider only locally finite, connected, non-oriented graphs which we identify with their geometric realization, i.e. simplicial complexes. This makes the concept of covering of the graph clear. Moreover, our graphs will be supplied with a natural distance metric.

Theorems 2 and 3 (see below) also have analogues in the cases of oriented graphs and coloured graphs when the number of colours is finite. This is important for constructions with Cayley graphs of finitely generated groups.

So let $X = (V_X, E_X)$ be a connected graph and V_X, E_X its set of vertices and edges respectively. Let the distance between two vertices $v_a \neq v_b \in V_X$ be the minimal number of edges needed to connect them, i.e.

$$\begin{aligned} \mathrm{dist}(v_a, v_b) &= \min\{n; \exists v_0, \ldots, v_n \in V_X, v_0 = v_a, v_n = v_b, \\ &\quad (v_i, v_{i+1}) \in E_X \text{ for } 0 \leq i \leq n-1\}. \end{aligned}$$

Let us consider a family $\{(X_n, v_n)\}$ of marked graphs, i.e. graphs with chosen vertices $v_n \in X_n$.

On the space of marked graphs there is a metric $Dist$ defined as follows

$$Dist((X_1, v_1), (X_2, v_2)) = \inf\left\{\frac{1}{n+1}; B_{X_1}(v_1, n) \text{ is isometric to } B_{X_2}(v_2, n)\right\}$$

where $B_X(v,n)$ is the ball of radius n in X centered on v.

For a sequence of marked graphs (X_n, v_n) we say that (X, v) is the limit graph if

$$\lim_{n\to\infty} Dist((X,v),(X_n,v_n)) = 0.$$

The limit graph is unique up to the isometry.

We will consider locally finite graphs, i.e. the degree $deg(v)$ of each vertex v is finite and we will always assume that the graphs are connected. Now we prove:

Lemma 1 *Let $\{(X_n, v_n)\}_{n=1}^{\infty}$ be a sequence of marked graphs whose degree is uniformly bounded, i.e. there exists $k > 0$, such that* $\deg(X_n) \leq k$ *for all $n \in I\!N$. Then there exists a subsequence $\{(X_{n_i}, v_{n_i})\}_{i=1}^{\infty}$ which converges to some marked graph (X, v).*

Proof Because the degrees of the graphs are uniformly bounded, we can use the diagonal argument. $\diamond$

Lemma 1 has as a corollary the following theorem:

Theorem 3 *The space of marked graphs of uniformly bounded degree is compact.*

Remark The analogous statement for the case of the space of Cayley graphs of groups with a fixed set of generators was mentioned in [8].

Proposition 1 *For any regular marked graph (X, v) there exists a sequence of finite marked regular graphs (X_n, v_n) converging to (X, v).*

Proof First of all let us suppose that the degree of X is even and equal to $2n$. Then X can be represented as the Schreier graph of F_n/H where F_n is a free group on n generators, H some subgroup of F_n and as generators for F_n/H we take the images of standard generators of F_n. We can suppose that the vertex v in X is the image of the identity element e in F_n. Now $H = \bigcup_{i=1}^{\infty} H_i$, where H_i is a sequence of subgroups of F_n such that for every i we have $H_i \subset H_{i+1}$ and H_i is finitely generated. By a theorem of M. Hall [9] every finitely generated subgroup of H_i can be represented as the intersection $\bigcap_{j=1}^{\infty} H_{ij}$ where H_{ij} are subgroups of F_n of finite index. By a diagonal process we can choose a sequence $\{H_{i_k j_k}\}_{k=1}^{\infty}$ such that the finite marked Schreier's graphs $\{F_n/H_{i_k j_k}, e\}_{k=1}^{\infty}$ converge to (X, v).

In the case when the degree of X is odd, the proof is similar but we have to use the version of Hall's theorem for $\mathbb{Z}_2 * \ldots * \mathbb{Z}_2$. $\diamond$

Remark There are other more direct proofs of Proposition 1. One of them was indicated to us by L. Bartholdi.

3 Continuity of the spectral measure of the Markov operator on graphs

On the locally finite, connected graph $X = (V_X, E_X)$ we can consider a random walk operator M acting on functions $f \in l^2(X, deg)$ as follows:

$$Mf(v) = \frac{1}{deg(v)} \sum_{w \sim v} f(w),$$

where $w \sim v$ means that w is a neighbor of v. This is a self-adjoint operator on $l^2(X, deg)$.

This random walk operator is related to the following simple random walk on X: starting at vertex v, we choose uniformly at random one of its neighbors w and then go to w.

By the random walk operator on marked graphs we will mean the random walk operator on the corresponding non-marked graph.

