The Ihara zeta function of infinite graphs, the KNS spectral measure and integrable maps

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Abstract. We define the Ihara zeta function for Cayley graphs of infinite finitely generated groups. We extend the definition of the Ihara zeta function to infinite graphs which are limits of sequences $\{X_n\}_{n=1}^{\infty}$ of finite k-regular graphs such that X_{n+1} covers X_n. We associate to such a graph a measure μ with support in $[-1,1]$ called the Kesten-von Neumann-Serre spectral measure. We present a few examples of computation of zeta function and a measure μ for Schreier graphs of some fractal groups generated by finite automata. These computations are closely related to the integrability of some 2-dimensional mappings which are also in focus of our considerations.

1. Introduction

Recently new results about spectral properties of infinite regular graphs were obtained. Namely in [1] an example of a graph with a spectrum which is a Cantor set was constructed, while in [14] an example of an infinite Cayley graph with a pure point spectrum of the Laplace operator was treated. The last result was used to answer a question of Atiyah about L^2 Betti numbers [18]. Some other results about spectra of Schreier graphs and amenability were obtained in [15, 16].

The idea of this paper is to use the techniques and results from the cited papers to introduce the Ihara zeta function using the residual finiteness of groups and graphs related to automata.

We also use the von Neumann trace to define the Ihara zeta function for Cayley graphs of infinite finitely generated groups.

The Ihara zeta function $\zeta_X(t)$ of a finite graph X is one of the analogues of the Riemann's zeta function

$$\zeta(s) = \sum_{n\geq 1} n^{-s} = \prod_{p \text{ prime}} (1-p^{-s})^{-1} \quad \text{for } \mathrm{Re} s > 1.$$

It was introduced in [23] (and studied in [3, 25, 26, 34, 35, 36, 39, 40, 41] and many other papers) by the relation

$$\zeta_X(t) = \prod_{[C]} (1 - t^{|C|})^{-1}$$

where the product is over all equivalence classes $[C]$ of primitive closed paths C in X and $|C|$ denotes the length of C.

The Ihara zeta function of a finite graph plays an important role in number theory, graph theory and mathematical physics [26, 40, 41, 44]. It is useful in the study of spectral properties of graphs and leads to the definition of Ramanujan graphs as graphs for which holds the analogue of the Riemann hypothesis about the zeros of zeta functions [23, 27, 37].

We show that the Ihara zeta function can be easily defined for some infinite graphs, first of all for Cayley graphs and for Schreier graphs of finitely generated groups. In Section 6 using von Neumann trace we define the Ihara zeta function for Cayley graphs of infinite finitely generated groups. Then, using a result of Serre [38], we extend the definition of the Ihara zeta function to infinite graphs which are limits of sequences $\{X_n\}_{n=1}^{\infty}$ of finite graphs such that X_{n+1} covers X_n. This definition works for instance for graphs which are Schreier graphs of a pair (G, H) where H is a subgroup of a group G which is an intersection of subgroups of finite index in G.

Also, we associate to such a graph a measure μ with support in $[-1, 1]$ which plays a role analogue to the Kesten spectral measure defined in [24] for Cayley graphs, and is called by us the Kesten-Neumann-Serre (KNS) spectral measure. This measure is used to get an integral presentation for the zeta function.

Finally we present a few examples of computation of the zeta function for Schreier graphs of some fractal groups considered in [1], [14], [15] and [16].

The possibility of such a computation is based on a trick of introducing extra parameters in the spectral problem and reducing it to the study of multidimensional rational mappings of the real Euclidean space $\mathbb{R}^d$ for $d \geq 2$. In case this map is integrable, i.e. there are functions $\psi : \mathbb{R}^d \to \mathbb{R}$ and $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\psi F = \alpha\psi$, the spectral problem usually can be completely treated and this is the case in most of our examples. The examples of integrable mappings of $\mathbb{R}^2$ obtained in [1], [14] and [16] come from finite automata and look mysterious. We hope that new examples of integrable mappings associated to finite automata and groups generated by them will be discovered and this will give a new impulse both to the spectral theory of graphs and groups and to the theory of dynamical systems.

2. The Markov operator and the spectral measure

The graphs $X = (V_X, E_X)$ we consider are connected, locally finite and the edges are not oriented. The graphs can have loops and multiple edges (such graphs are often called multigraphs). For a vertex v we denote by $deg(v)$ the degree of v, i.e. the number of edges containing v (a loop counts twice). For more on such a definition of graphs see [4]. Let $l^2(V_X, deg)$ denote the space of real functions on the set of vertices which are square summable with the weight deg.

On such a graph X one can consider a random walk operator M acting on functions $f \in l^2(V_X, deg)$ as follows:

$$Mf(v) = \frac{1}{deg(v)} \sum_{(w,v)} f(w),$$

where the sum is taken over all edges (w, v) containing the vertex v. If there are k loops at the vertex v, $f(v)$ will appear $2k$ times in the sum and if between the distinct vertices v and w there are n edges, $f(w)$ will appear n times in the sum.

This is a self-adjoint operator on $l^2(X, deg)$ with the spectrum in $[-1, 1]$.

The motivation for such a definition becomes clear if we consider the Hecke type operator

$$\mathcal{H} = \frac{1}{2|S|} \sum_{s \in S} \left(\pi(s) + \pi(s^{-1})\right)$$

acting on $l^2(G/H)$ where $H < G$ is a subgroup, S is a set of generators and π is a quasi-regular representation $\lambda_{G/H}$. The realization of $\mathcal{H}$ as a Markov operator for a simple random walk on the Schreier graph $\mathcal{S}(G, H, S)$ (about Schreier graphs see Section 7) leads to the above definition of M.

As the operator M is a bounded ($||M|| \leq 1$) and self-adjoint operator, it has the spectral decomposition

$$M = \int_{-1}^{1} \lambda dE(\lambda),$$

where E is the spectral measure. This spectral measure is defined on Borel subsets of the interval $[-1, 1]$ and takes values in the set of projections on the Hilbert space $l^2(X, deg)$. The matrix μ^X of measures μ^X_{xy}, $x, y \in V_X$ can be associated with E as follows:

$$\mu^X_{xy}(B) = \langle E(B)\delta_x, \delta_y \rangle,$$

where B is a Borel subset of $[-1, 1]$ and δ_x is the function which equals 1 at x and 0 elsewhere.

In general, $\lambda \in Sp(M)$ if and only if for every $\varepsilon > 0$ there exists μ^X_{xy} such that $|\mu^X_{xy}((\lambda - \varepsilon, \lambda + \varepsilon))| > 0$. But in fact (see [24]) $\lambda \in Sp(M)$ if and only if for every $\varepsilon > 0$ there exists $x \in X$ such that $\mu^X_{xx}((\lambda - \varepsilon, \lambda + \varepsilon)) > 0$.

Indeed let us show that if, for $B = (\lambda - \varepsilon, \lambda + \varepsilon)$, we have $|\mu^X_{xy}(B)| > 0$ then $\mu^X_{xx}(B) > 0$. As $E(B)$ is a projection, one has

$$\begin{aligned} 0 &< (\mu^X_{xy}(B))^2 = \langle E(B)\delta_x, \delta_y \rangle^2 \leq \langle E(B)\delta_x, E(B)\delta_x \rangle \langle \delta_y, \delta_y \rangle \\ &= \langle E(B)\delta_x, \delta_x \rangle deg(y) = \mu^X_{xx}(B) deg(y), \end{aligned}$$

as needed.

We call the measures $\mu_x^X = \mu_{xx}^X$ the Kesten spectral measures.

If the graph X is vertex transitive the measures μ_x^X are independent of x. In particular this is the case when X is a Cayley graph of a group. In the latter case the measure μ can be alternatively defined via von Neumann's approach as $\mu(B) = trE(B)$ where tr is the trace in the von Neumann algebra of a group generated by the left regular representation [31].

In Section 4 we will define an analogue of the Kesten-Neumann spectral measure for graphs which are not vertex transitive which would be an average of the Kesten spectral measures over the set of vertices in a situation when the graph is the limit of a covering sequence of finite graphs in the sense of the next section (more details in Section 4).

3. Convergence of graphs and coverings

In this section we recall few well known facts about convergence of graphs. For details see [13].

The distance between two vertices $v_a \neq v_b \in V_X$ is the minimal number of edges needed to connect them, i.e.

$$\begin{aligned}\mathrm{dist}(v_a, v_b) &= \min\{n; \exists v_0, \ldots, v_n \in V_X, v_0 = v_a, v_n = v_b,\\ &\quad (v_i, v_{i+1}) \in E_X \text{ for } 0 \leq i \leq n-1\}.\end{aligned}$$

Let us consider a family $\{(X_n, v_n)\}$ of marked graphs, i.e. graphs with chosen vertices $v_n \in V_{X_n}$.

On the space of marked graphs there is a metric $Dist$ defined as follows

$$Dist((X_1, v_1), (X_2, v_2)) = \inf\left\{\frac{1}{n+1}; B_{X_1}(v_1, n) \text{ is isometric to } B_{X_2}(v_2, n)\right\}$$

where $B_X(v, n)$ is the ball of radius n in X centered on v.

For a sequence of marked graphs (X_n, v_n) we say that (X, v) is the limit graph if

$$\lim_{n\to\infty} Dist((X, v), (X_n, v_n)) = 0.$$

The limit graph is unique up to the isometry.

For any sequence $\{X_n, v_n\}$ of finite marked graphs of uniformly bounded degree there exists an infinite marked graph (X, v) which is the limit of a subsequence of $\{(X_n, v_n)\}$ (see [13] Theorem 3).

The graphs we consider are naturally equipped with a structure of one dimensional CW complexes and we can use a general theory of coverings (see for instance [32]).

Definition 3.1. The sequence of finite graphs $\{X_n\}$ is a covering sequence if X_{n+1} covers X_n.

Any covering sequence has a limit $\lim X_n$ in the sense of the above definition ([13]).

In Section 7 we will consider special cases of covering sequences of algebraic origin.

4. The definition of the Ihara zeta function for infinite graphs

The Ihara zeta function $\zeta_X(t)$ for a finite regular graph X satisfies the relation

$$\zeta_X(t) \quad = \quad \exp\left(\sum_{r=1}^{\infty} c_r t^r/r\right), \tag{4.1}$$

where c_r is the number of closed, oriented loops of length r in the graph X [23].

Indeed

$$\begin{aligned}\ln\zeta_X(t) \quad &= \quad -\sum_{[C]}\ln(1-t^{|C|}) = \sum_{[C]}\sum_{j\geq 1}\frac{1}{j}t^{|C|j} = \sum_{j\geq 1}\sum_{d\geq 1}\sum_{[C],|C|=d}\frac{1}{j}t^{dj} \\ &= \quad \sum_{j\geq 1}\sum_{d\geq 1}\sum_{C,|C|=d}\frac{1}{dj}t^{dj} = \sum_{r\geq 1}\frac{c_r}{r}t^r\end{aligned}$$

because there are d representatives in $[C]$ if the length of C is d and $\sum_{d|r}\sum_{C,|C|=d} = c_r$ as C is a primitive loop. And thus

$$\ln\zeta_X(t) = \sum_{r=1}^{\infty}\frac{c_r}{r}t^r.$$

In [23] it was also shown that

Theorem 4.1 ([23]). *The Ihara zeta function $\zeta_X(t)$ for a finite regular graph X of degree k satisfies:*

$$\zeta_X(t) = (1-t^2)^{-\frac{1}{2}(k-2)|X|}\det(1-tkM+(k-1)t^2)^{-1}, \tag{4.2}$$

where M is the Markov operator on X.

We are going to use these two basic facts and the following result of Serre to define the zeta function for infinite graphs, which are limits of sequences of finite k-regular graphs.

Following Serre [38], we say that the eigenvalues $\lambda_i(X_n)$ of the Markov operators M_n on the sequence of finite graphs X_n are equidistributed with respect to a measure μ which has support in $[-1,1]$ if the sequence of measures μ_n

$$\mu_n = \sum \delta_{\lambda_i(X_n)}/|X_n|$$

which we call counting measures and where δ_x is the Dirac measure at x, converges weakly to the measure μ.

Theorem 4.2 ([38]). *Let X_n be a family of finite graphs of degree k. The following conditions are equivalent:*

1) The eigenvalues of $M(X_n)$ are equidistributed with respect to some measure μ.

2) The formal power series

$$\zeta_{X_n}(t)^{1/|X_n|}$$

has a limit in $\mathbb{R}[[t]]$.

3) For every $r \geq 1$, the sequence

$$c_r(X_n)/|X_n| \tag{4.3}$$

where c_r is the number of closed loops of length r has a limit denoted by $\widetilde{c_r}$.

As a corollary of Theorem 4.2 we obtain

Corollary 4.3. *Let $\{X_n\ , v_n\}$ be a covering sequence of finite k-regular graphs and M_n the sequence of the corresponding Markov operators. Then the eigenvalues of M_n are equidistributed with respect to some measure μ.*

Proof. The ratio $c_r(X_n)/|X_n|$ decreases with n. Thus the limit $\widetilde{c_r}$ of $c_r(X_n)/|X_n|$ exists and therefore all statements of Theorem 4.2 hold, in particular the eigenvalues of M_n are equidistributed with respect to some measure μ. □

Definition 4.4. The above limit measure μ will be called the Kesten-Neumann-Serre (KNS) spectral measure of X and $\{X_n\}$, where X is the limit of the sequence $\{X_n, v_n\}$.

The measure μ is an average of Kesten spectral measures $\mu_{v,v}$ over the set of vertices v of the graph, using the approximating sequence $\{X_n\}$.