Let $\rho(M)$ be the spectral radius of M. Then it is known and easy to see that

$$\lim_{n\to\infty} \sqrt[2n]{p^{2n}(v,v)} = \rho(M) = ||M||_{l^2(X,deg)}$$

for any vertex $v \in X$, where $p^{2n}(v,v)$ is the probability of going from v to v in $2n$ steps. In particular $||M|| = \rho(M) \leq 1$ and $\rho(M)$ is in the spectrum of M. If X is finite then $||M|| = 1$ and 1 is the largest eigenvalue of M. For a finite connected graph X let $\lambda_1(X)$ denote the second eigenvalue after 1 in a natural decreasing ordering of points in the spectrum of M.

Lemma 2 *Let $f : X_1 \to X_2$ be a covering between two graphs X_1 and X_2. Then*

$$\rho(X_2) \geq \rho(X_1).$$

Proof By definition of the cover, different loops in X_1 are projected onto different loops in X_2. The loops in X_1 which start and finish in v are projected onto loops in X_2 which start and finish in $f(v)$. Thus

$$p^n(v,v) \leq p^n(f(v), f(v)),$$

which implies that $\rho(X_1) \leq \rho(X_2)$. $\diamond$

Since M is a bounded ($||M|| \leq 1$) and self-adjoint operator, it has the spectral decomposition

$$M = \int_{-1}^{1} \lambda E(\lambda),$$

where E is the spectral measure. This spectral measure is defined on Borel subsets of the interval $[-1, 1]$ and takes the values in projections on the Hilbert space $l^2(X, deg)$. The matrix μ^X of measures μ^X_{xy} for vertices $x, y \in X$ can be associated with E as follows:

$$\mu^X_{xy}(B) = \langle E(B)\delta_x, \delta_y \rangle,$$

where B is a Borel subset of $[-1, 1]$ and δ_x is the function which equals 1 at x and 0 elsewhere.

Now, in general, $\lambda \in Sp(M)$ if and only if for every $\varepsilon > 0$ there exists μ^X_{xy} such that $|\mu^X_{xy}((\lambda - \varepsilon, \lambda + \varepsilon))| > 0$. But we also have the following result (see [12]):

Lemma 3 $\lambda \in Sp(M)$ *if and only if for every* $\varepsilon > 0$ *there exists* $x \in X$ *such that* $\mu^X_{xx}((\lambda - \varepsilon, \lambda + \varepsilon)) > 0$.

Proof We need only show that if, for $B = (\lambda - \varepsilon, \lambda + \varepsilon)$, we have $|\mu^X_{xy}(B)| > 0$ then $\mu^X_{xx}(B) > 0$. As $E(B)$ is a projection, one has

$$\begin{aligned} 0 &< (\mu^X_{xy}(B))^2 = \langle E(B)\delta_x, \delta_y \rangle^2 \leq \langle E(B)\delta_x, E(B)\delta_x \rangle \langle \delta_y, \delta_y \rangle \\ &= \langle E(B)\delta_x, \delta_x \rangle deg(y) = \mu^X_{xx}(B) deg(y), \end{aligned}$$

which ends the proof. $\diamond$

Our main tool will be the weak convergence of the measures $\mu^{X_n}_{v_n v_n}$ to the measure μ^X_{vv} (provided that the sequence of marked graphs (X_n, v_n) converges to the marked graph (X, v)), i.e.

$$\lim_{n \to \infty} \int_{-1}^{1} f \mu^{X_n}_{v_n v_n} = \int_{-1}^{1} f \mu^X_{vv}$$

for any $f \in C[-1, 1]$. The weak convergence implies (see for instance [3]) that for any open interval $B \subset [-1, 1]$:

$$\liminf_{n \to \infty} \mu^{X_n}_{v_n v_n}(B) \geq \mu^X_{vv}(B).$$

Lemma 4 *Let us suppose that the sequence of marked graphs* (X_n, v_n) *converges to the marked graph* (X, v)*. Then the measures* $\mu^{X_n}_{v_n v_n}$ *converge weakly to the measure* μ^X_{vv}*.*

Proof We are going to prove that the moments of the measures $\mu^{X_n}_{v_n v_n}$ converge to the moments of the measure μ^X_{vv}. As it is easy to see, in our situation this implies weak convergence of corresponding measures (see for example [3]). The l-th moment of the measure μ^Y_{yy} for a graph Y and $y \in Y$ is given by

$$(\mu^Y_{yy})^{(l)} = \int_{-1}^{1} \lambda^l \mu^Y_{yy}(\lambda) = \int_{-1}^{1} \lambda^l \langle E(\lambda)\delta_y, \delta_y\rangle = \langle M^l \delta_y, \delta_y\rangle.$$

Thus the l-th moment of the measure μ^Y_{yy} is equal to the probability of going from y to y in l steps. But for n sufficiently large, the balls $B_{X_n}(v_n, l)$ and $B_X(v, l)$ are isometric and the l-th moment of the measure $\mu^{X_n}_{v_n v_n}$ is the same as the l-th moment of the measure μ^X_{vv}.