Let $\{X_n\}$ be a sequence of graphs converging to X. Using Theorem 4.1 we have

$$\begin{aligned} \frac{1}{|X_n|} \ln \zeta_{X_n}(t) &= -\frac{k-2}{2} \ln(1-t^2) - \frac{1}{|X_n|} \ln \det(1 - tkM_n + (k-1)t^2) \qquad (4.4) \\ &= \sum_{r=1}^{\infty} \frac{c_r(X_n)}{|X_n|} t^r/r. \qquad (4.5) \end{aligned}$$

Passing to the limit we come to the following

Definition 4.5. Let $X = \lim_{n\to\infty} X_n$ where X_n is a sequence of k regular marked graphs such that the limit of (4.3) exists when $n \to \infty$. The zeta function $\zeta_X(t)$ of the graph X, with respect to the sequence $\{X_n\}$, is defined by

$$\ln \zeta_X(t) = \lim_{n\to\infty} \frac{1}{|X_n|} \ln \zeta_{X_n}(t) = \sum_{r=1}^{\infty} \widetilde{c_r} t^r/r. \tag{4.6}$$

The series on the right-hand side of (4.6) has a nontrivial interval of convergence around the point 0. Namely an easy estimate gives $\widetilde{c_r} \leq k^r$ from which it follows that the radius of convergence is at least $\frac{1}{k}$. Thus function $\zeta_X(t)$ is well defined and analytic in a neighborhood of 0. In Section 5 we give a better estimate on the radius of convergence.

The definition of $\zeta_X(t)$ depends on the approximating sequence X_n but we will usually omit writing $\{X_n\}$.

Remark Form formula (4.6) it follows that for every tree X which admits a finite quotient (for instance a homogenous tree), the zeta function $\zeta_X(t)$ is equal to 1. Indeed for such X one can construct a sequence of finite graphs X_n such that $X = \lim_{n\to\infty} X_n$ and such that the girths of X_n tend to infinity. In this case $\widetilde{c_r} = 0$.

5. The integral presentation and the problem of moments

In this section we show that the moments of the KNS spectral measure are determined by the zeta function. We prove

Theorem 5.1. *Let $X = \lim X_n$ under assumption that the limit (4.3) exists and let μ be as in Definition 4.4. The radius of convergence of the series (4.6) is at least $\frac{1}{k-1}$ and for any t such that $|t| < \frac{1}{k-1}$ the following relation holds*

$$\ln \zeta_X(t) = -\frac{k-2}{2}\ln(1-t^2) - \int_{-1}^{1} \ln(1 - tk\lambda + (k-1)t^2)d\mu(\lambda). \tag{5.7}$$

The KNS spectral measure μ is uniquely determined by the zeta function ζ_X.

Proof. Passing to the limit ($n \to \infty$) in (4.4) and (4.5) and using the notation from Theorem 4.2 we get

$$\begin{aligned}
\ln \zeta_X(t) &= -\frac{k-2}{2}\ln(1-t^2) - \lim_{n\to\infty}\frac{1}{|X_n|}\ln\det(-ktM_n + 1 + (k-1)t^2)\\
&= -\frac{k-2}{2}\ln(1-t^2) - \lim_{n\to\infty}\int_{-1}^{1}\ln(1 - tk\lambda + (k-1)t^2)d\mu_n(\lambda)\\
&= -\frac{k-2}{2}\ln(1-t^2) - \int_{-1}^{1}\ln(1 - tk\lambda + (k-1)t^2)d\mu(\lambda)
\end{aligned}$$

as $1 - tk\lambda + (k-1)t^2 > 0$ on $[-1, 1]$, for $|t| < \frac{1}{k-1}$.

We will show that the function $\zeta_X(t)$ given by the formula (5.7) determines all moments of the measure μ.

We have

$$1 - tk\lambda + (k-1)t^2 = (1 - a(\lambda)t)(1 - b(\lambda)t),$$

where

$$\begin{aligned} a(\lambda) &= \frac{k\lambda - \sqrt{k^2\lambda^2 - 4k + 4}}{2}, \\ b(\lambda) &= \frac{k\lambda + \sqrt{k^2\lambda^2 - 4k + 4}}{2}. \end{aligned}$$

Thus for $|t| < \frac{1}{k-1}$ we get

$$\begin{aligned} \ln \zeta_X(t) + \frac{k-2}{2}\ln(1-t^2) &= -\int_{-1}^{1} \ln(1 - tk\lambda + (k-1)t^2) d\mu(\lambda) \\ &= -\int_{-1}^{1} \ln(1 - a(\lambda)t)(1 - b(\lambda)t) d\mu(\lambda) \\ &= -\int_{-1}^{1} \left(\ln(1 - a(\lambda)t) + \ln(1 - b(\lambda)t)\right) d\mu(\lambda) \\ &= -\int_{-1}^{1} \sum_{n=1}^{\infty} \frac{(a^n(\lambda) + b^n(\lambda))t^n}{n} d\mu(\lambda) \\ &= -\sum_{n=1}^{\infty} \frac{t^n}{n} \int_{-1}^{1} (a^n(\lambda) + b^n(\lambda)) d\mu(\lambda) \qquad (5.8) \end{aligned}$$

as the above integrals are non-singular for $|t| < \frac{1}{k-1}$.

From the expression of $a(\lambda)$, $b(\lambda)$ we get for $|\lambda| \le 1$

$$\max\{|a(\lambda)|, |b(\lambda)|\} \le \frac{k + \sqrt{k^2 - 4k + 4}}{2} = k - 1.$$

Thus From (5.8) it follows that the radius of convergence of $\ln \zeta_X(t)$ is at least $\frac{1}{k-1}$.

We remark that

$$a^n(\lambda) + b^n(\lambda)$$

is a polynomial in λ of degree n as can be easily checked.

For $\ln \zeta_X(t) + \frac{k}{2}\ln(1-t^2) = \sum_{n=1}^{\infty} d_n t^n$ we have

$$\int_{-1}^{1} (a^n(\lambda) + b^n(\lambda)) d\mu(\lambda) = -n d_n,$$

which shows that for any n, the n-th moment of μ is determined by $\ln \zeta_X(t)$. One can also express the n-th moment using Chebyshev polynomials.

□

6. The Ihara zeta function for groups

Let G be a finitely generated group with a finite set of generators S and let $\mathcal{S} = \mathcal{S}(G, S)$ be the corresponding Cayley graph. We are going to define the Ihara zeta function for $\mathcal{S}$ with no restriction on G.

Let us use the relation

$$\ln \zeta_X(t) = -\frac{k-2}{2} \ln(1-t^2) - \ln \det(1 - tkM + (k-1)t^2)$$

which holds for finite k-regular graphs X and which can be rewritten

$$\ln \zeta_X(t) = -\frac{k-2}{2} \ln(1-t^2) - tr \ln(1 - tkM + (k-1)t^2).$$

Recall that the von Neumann algebra $\mathcal{N}(G)$ generated by the left regular representation in $l^2(G)$ is of finite type i.e. there is a canonical trace tr defined by the relation

$$tra = \langle a\delta_1, \delta_1 \rangle$$

where $\delta_1 \in l^2(G)$ is the delta function at the unit.