This ends the proof of Lemma 4. ◇

Proof of Theorem 2 This is a consequence of Lemma 4 and the fact that the space of graphs that we are considering is a metric space. ◇

4 Some applications

Proposition 2 *Let* $\{(X_n, v_n)\}_{n=1}^{\infty}$ *be a sequence of marked graphs which converges to the marked graph* (X, v)*. Then for every* α *in the spectrum of the random walk operator* M_X *on* X*, i.e.* $\alpha \in Sp(M_X)$*, and for every* $\varepsilon > 0$*, there exists* $N \in I\!N$ *such that for* $n \geq N$ *there is always a spectral value of* M_{X_n} *which is in the interval* $(\alpha - \varepsilon, \alpha + \varepsilon)$*.*

Proof Let x be any vertex in X, and let d be its distance from $v \in X$, i.e. $d = \text{dist}_X(v, x)$. For n sufficiently large the balls $B_{X_n}(v_n, d)$ and $B_X(v, d)$ are isometric. Let x_n be the image of x in X_n. Now, for any $\lambda \in Sp(M_X)$ and for every $\varepsilon > 0$, there exists $x \in X$ such that

$$\mu^X_{xx}((\alpha - \varepsilon, \alpha + \varepsilon)) = c > 0.$$

This means that for any $\varepsilon' > 0$ and n sufficiently large, by Lemma 4 we have

$$\mu^{X_n}_{x_n x_n}((\alpha - \varepsilon, \alpha + \varepsilon)) \geq c - \varepsilon',$$

which ends the proof of Proposition 2. ◇

We can prove the following proposition:

Proposition 3 *Let* $\{(X_n, v_n)\}_{n=1}^{\infty}$ *be a sequence of finite marked graphs convergent to the marked graph* (X, v)*. If the number of vertices of the graph* X_n *tends to infinity, then*

$$\liminf_{n\to\infty} \lambda_1(X_n) \geq \rho(X).$$

Proof As $\rho(X) \in Sp(M_X)$, Proposition 3 follows immediately from Proposition 2 in the case where $\rho(X) < 1$. In the case $\rho(X) = 1$ we need the following:

Lemma 5 *Let* X *be an infinite, connected graph such that* $\rho(X) = 1$*. Then 1 is not an isolated eigenvalue in* $Sp(M_X)$*.*

Proof If 1 were an isolated eigenvalue then there would be an eigenfunction in $l^2(X, deg)$ with the eigenvalue 1. Because the l^2 norm of this function is finite, it has to attain either a maximum or minimum. But as it corresponds to the eigenvalue 1, it attains the extremum also on the neighbours of the vertex on which it attains the extremum. And as the graph is connected, this implies that this eigenfunction is constant. However, as the graph is infinite this means that l^2 norm is infinite and we obtain a contradiction. ◇

Thus if $\rho(X) = 1$, Proposition 2 and Lemma 5 imply that for n sufficiently large there are eigenvalues of X_n different from 1 but arbitrarily close to 1. This ends the proof of Proposition 3. ◇

As a corollary we obtain the following generalization of Alon-Boppana's theorem:

Theorem 4 *Let* $\{X_n\}_{n=1}^{\infty}$ *be a sequence of finite connected graphs. If the degree of each graph (which does not have to be regular) is bounded by* k *and the number of vertices of* X_n *tends to infinity, then*

$$\liminf_{n\to\infty} \lambda_1(X_n) \geq \frac{2\sqrt{k-1}}{k}.$$

Proof Suppose this is not true, i.e.

$$\liminf_{n\to\infty} \lambda_1(X_n) < \frac{2\sqrt{k-1}}{k}.$$

Let v_n be any vertex in the graph X_n. By Lemma 2, there exists a subsequence of marked graphs $\{(X_{n_i}, v_{n_i})\}_{i=1}^{\infty}$ convergent to the marked graph (X, v). The degree of the graph X is bounded by k. So one knows (see [6] for the case when X is a k-regular graph and see [24] for the general case) that

$$\rho(X) \geq \frac{2\sqrt{k-1}}{k}.$$

Thus by Proposition 3 we have

$$\liminf_{i \to \infty} \lambda_1(X_{n_i}) \geq \frac{2\sqrt{k-1}}{k}$$

which gives a contradiction. ◇

If $\alpha \in Sp(M_X)$ is not an isolated eigenvalue in $Sp(M_X)$ then Proposition 2 has as a corollary the following proposition:

Proposition 4 *Let $\{(X_n, v_n)\}_{n=1}^{\infty}$ be a sequence of marked graphs which converges to the marked graph (X, v). If $\alpha \in Sp(M_X)$ is not an isolated point in $Sp(M_X)$ then for every $\varepsilon > 0$:*

$$\#\{\lambda \in Sp(M_{X_n}); \lambda \in (\alpha - \varepsilon, \alpha + \varepsilon)\} \to_{n \to \infty} \infty.$$

Below is one more example (besides the one given in Lemma 5) of infinite graphs for which the spectral radius is not an isolated point in the spectrum:

Proposition 5 *If there exists a vertex in X whose orbit under the group $Aut(X)$ of automorphisms of X is infinite then $\rho(X)$ is not an isolated point in the spectrum $Sp(M_X)$.*

Proof If $\rho(X)$ were an isolated eigenvalue then there would be an eigenfunction f in $l^2(X, deg)$ corresponding to $\rho(X)$. As the norm of M is attained on f and because for any $g \in l^2(X, deg)$ one has $\|Mg\| \leq \|M|g|\|$, we can suppose that f is positive. The function f must be strictly positive, because if it were zero somewhere, it would be zero at its neighbours, and as X is connected this gives a contradiction. Now, using the group $Aut(X)$ we can translate the function f and obtain a new eigenfunction f'. As there is a vertex whose orbit under $Aut(X)$ is infinite, we can obtain the eigenfunction f' which is different from f. Now, let us consider the function F on X:

$$F(x) = \max\{f(x), f'(x)\}.$$

For F one has

$$MF(x) \geq \rho(X)F(x)$$

for each $x \in X$ and F belongs to $l^2(X, deg)$. As f and f' have the same l^2 norms, there exist v and w in X such that $f(v) > f'(v)$ and $f(w) < f'(w)$. There is a vertex $z \in X$ such that the strict inequality

$$MF(z) > \rho(X)F(z), \tag{1}$$

holds. Indeed, if $MF(v) = \rho(X)F(v)$ then $f(u) \geq f'(u)$ for all vertices u at distance 1 from v. If $MF(u) = \rho(X)F(u)$ for every such vertex then $f(t) \geq f'(t)$ for all vertices t at distance 2 from v, etc. As $f(w) < f'(w)$ and X is connected, there must be a vertex z such that the inequality (1) holds.

But this means that $||M|| > \rho(X)$ which gives a contradiction. ◇

Now we will consider graphs with a finite number of orbits, i.e. graphs X such that $\#(X/Aut(X))$ is finite.

Theorem 5 *Let $\{(X_n, v_n)\}_{n=1}^{\infty}$ be a sequence of finite marked graphs with a degree uniformly bounded by k and such that $\#(X_n/Aut(X_n)) \leq K$ for each graph X_n. If (X, v) is the limit graph for the sequence $\{(X_n, v_n)\}_{n=1}^{\infty}$ then for any open interval $B \subset [-1, 1]$ and any vertex $v \in X$ we have:*

$$\liminf_{n\to\infty} \frac{\#\{\lambda \in Sp(M_{X_n}); \lambda \in B\}}{\#X_n} \geq \frac{(\mu_{vv}^X(B))^2}{K \cdot k^2},$$

where the eigenvalues are counted with multiplicities.

Proof We will prove Theorem 5 in the case where $K = 1$. The proof in the general case is similar. Thus we will consider homogenous graphs, i.e. graphs X such that $\#(X/Aut(X)) = 1$. For a homogenous graph, the spectral measure μ_{xx}^X does not depend on $x \in X$ and we will write for simplicity μ^X. Also, if we consider a sequence of marked homogenous graphs $\{(X_n, v_n)\}_{n=1}^{\infty}$ convergent to the marked graph (X, v), the limit space does not depend on the sequence v_n and the limit graph is homogenous. We can suppose that the degree of each graph X_n is k.

Let $d(n) = \#\{\lambda \in Sp(M_{X_n}); \lambda \in B\}$.

For $\lambda_i \in Sp(M_{X_n})$, let E_{λ_i} be the eigenspace in $l^2(X_n, deg)$ corresponding to λ_i. The $l^2(X_n, deg)$ admits the following decomposition

$$l^2(X_n, deg) = \bigoplus_{i=1}^{\#X_n} E_{\lambda_i},$$

where we suppose E_{λ_i} to be one-dimensional and thus we take as many one-dimensional eigenspaces as multiplicity indicates. So

$$E_{\lambda_i} = \{\lambda f_i; \lambda \in I\!R, ||f_i||_{l^2(X_n,deg)} = 1\}.$$

Let us choose a vertex $x \in X_n$. Then we have

$$\delta_x = \sum_{i=1}^{\#X_n} a_i f_i,$$

where a_i are real numbers such that $\sum_{i=1}^{\#X_n} a_i^2 = ||\delta_x||_2^2 = k$. And by definition of the projection $E(B)$

$$E(B)\delta_x = \sum_{j=1}^{d(n)} a_{i_j} f_{i_j}.$$

Because of the weak convergence of the measures μ^{X_n} to the measure μ^X (Lemma 4), for any $\varepsilon > 0$ there exists $N \in I\!N$, such that for $n > N$ we have

$$\mu^X(B) - \varepsilon \le \mu^{X_n}(B).$$

By definition we have

$$\mu^{X_n}(B) = \mu^{X_n}_{xx}(B),$$

where x is any vertex in X_n. So

$$\begin{aligned}
\mu^X(B) - \varepsilon &\le \mu^{X_n}(B) = \mu^{X_n}_{xx}(B) = \langle E(B)\delta_x, \delta_x\rangle \\
&= \left\langle \sum_{j=1}^{d(n)} a_{i_j} f_{i_j}, \delta_x \right\rangle = \sum_{j=1}^{d(n)} a_{i_j} f_{i_j}(x)k \\
&\le \sqrt{\sum_{j=1}^{d(n)} a_{i_j}^2}\sqrt{\sum_{j=1}^{d(n)} f_{i_j}^2(x)k} \le ||\delta_x||_2\sqrt{\sum_{j=1}^{d(n)} f_{i_j}^2(x)k} \\
&= \sqrt{\sum_{j=1}^{d(n)} f_{i_j}^2(x)k}\sqrt{k}.
\end{aligned}$$

Thus

$$(\mu^X(B) - \varepsilon)^2 \le \sum_{i=1}^{d(n)} f_{i_j}^2(x)k^3.$$

This gives us

$$\begin{aligned}
\#X_n(\mu^X(B)-\varepsilon)^2 &= \sum_{x\in X_n}(\mu^X(B)-\varepsilon)^2 \\
&\leq \sum_{x\in X_n}\sum_{j=1}^{d(n)} f_{i_j}^2(x)k^3 \\
&= \sum_{j=1}^{d(n)}\left(\sum_{x\in X_n} f_{i_j}^2(x)k\right)k^2 \\
&= d(n)k^2,
\end{aligned}$$

which ends the proof. $\diamond$

For future reference, we notice that the above proof shows:

$$\sum_{x\in X_n}\left(\mu_{xx}^{X_n}(B)\right)^2 \leq d(n)k^2. \tag{2}$$

We are going to prove two statements analogous to Proposition 3 and Theorem 5 but in a slightly different situation.

Proposition 6 *Let $\{X_n\}_{n=1}^\infty$ be a sequence of finite graphs such that*

1. *each graph X_n is covered by a graph X and*
2. *the number of vertices of X_n tends to infinity, i.e. $\#X_n \to_{n\to\infty} \infty$.*

Then

$$\liminf_{n\to\infty}\lambda_1(X_n) \geq \rho(X).$$

Proof Suppose that this is not true, i.e. there exists $\varepsilon > 0$ and a subsequence $\{X_{n_i}\}_{i=1}^\infty$ such that

$$\lambda_1(X_{n_i}) \leq \rho(X) - \varepsilon.$$

Because X covers a finite graph, the degree of graphs $\{X_n\}_{n=1}^\infty$ is uniformly bounded. Let us choose a vertex v in X, and let v_n be its image in X_n. By Lemma 1 there exists a subsequence $\{X_{n_{i_j}}, v_{n_{i_j}}\}_{j=1}^\infty$ of marked graphs which is convergent to a marked graph (X', v'). Now we prove:

Lemma 6 *The limit graph X' is covered by the graph X.*

Proof We will construct a covering of X' by X such that v is mapped onto v'. For $m > 0$, let $\delta(m)$ be a constant such that the ball $B_{X_{n_{i_{\delta(m)}}}}(v_{n_{i_{\delta(m)}}}, m)$ is isometric to the ball $B_{X'}(v', m)$. Let $p_{n_{i_{\delta(m)}}} : X \to X_{n_{i_{\delta(m)}}}$ be a cover mapping between the graph X and the graph $X_{n_{i_{\delta(m)}}}$ such that $p_{n_{i_{\delta(m)}}}(v) = v_{n_{i_{\delta(m)}}}$. By definition of the covering, it follows that the ball $B_X(v, m)$ covers the ball $B_{X_{n_{i_{\delta(m)}}}}(v_{n_{i_{\delta(m)}}}, m)$. As the latter is isometric to $B_{X'}(v', m)$ let

$$f_m : B_X(v, m) \to B_{X'}(v', m)$$

be a covering of $B_{X'}(v', m)$ by $B_X(v, m)$ such that $f_m(v) = v'$.