The last relation can be used to define

$$\ln \zeta_G(t) = -(m-1)\ln(1-t^2) - tr \ln(1 - 2tmM + (2m-1)t^2) \tag{6.9}$$

where m is the number of generators (thus $k = 2m$) and $M \in \mathcal{N}(G)$ is the Markov operator on $\mathcal{S}$.

Let G be a residually finite group, with a finite set of generators S, $|S| = m$ and let $H_n \triangleleft G$, $n = 1, \dots$ be a descending sequence of normal subgroups with the trivial intersection. Let $\mathcal{S}$ and $\mathcal{S}_n$ by the Cayley graphs of the groups G and $G_n = G/H_n$ with respect to the system of generators S and its images S_n in G_n. Then $\mathcal{S}_n$, $n = 1, \dots$, constitute the covering system of finite graphs and $\mathcal{S} = \lim_n \mathcal{S}_n$.

Proposition 6.1. *The Ihara zeta function of $\mathcal{S}$ defined by the relation (6.9) coincides with the Ihara zeta function of $\mathcal{S}$ with respect to the approximating sequence given by Definition 4.5.*

Moreover the function given by (6.9) coincides with the function

$$-(m-1)\ln(1-t^2) - \int_{-1}^{1} \ln(1 - 2tm\lambda + (2m-1)t^2) d\mu(\lambda)$$

where μ is the Kesten spectral measure associated to the random walk on $\mathcal{S}$.

Corollary 6.2. *Definition 4.5 of the Ihara function on $\mathcal{S}$ does not depend on the choice of the above sequence H_n of normal subgroups.*

Proof. It is known that $\mu = \lim_n \mu_n$, where μ_n are uniform measures on S_n [13]. Thus the integral presentation leads to the relation for the Ihara zeta function for $\mathcal{S}$ and $\mathcal{S}_n$. In particular we obtain independence from $\{H_n\}$.

By definition of the spectral measure μ

$$tr \ln(1 - 2tmM + (2m-1)t^2) = \int_{-1}^{1} \ln(1 - 2tm\lambda + (2m-1)t^2) d\mu(\lambda).$$

□

For $\mathbb{Z}$, and more generally for $\mathbb{Z}^d$, the spectral measure with respect to the standard generators is well known [33]. This leads to the following formula for the corresponding Ihara zeta function for $\mathbb{Z}^d$

$$\ln \zeta_{\mathbb{Z}^d}(t) = -(d-1)\ln(1-t^2) - \int_{-1}^{1} \ln(1-2dt\lambda+(2d-1)t^2)d\mu^{*d}(\lambda),$$

where $d\mu^{*d}$ is a d-fold convolution of the measure $d\mu(\lambda) = \frac{1}{\pi\sqrt{1-\lambda^2}}d\lambda$.

In particular for $\mathbb{Z}$ we get

$$\ln \zeta_{\mathbb{Z}}(t) = -\int_{-1}^{1} \ln(1-2t\lambda+t^2)\frac{1}{\pi\sqrt{1-\lambda^2}}d\lambda = 0.$$

One can easily obtain the same result from formula (4.6) as follows from the remark at the end of Section 4.

Also for the free group F_m of rank m, the spectral measure with respect to the standard generators was computed (see [24]). This leads to the following formula for the corresponding Ihara zeta function

$$\begin{aligned}\ln \zeta_{F_m}(t) = & -(m-1)\ln(1-t^2) \\ & - \int_{-\frac{\sqrt{2m-1}}{m}}^{\frac{\sqrt{2m-1}}{m}} \ln(1-2tm\lambda+(2m-1)t^2)\frac{\sqrt{2m-1-m^2\lambda^2}}{\pi(1-\lambda^2)}d\lambda = 0.\end{aligned}$$

The above result can be also obtained from formula (4.6). Indeed the remark at the end of Section 4 shows that zeta functions of trees which admit finite quotients (for instance homogenous trees) are equal to 1.

7. Schreier graphs and subgroup separability

The important examples of graphs we consider are the following. Let G be a finitely generated group and $H < G$ be a subgroup of infinite index. H is called separable if there exists a sequence H_n of finite index subgroups of G such that $\bigcap H_n = H$. The group G is called subgroup separable if all finitely generated subgroups of G are separable. The examples of such groups include free groups [21], polycyclic groups, some fundamental groups of 3 manifolds and the group from [10] as was shown recently in [20].

The classical construction due to Schreier associates to the triple (G, H, S) where S is system of generators of G a graph $\mathcal{S}(G, H, S)$ often denoted as $\mathcal{S}(G/H)$ as follows. The set of vertices for $\mathcal{S}$ is the set G/H of left cosets, and two cosets gH, hH are joined by an edge, labeled by a generator $s \in S$ if $sgH = hH$. The Cayley graph of a group is a particular case of the Schreier graph, corresponding to the case $H = 1$. We consider $\mathcal{S}(G/H)$ as a marked graph with H as a distinguished vertex (forgetting about the labeling).

Proposition 7.1. *Let H be a separable subgroup of G. Then the Schreier graphs $X_n = \mathcal{S}(G/H_n, S)$ converge to the Schreier graph $X = \mathcal{S}(G/H, S)$ and the counting measures μ_n converge to the measure μ.*

Proof. Consider more general situation when the sequence of marked graphs (X_n, v_n) converges to the marked graph (X, v). Then the measures $\mu^{X_n}_{v_n v_n}$ converge weakly to the measure μ^X_{vv} (see [13]). Indeed in this case the moments of the measures $\mu^{X_n}_{v_n v_n}$ converge to the moments of the measure μ^X_{vv}. As it is easy to see, in our situation this implies weak convergence of corresponding measures. Indeed the l-th moment of the measure μ^Y_{yy} for a graph Y and $y \in Y$ is given by

$$(\mu^Y_{yy})^{(l)} = \int_{-1}^{1} \lambda^l \mu^Y_{yy}(\lambda) = \int_{-1}^{1} \lambda^l \langle E(\lambda)\delta_y, \delta_y\rangle = \langle M^l \delta_y, \delta_y\rangle.$$

Thus the l-th moment of the measure μ^Y_{yy} is equal to the probability of going from y to y in l steps. But for n sufficiently large, the balls $B_{X_n}(v_n, l)$ and $B_X(v, l)$ are isometric and the l-th moment of the measure $\mu^{X_n}_{v_n v_n}$ is the same as the l-th moment of the measure μ^X_{vv}. □

Thus if H is a separable subgroup of G and $H = \cap H_n$, then to this data and the set of generators S we can associate a graph $\mathcal{S}(G/H, S)$ with a KNS spectral measure.

In fact up to the cover of degree 2 any regular graph can be realized as a Schreier graph [27].

In the latter sections we shall consider examples of triples (G, H, S) which come from finite automata.

8. The spectral problem and integrability

In the following sections we will describe several examples of computation of the Ihara function of Cayley graphs and Schreier graphs related to some fractal groups. These examples come from [1, 14, 16] and correspond to groups generated by finite automata. In order to make our computations more clear for the reader we would like to recall a few facts on which the computation is based.

We omit the definition of the group $G = G(A)$, generated by a finite noninitial automaton A and basic facts about such groups. This can be found in [19, 14, 15, 16]. Let us just remind that G naturally acts on a d regular rooted tree T by automorphisms, where d is the cardinality of the alphabet and there is a natural embedding $G \to G \wr_D S_d$ in the permutational wreath product, where the symmetric group S_d acts on the alphabet D of the automaton A and $d = |D|$.

To any point $\xi \in \partial T$ one associates a parabolic subgroup $P = St_G(\xi)$ and a Schreier graph $\mathcal{S} = \mathcal{S}(G, P, S)$ where S is a system of generators of G. If $P = \{1\}$ then $\mathcal{S}$ is the Cayley graph of G. The graph $\mathcal{S}$ is the limit of finite Schreier graphs $\mathcal{S}_n = \mathcal{S}(G, P_n, S)$ where $P_n = St_n(u_n)$ and u_n is the vertex on the path ξ on the level n. The spectrum of the Markov operator M corresponding to $\mathcal{S}$ is approximated by spectra of Markov operators M_n corresponding to the graphs $\mathcal{S}_n$. The KNS spectral measure μ on $\mathcal{S}$ is the limit of counting measures μ_n of the spectrum of M_n.

Let $S = \{a_1, \ldots, a_m\}$ and

$$M = \frac{1}{2m}\sum_{i=1}^{m}(\rho(a_i) + \rho(a_i^{-1}))$$

be a Markov operator in $l^2(G/P)$ corresponding to the simple random walk on G/P where $\rho = \lambda_{G/P}$ is a quasi-regular representation. This operator is approximated by the operators

$$M_n = \frac{1}{2m}\sum_{i=1}^{m}(\rho_n(a_i) + \rho_n(a_i^{-1}))$$

where $\rho_n = \lambda_{G/P_n}$.

The idea used in [1, 14] for the computation of the spectrum is to consider a multi-parametric family of operators

$$M(\eta_1, \ldots, \eta_m, \lambda) = \frac{1}{2m}\sum_{i=1}^{m}\eta_i(\rho(a_i) + \rho(a_i^{-1})) - \lambda I,$$

$$M_n(\eta_1, \ldots, \eta_m, \lambda) = \frac{1}{2m}\sum_{i=1}^{m}\eta_i(\rho_n(a_i) + \rho_n(a_i^{-1})) - \lambda I_n$$

where $\eta_1, \ldots, \eta_m, \lambda$ are parameters taking values in $\mathbb{R}$ (the number of parameters can be reduced by 1) and I, I_n are the unit operators in the spaces $l^2(G/P)$, $l^2(G/P_n)$. It may happen, and this is the case in our examples, that for the determinant

$$Q_n(\eta_1, \ldots, \eta_m, \lambda) = \det(M_n(\eta_1, \ldots, \eta_m, \lambda))$$

the following recursion holds

$$Q_{n+1}(\eta_1, \ldots, \eta_m, \lambda) = f(\eta_1, \ldots, \eta_m, \lambda)g^{d^n}(\eta_1, \ldots, \eta_m, \lambda)Q_n(F(\eta_1, \ldots, \eta_m, \lambda))$$

where $f, g : \mathbb{R}^{m+1} \to \mathbb{R}$ are some functions and F is a rational map $\mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$. Thus

$$Q_{n+1}(\overline{\eta}, \lambda)$$

depends on

$$Q_1(F^{n-1}(\overline{\eta}, \lambda)),$$

where $\overline{\eta} = (\eta_1, \dots, \eta_m)$. Therefore the problem of the determination of the spectrum

$$Sp(M_n(\eta_1, \dots, \eta_m, \lambda)) = \{(\eta_1, \dots, \eta_m, \lambda); Q_n(\eta_1, \dots, \eta_m, \lambda) = 0\}$$

(the spectrum of M_n is $Sp(M_n(1, \dots, 1, \lambda))$), is closely related to the properties of the dynamical system, generated by the mapping F. The situation is much simpler in case the dynamical system is integrable, i.e. satisfies the following

Definition 8.1. We call a rational map $F : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ integrable if there exist a rational map $\psi : \mathbb{R}^{m+1} \to \mathbb{R}$ and a polynomial $\alpha : \mathbb{R} \to \mathbb{R}$ such that

$$\alpha\psi = \psi F. \tag{8.10}$$

In this case F is semi-conjugated to a one-dimensional map α. In such a situation Q_n can be factorized into factors (the nature of which depends on ψ) and the problem of the computation of $\zeta_X(t)$ can be solved in two ways.

First, by computation of the KNS spectral measure and use of the formula (5.7) and second by use of the factorization of Q_n (see the examples below).

The idea of integrability based on the relation (8.10) is not new in dynamical systems and goes back perhaps to Poincaré (we heard this from E. Ghys). In [42] a stronger version of the definition, based on the relation of commutation

$$\alpha\beta = \beta\alpha$$

is discussed. For the case when F is a meromorphic function and ψ and α are rational mappings of $\mathbb{C}$, the equation (8.10) is completely treated in [6]. It would be nice to develop theory of integrable multidimensional rational mappings of $\mathbb{R}^d$, in particular for dimension two.

9. The examples with absolutely continuous KNS spectral measure

In this section we will consider two examples of groups generated by finite automata whose Schreier graphs have absolutely continuous KNS spectral measure.

Note that often in our examples below we use non-normalized Markov operators in computations, statements and figures.

The groups $G = \langle a, b, c, d\rangle$ and $\widetilde{G} = \langle a, \widetilde{b}, \widetilde{c}, \widetilde{d}\rangle$ are generated by automata given in Figures 1 and 2. They have intermediate growth between polynomial and exponential. The group G embeds in $\widetilde{G}$ via the map $a \to a$, $b \to \widetilde{b}\widetilde{d}$, $c \to \widetilde{c}\widetilde{b}$, $d \to \widetilde{d}\widetilde{c}$.

Let $P = St_G(1^\infty)$ and $\widetilde{P} = St_{\widetilde{G}}(1^\infty)$ be parabolic subgroups, where 1^∞ corresponds to the infinite path $1, 1, 1, \dots$. Let $\mathcal{S} = \mathcal{S}(G, P, S)$ and $\widetilde{\mathcal{S}} = \mathcal{S}(\widetilde{G}, \widetilde{P}, \widetilde{S})$ denote the Schreier graphs where $S = \{a, b, c, d\}$ and $\widetilde{S} = \{a, \widetilde{b}, \widetilde{c}, \widetilde{d}\}$. The Schreier graphs $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are described in [19].