By the diagonal argument, there exists a subsequence $\{f_{m_j}\}_{j=1}^{\infty}$ of coverings such that f_{m_j}, restricted to $B_X(v, m_{j'})$ for $m_{j'} \leq m_j$, coincides with $f_{m_{j'}}$, i.e.

$$f_{m_j}|_{B_{X(v, m_{j'})}} = f_{m_{j'}}.$$

Thus there exists a covering

$$f : X \to X',$$

such that $f|_{B_X(v, m_j)} = f_{m_j}$, which ends the proof of Lemma 6. ◇

Thus by Lemma 2

$$\rho(X') \geq \rho(X).$$

Hence Proposition 3 gives us the desired contradiction. This ends the proof of Proposition 6. ◇

Theorem 6 *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of finite connected graphs with uniformly bounded degree, such that the number of vertices of X_n tends to infinity.*

1. *Then there exists a subsequence $\{X_{n_i}\}_{i=1}^{\infty}$ and a graph X' such that for any open interval $B \subset [-1, 1]$ and any vertex $x' \in X'$ there exists a constant $c_{x'} > 0$ such that*

$$\liminf_{n \to \infty} \frac{\#\{\lambda \in Sp(M_{X_{n_i}}); \lambda \in B\}}{\# X_n} \geq c_{x'} \cdot (\mu_{x'x'}^{X'}(B))^2.$$

2. *If the graphs X_n are covered by a graph X such that there exists a group Γ acting cocompactly on X then X' can be chosen in such a way that X covers X'.*

Proof As the graphs X_n have uniformly bounded degree, there exist positive integers $N^1, \ldots, N^R, \ldots$ with the following property:

Property (P) *Let us consider any graph X_n. Then up to the isometry there are at most N^R different balls of radius R in X_n.*

Let $\{B_R^{X_n}\}_{R=1}^{\infty}$ be any sequence of sets of balls in X_n of radius R with the following properties:

1. For fixed n and R the balls in $B_R^{X_n}$ are isometric to each other;
2. If $B(v, R) \in B_R^{X_n}$ then $B(v, R-1) \in B_{R-1}^{X_n}$.

Now we prove:

Lemma 7 *It is possible to find a sequence $\{B_R^{X_n}\}_{R=1}^{\infty}$ in such a way that*

$$\# B_R^{X_n} \geq \# X_n \cdot c_R,$$

where $c_R = (N^1 \cdot \ldots \cdot N^R)^{-1}$.

Proof We construct the sequence $\{B_R^{X_n}\}_{R=1}^{\infty}$ by induction:

1. There are $\# X_n$ balls of radius 1 in X_n. According to Property (P) there are at least $\# X_n \cdot (N^1)^{-1}$ among these balls which are isometric to each other. So let $B_1^{X_n}$ be any set of cardinality at least $\# X_n \cdot (N^1)^{-1}$ consisting of balls of radius 1 which are isometric to each other.

2. Suppose that we constructed the sets $B_1^{X_n}, \ldots, B_{R-1}^{X_n}$ with the desired properties. Let us consider the balls of radius R with the same centers as the balls in $B_{R-1}^{X_n}$. According to Property (P) there are at least $\# B_{R-1}^{X_n} \cdot (N^R)^{-1}$ among these balls which are isometric to each other and let $B_R^{X_n}$ be any set of them. Thus by induction

$$\# B_R^{X_n} \geq \# B_{R-1}^{X_n} \cdot (N^R)^{-1} \geq \# X_n \cdot c_{R-1} \cdot (N^R)^{-1} = \# X_n \cdot c_R,$$

which ends the proof of Lemma 7. $\diamond$

Because the graphs X_n are finite, for each graph X_n there exists an integer $R(n)$ such that for $R > R(n)$ one has

$$B_R^{X_n} = B_{R(n)}^{X_n}.$$

So for each X_n there exists a vertex $v_n \in X_n$ such that for every R there is at least one ball in $B_R^{X_n}$ centered on v_n. By Lemma 1 there exists a subsequence of marked graphs $\{(X_{n_j}, v_{n_j})\}_{j=1}^{\infty}$ which is convergent to some marked graph (X', v). We will show that the sequence of graphs $\{X_{n_j}\}_{j=1}^{\infty}$ and the graph X' have the properties stated in Theorem 6.

Let $C\left(B_R^{X_n}\right)$ be the set of centers of the balls in $B_R^{X_n}$, i.e.

$$C\left(B_R^{X_n}\right) = \left\{v \in X_n; B(v, R) \in B_R^{X_n}\right\}.$$

We need the following:

Lemma 8 *For any interval $B \subset [-1, 1]$ and $\varepsilon > 0$ there exist integers $N > 0$ and $R > 0$ such that for any $n_j > N$ and for any vertex $x \in C(B_R^{X_{n_j}})$, we have*

$$\mu_{xx}^{X_{n_j}}(B) \geq \mu_{vv}^{X'}(B) - \varepsilon.$$

Proof Suppose this is not true, i.e. there exist an interval $B \subset [-1, 1]$, $\varepsilon > 0$, a subsequence of graphs $\{X_{n_{j_i}}\}_{i=1}^{\infty}$, a sequence of integers $\{R_i\}_{i=1}^{\infty}$ ($\lim_{i\to\infty} R_i = \infty$) and a sequence of vertices $x_i \in X_{n_{j_i}}$ ($x_i \in C\left(B_{R_i}^{X_{n_{j_i}}}\right)$) such that

$$\mu_{x_i x_i}^{X_{n_{j_i}}}(B) < \mu_{vv}^{X'}(B) - \varepsilon. \tag{3}$$

By construction, the sequence of marked graphs $\{(X_{n_{j_i}}, x_i)\}_{i=1}^{\infty}$ is convergent to the marked graph (X', v). So by Lemma 4 there exists an integer i such that

$$\mu_{x_i x_i}^{X_{n_{j_i}}}(B) \geq \mu_{vv}^{X'}(B) - \varepsilon,$$

which contradicts (3). ◇

We are now in a position to prove the first statement of Theorem 6. We will prove this for $x' = v$ and later remark that the same proof works for other vertices. First of all if $\mu_{vv}^{X'}(B) = 0$ there is nothing to prove.

Let us consider an open interval $B \subset [-1, 1]$ such that $\mu_{vv}^{X'}(B) > 0$. Let $\varepsilon = \frac{1}{2}\mu_{vv}^{X'}(B)$. By Lemma 8 there exist positive integers N and R such that for $n_j > N$ and for any vertex $x \in C\left(B_R^{X_{n_j}}\right)$ we have:

$$\mu_{xx}^{X_{n_j}}(B) \geq \frac{1}{2}\mu_{vv}^{X'}(B).$$

Let $k = deg(x)$ (there exists a positive integer N' such that for $n_j > N'$ the degree of vertices in $C\left(B_R^{X_{n_j}}\right)$ is the same for all X_{n_j}). Then for $n_j > max\{N, N'\}$ by Lemma 7 and (2) we have:

$$
\begin{aligned}
\#X_{n_j}\left(\frac{1}{2}\mu^{X'}_{vv}(B)\right)^2 &\leq (c_R)^{-1}\#B_R^{X_{n_j}}\cdot\left(\frac{1}{2}\mu^{X'}_{vv}(B)\right)^2\\
&= (c_R)^{-1}\sum_{x\in C\left(B_R^{X_{n_j}}\right)}\left(\frac{1}{2}\mu^{X'}_{vv}(B)\right)^2\\
&\leq (c_R)^{-1}\sum_{x\in C\left(B_R^{X_{n_j}}\right)}\left(\mu^{X_{n_j}}_{xx}(B)\right)^2\\
&\leq (c_R)^{-1}d(n_j)k^2.
\end{aligned}
$$

As c_R and k are constants which do not depend on X_{n_j}, this ends the proof of Theorem 6 for $x' = v$. In the case when $x' \neq v$ the proof is almost the same, because for R sufficently large, x' is in the ball $B(v, R)$ and therefore in our considerations we can replace v by x'.

Now we prove the second statement in Theorem 6. Because $\Gamma \setminus X$ is finite, there exists a vertex $w \in X$ and a subsequence of marked graphs $\{(X_{n_{j_i}}, v_{n_{j_i}})\}_{i=1}^{\infty}$ such that a covering of $X_{n_{j_i}}$ by X maps w onto $v_{n_{j_i}}$. As the sequence $\{(X_{n_{j_i}}, v_{n_{j_i}})\}_{i=1}^{\infty}$ is convergent to the marked graph (X', v), by Lemma 6 we have that X' is covered by X. $\diamond$

Theorem 6 has the following corollary:

Corollary 1 *Let X be an infinite connected graph and X_n the family of finite connected graphs covered by X. (Where here we mean that for each X_n there exists a group Γ_n acting on X such that $\Gamma_n \setminus X = X_n$.) Then for every $\varepsilon > 0$ there exists $c = c(X, \varepsilon)$, $0 < c < 1$, such that at least $c\#X_n$ eigenvalues λ of X_n satisfy $\lambda \geq \rho(X) - \varepsilon$. In particular for any $r \in I\!N$ we have*

$$\liminf_{n\to\infty} \lambda_r(X_n) \geq \rho(X),$$

where $\lambda_1(X_n) \geq \lambda_2(X_n) \geq \ldots$ are the eigenvalues of M_{X_n} listed in ordered way.