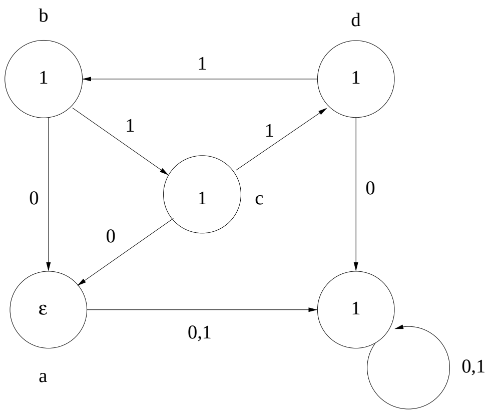

Figure 1: The automaton generating the group G

Theorem 9.1 ([1]). *The spectrum of the graph $\mathcal{S}$ coincides with the set*

$$\left[-\frac{1}{2},0\right]\cup\left[\frac{1}{2},1\right]$$

and the KNS spectral measure is equal to

$$\frac{|1-4x|}{2\pi\sqrt{x(2x-1)(2x+1)(1-x)}}dx.$$

The spectrum of the graph $\widetilde{\mathcal{S}}$ coincides with the set

$$[0,1]$$

and the KNS spectral measure is equal to

$$\frac{1}{\pi\sqrt{x-x^2}}dx.$$

Corollary 9.2. *The Ihara zeta functions for the Schreier graphs $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ of the groups G and $\widetilde{G}$ are equal to*

$$\begin{aligned}\ln\zeta_{\mathcal{S}}(t) &= -3\ln(1-t^2)-\int_{-\frac{1}{2}}^{0}\frac{\ln(1-t8x+7t^2)|1-4x|}{2\pi\sqrt{x(2x-1)(2x+1)(1-x)}}dx\\ &\quad -\int_{\frac{1}{2}}^{1}\frac{\ln(1-t8x+7t^2)|1-4x|}{2\pi\sqrt{x(2x-1)(2x+1)(1-x)}}dx,\end{aligned}$$

$$\ln\zeta_{\widetilde{\mathcal{S}}}(t)=-3\ln(1-t^2)-\int_{0}^{1}\frac{\ln(1-8xt+7t^2)}{\pi\sqrt{x^2-x}}dx.$$

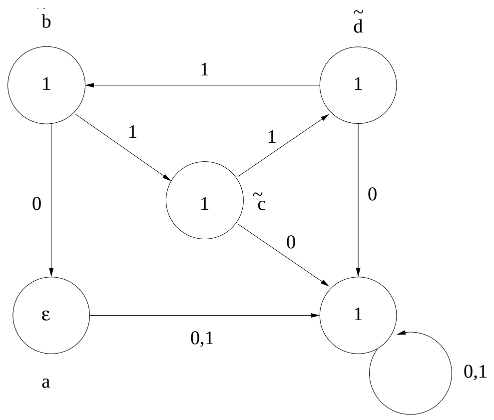

Figure 2: The automaton generating the group $\widetilde{G}$

Now we are going to recall some steps of the computation of the measure μ for these examples, using the integrability of the corresponding map F. We will also get an alternative integral representation for $\ln\zeta(t)$.

Let

$$M_n = \rho_n(a) + \rho_n(b) + \rho_n(c) + \rho_n(d)$$

be an $2^n \times 2^n$ matrix representing the normalized Markov operator on the Schreier graph $\mathcal{S} = (G, P_n, S)$ $(P_n = St_G(1^n))$. As $a^2 = b^2 = c^2 = d^2 = 1$ we do not need inverses. Let

$$M_n(\lambda,\eta) = M_n - (\lambda+1)\rho(a) - (\eta+1)I_n,$$

$$Q_n(\lambda,\eta) = \det M_n(\lambda,\eta).$$

For $n \geq 2$, as was shown in [1], we have

$$Q_n(\lambda,\eta) = (4-\eta^2)^{2^{n-2}} Q_{n-1}(F(\lambda,\eta)), \tag{9.11}$$

where

$$F(\lambda,\eta) = \left(\frac{2\lambda^2}{4-\eta^2}, \eta + \frac{\eta\lambda^2}{4-\eta^2}\right).$$

The map F admits the following inverse

$$F^{-1}(\lambda,\eta) = \left(\pm\sqrt{\frac{\lambda}{2}\left(4 - \left(\frac{2\eta}{2+\lambda}\right)^2\right)}, \frac{2\eta}{2+\lambda}\right).$$

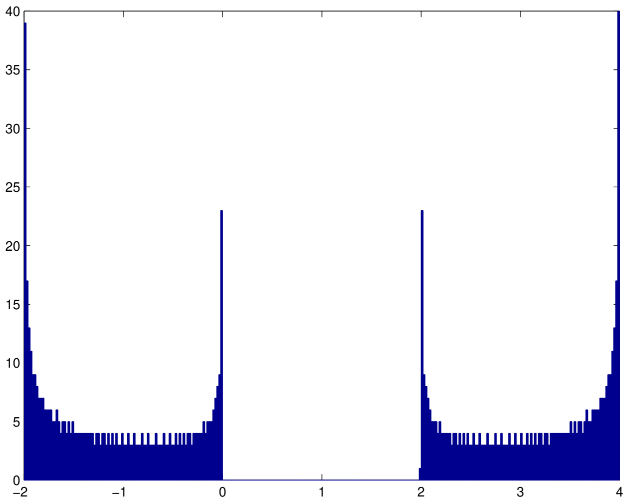

Figure 3: The histogram of the spectrum in the case of the group G

The set of F-preimages of a point drawn with the help of a computer looks like the set given by Figure 5. It is natural to conjecture that the map F has this set as the attractor, but this is only our guess.

Proposition 9.3. *The map F is integrable with*

$$\begin{aligned}\psi(\lambda,\eta) &= \frac{4-\eta^2+\lambda^2}{\lambda},\\ \alpha(x) &= 2x^2-1.\end{aligned}$$

Proof. Indeed the direct computation gives

$$\alpha\psi = \frac{1}{8}(-4+\eta^2-\lambda^2)^2/\lambda^2-1=\psi F.$$

□

The mapping $x \to 2x^2-1$ is known as the von Neumann-Ulam map [28]. It is conjugated to the map $x \to x^2-2$. Outside the interval the dynamic of this map is simple: any point x, $|x|>1$ goes to infinity. On the other hand in the interval $[-1,1]$ the dynamic is chaotic. Indeed, the map α is conjugated to the expanding map of S^1, namely $z \to z^2$ and to the one-sided Bernoulli shift on two symbols. The measure $\frac{dx}{\pi\sqrt{1-x^2}}$ is invariant with respect to α and is the measure of the maximal entropy, because it corresponds to $\{\frac{1}{2},\frac{1}{2}\}$ Bernoulli shift.