The first statement of Corollary 1 was proved by Greenberg [5] (see also [14]) and the second statement was proved by Burger [2]. In the case when X is a regular tree Corollary 1 was observed by Serre [21]. The exposition of all statements contained in Corollary 1 can be also found in [13] (Theorem 13 in Chapter 9).

Proof of Corollary 1 Suppose this is not true, i.e. there exists $\varepsilon > 0$ and a subsequence of graphs $\{X_{n_i}\}_{i=1}^{\infty}$ such that

$$\lim_{n\to\infty} \frac{\#\{\lambda \in Sp(M_{X_{n_i}}); \lambda \geq \rho(X) - \varepsilon\}}{\#X_{n_i}} = 0. \tag{4}$$

Clearly $\#X_n$ tends to infinity in this case. By Theorem 6 there exist a subsequence $\{X_{n_{i_j}}\}_{j=1}^{\infty}$ and a graph X' which is covered by the graph X, such that for any vertex $x' \in X'$ there is a constant $c_{x'} > 0$ such that

$$\liminf_{j\to\infty} \frac{\#\{\lambda \in Sp(M_{X_{n_{i_j}}}); \lambda \geq \rho(X') - \varepsilon\}}{\#X_{n_{i_j}}} \geq c_{x'} \mu_{x'x'}^{X'} \left((\rho(X') - \varepsilon, 1]\right).$$

As $\rho(X')$ belongs to $Sp(M_{X'})$, by Lemma 3 there exists $x' \in X'$ such that $\mu_{x'x'}^{X'}((\rho(X') - \varepsilon, 1]) = c > 0$. By Lemma 2, $\rho(X') \geq \rho(X)$. Thus

$$\liminf_{j\to\infty} \frac{\#\{\lambda \in Sp(M_{X_{n_{i_j}}}); \lambda \geq \rho(X) - \varepsilon\}}{\#X_{n_{i_j}}} > 0,$$

which contradicts (4). $\diamond$

5 Conclusions

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of finite graphs covered by a graph X or converging to X. According to our results there is an extremal case when $\lambda_1(X_n) \leq \rho(X)$ for all n and this leads to the following definitions, the first of which generalizes the ones given in [5] and [14]:

Definition 1 *Let $\{X_n\}$ be an infinite family of finite graphs, covered by an infinite graph X. The family $\{X_n\}$ is called X-**Ramanujan** if every eigenvalue not equal to 1 of the simple random walk on X_n is less than the spectral radius $\rho(X)$ of the simple random walk on X, i.e.*

$$\lambda_1(X_n) \leq \rho(X),$$

for each graph X_n.

Definition 2 *Let $\{(X_n, v_n)\}_{n=1}^{\infty}$ be a sequence of finite marked graphs, convergent to the infinite marked graph (X, v). The sequence $\{(X_n, v_n)\}_{n=1}^{\infty}$ is* *called* (X, v)**-Ramanujan** *if every eigenvalue not equal to 1 of the simple random walk on* X_n *is less than the spectral radius* $\rho(X)$ *of the simple random walk on* X*, i.e.*

$$\lambda_1(X_n) \leq \rho(X),$$

for each graph X_n*.*

Ramanujan graphs were defined in [16], [17] and implicitly in [11] as finite k-regular graphs with $\lambda_1 \leq \frac{2\sqrt{k-1}}{k}$. For a given k of the form $k = q + 1$ where q is a power of a prime number, the constructions of the infinite families of Ramanujan graphs covered by a regular tree (namely a Cayley graph of a free group of corresponding rank) were constructed by Lubotzky, Phillips and Sarnak in [16] and by Margulis in [17].

It would be interesting to find other examples of a graph X (not a tree) with the spectral radius $\frac{2\sqrt{k-1}}{k}$ and a sequence X_n which is X-Ramanujan or at least (X, v)-Ramanujan for some $v \in X$.

In [15] an example of an infinite non-regular tree X was constructed such that X admits infinitely many finite quotients and none of its quotients is X-Ramanujan.

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