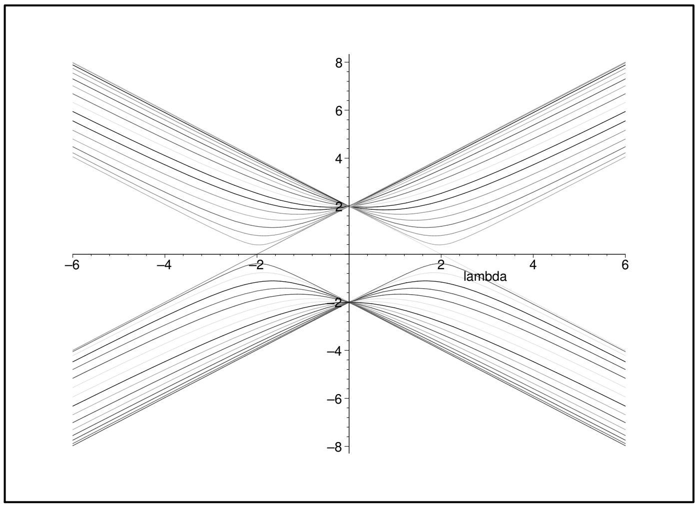

Figure 4: The zeros of Q_6 associated to the group G

In [1] the following factorization for $Q_n(\lambda,\eta)$ was proven

$$Q_n(\lambda,\eta)=(2-\eta-\lambda)(2-\eta+\lambda)\prod_{\substack{\phi=\frac{2\pi j}{2^n}\\ j=1,\dots,2^{n-1}-1}} H_{\cos(\phi)}(\lambda,\eta)$$

where

$$H_\theta(\lambda,\eta)=4-\eta^2+\lambda^2+4\lambda\theta.$$

It can be alternatively written as

$$Q_n(\lambda,\eta)=(2-\eta-\lambda)(2-\eta+\lambda)\prod_{\substack{\theta\in\cup_{i=0}^{n-2}\\ \alpha^{-i}(0)}} H_\theta(\lambda,\eta).$$

Indeed, from the integrability of F it follows that

$$H_\theta(F(\lambda,\eta))=H_{\theta_1}(\lambda,\eta)H_{\theta_2}(\lambda,\eta),$$

where θ_1, θ_2 are the preimages of θ under α and to get a factorization of Q_n one can use relation (9.11).

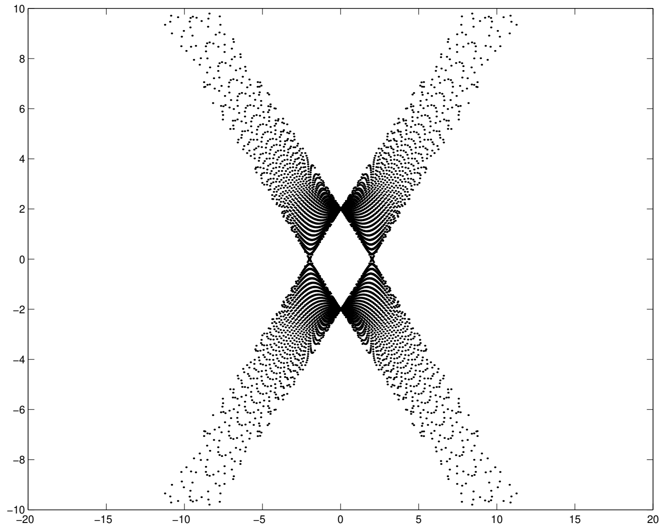

Figure 5: The attractor associated to the group G

The map F maps the hyperbola $H_\theta(\lambda,\eta)=0$ onto the hyperbola $H_{\alpha(\theta)}(\lambda,\eta)=0$. The spectrum of the 2-parametric family $M(\lambda,\eta)$ is a union of hyperbola.

Proposition 9.4. *The following relation holds for the graph associated to the group G*

$$\ln\zeta(t) \quad = \quad -3\ln(1-t^2)-\frac{1}{2}\int_{-1}^{1}\ln\left(1-4t-2t^2-28t^3+49t^4+16t^2x\right)d\nu(x),$$

where

$$d\nu(x)=\frac{dx}{\pi\sqrt{1-x^2}}.$$

Proof. Using the factorization of Q_n we get

$$\begin{aligned}
\zeta_n(t) &= (1-t^2)^{-3\cdot 2^n}\det(1-t2M_n+7t^2)^{-1}\\
&= (1-t^2)^{-3\cdot 2^n}(2t)^{-2^n}\det\left(M_n+\frac{1+7t^2}{-2t}\right)^{-1}\\
&= (1-t^2)^{-3\cdot 2^n}(2t)^{-2^n}Q_n^{-1}\left(-1,\frac{1+7t^2}{2t}-1\right)\\
&= (1-t^2)^{-3\cdot 2^n}(2t)^{-2^n}\left(8t-1-7t^2\right)^{-1}\left(4t-1-7t^2\right)^{-1}\prod_{\substack{\theta\in\cup_{i=0}^{n-2}\\ \alpha^{-i}(0)}}\left(-1,\frac{1+7t^2}{2t}-1\right)^{-1}\\
&= (1-t^2)^{-3\cdot 2^n}\left(8t-1-7t^2\right)^{-1}\left(4t-1-7t^2\right)^{-1}\\
&\quad \prod_{\substack{\theta\in\cup_{i=0}^{n-2}\\ \alpha^{-i}(0)}}\left(5(2t)^2-\left(1+7t^2-2t\right)^2-4(2t)^2\theta\right)^{-1}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{2^n}\ln\zeta_n(t) &= -3\ln(1-t^2)\\
&\quad -\lim_{n\to\infty}\frac{1}{2^n}\sum_{\theta\in\cup_{i=0}^{n-2}\alpha^{-i}(0)}\ln\left(1-4t-2t^2-28t^3+49t^4+16t^2\theta\right)\\
&= -3\ln(1-t^2)\\
&\quad -\frac{1}{2}\lim_{n\to\infty}\int_{-1}^{1}\ln\left(1-4t-2t^2-28t^3+49t^4+16t^2x\right)d\nu_n(x)
\end{aligned}$$

where ν_n is a discrete measure uniformly distributed on the set $\cup_{i=0}^{n-2}\alpha^{-i}(0)$ of $2^{n-1}-1$ points. The measures ν_n converge to the measure of the maximal entropy of α which we call ν. This measure corresponds to $(\frac{1}{2},\frac{1}{2})$ Bernoulli measure in realization of α as the Bernoulli shift and is a projection on $[-1,1]$ via the map $x\to\cos x$ of the Lebesgue measure on S^1. This ends the proof. $\square$

For the group $\widetilde{G}$ we use the same notations but with tilde. We have the relation

$$\frac{1}{2}Q_n(\lambda,\eta)=\widetilde{Q}_n(\lambda,\eta).$$

In this case

$$F(\lambda,\eta)=\left(\frac{\lambda^2}{1-\eta^2},\eta+\frac{\mu\lambda^2}{1-\eta^2}\right)$$

and the following proposition holds

Proposition 9.5. *For*

$$\psi(\lambda,\eta)=\frac{2(1-\eta^2+\lambda^2)}{\lambda}$$

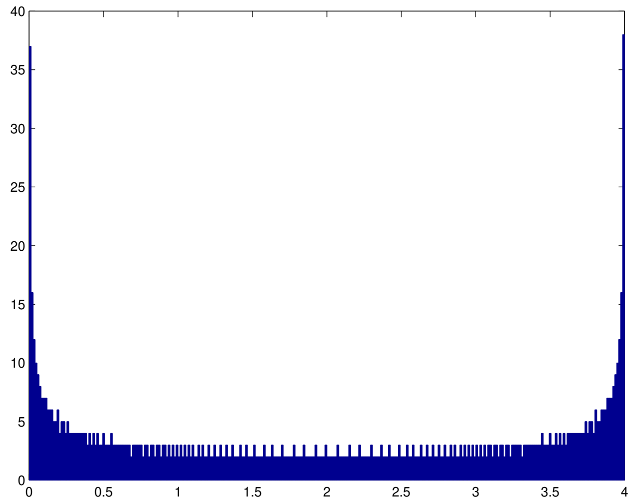

Figure 6: The histogram of the spectrum of the group $\widetilde{G}$

and

$$\alpha(x) = 2x^2 - 1$$

we have (8.10).

Proof. This follows from Proposition 9.3. Indeed from the relation $\frac{1}{2}Q_n = \widetilde{Q}_n$, it follows that the maps F for G and $\widetilde{G}$ are conjugated by the map $(\lambda, \eta) \to (2\lambda, 2\eta)$. □

In the same way as for G one can also compute the zeta function for the group $\widetilde{G}$ using the factorization of Q_n.

10. The examples with totally discontinuous spectrum

In this and in the following sections we will consider the examples of fractal groups for which the associated Schreier graphs have discrete spectral measure. These are Fabrikowski-Gupta group from Figure 7, Gupta-Sidki 3-group from Figure 8 and a group generated by the automaton from Figure 9 studied in [1]. Although the number of different groups is 3, the number of different spectra, KNS measures and Ihara zeta functions is only 2 because the last two cases lead to the same Schreier graphs (if we forget the labeling of the edges).

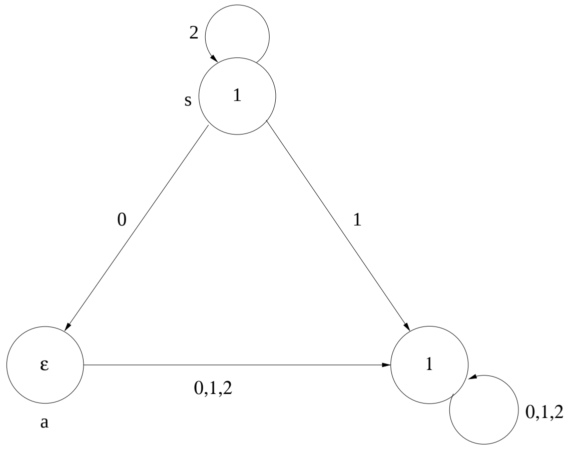

Figure 7: The automaton generating the group Γ

Let Γ, $\overline{\Gamma}$, $\overline{\overline{\Gamma}}$ be the groups generated by the automata from Figures 7, 8 and 9 over the alphabet $\{0,1,2\}$ where ε is the cyclic permutation.

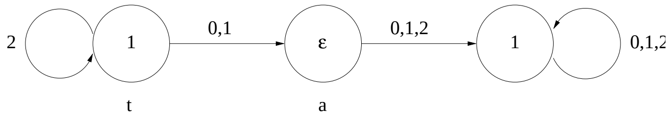

Figure 8: The automaton generating the group $\overline{\Gamma}$

Let $P = St_{\Gamma}(2^{\infty})$, $\overline{P} = St_{\overline{\Gamma}}(2^{\infty})$ and $\overline{\overline{P}} = St_{\overline{\overline{\Gamma}}}(2^{\infty})$ be parabolic subgroups, where 2^{∞} corresponds to the infinite path $2,2,2,\ldots$. Let $\mathcal{S}$, $\overline{\mathcal{S}}$, $\overline{\overline{\mathcal{S}}}$ be the the corresponding Schreier graphs.

All groups Γ, $\overline{\Gamma}$ and $\overline{\overline{\Gamma}}$ are fractal, the groups Γ and $\overline{\overline{\Gamma}}$ are branch, the group $\overline{\Gamma}$ is weakly branch, the groups Γ and $\overline{\Gamma}$ are virtually torsion free and the group $\overline{\Gamma}$ is a torsion 3-group.

Let us first consider the group $\Gamma = \langle a, s \rangle$. The group Γ embeds into permutational wreath product $\Gamma \wr_D S_3$ via the map $a \to (1,1,1)\varepsilon$, $s \to (a,1,s)1$, where $\varepsilon \in S_3$ is the cyclic permutation. Let

$$M = \rho(a) + \rho(a^{-1}) + \rho(s) + \rho(s^{-1})$$

be the Markov operator on $l^2(G/P)$ and

$$M_n = \rho_n(a) + \rho_n(a^{-1}) + \rho_n(s) + \rho_n(s^{-1})$$

the approximation on the n-th level.

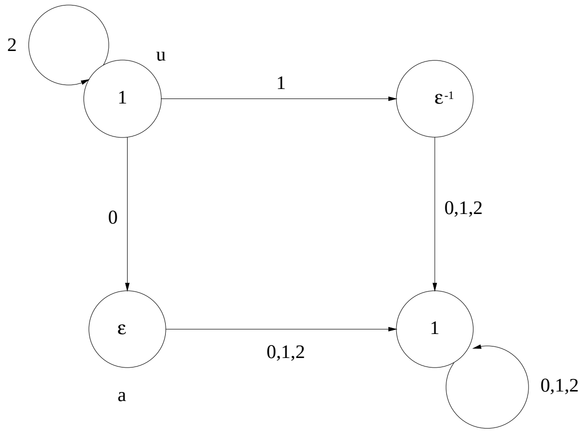

Figure 9: The automaton generating the group $\overline{\overline{\Gamma}}$

Theorem 10.1 ([1]). *The spectrum of the graph $\mathcal{S}$ is the closure of the set of points*

$$1 \pm \underbrace{\sqrt{6 \pm \sqrt{6 \pm \ldots \pm \sqrt{6}}}}_{n \text{ times}}$$

for $n = 0, 1, \ldots$.

The above set is a union of a Cantor set of null Lebesgue measure and a countable collection of isolated points supporting the KNS spectral measure μ which is discrete and which has value $\frac{1}{3^n+1}$ at the point whose definition involves n radicals.

From Theorem 5.1 we obtain the following

Corollary 10.2. *The Ihara zeta functions for the above graph is equal to*

$$\begin{aligned}\ln \zeta_{\mathcal{S}}(t) &= -\ln(1-t^2) - \frac{1}{3}\ln(1-t-3t^2) \\ &\quad - \sum_{j=1}^{\infty} \ln\left(1 - t\left(1 \pm \underbrace{\sqrt{6 \pm \sqrt{6 \pm \ldots \pm \sqrt{6}}}}_{j \text{ times}}\right) + 3t^2\right)\frac{1}{3^{j+1}},\end{aligned}$$

where all possible combinations of the sign $\pm$ appear in the above sum.

Now we are going to recall some steps of the computation of the spectrum and the KNS spectral measure μ in order to obtain an alternative description of $\zeta_{\mathcal{S}}(t)$.

Let

$$Q_n(\lambda, \eta) = \det\left(\lambda(\rho_n(a) + \rho_n(a^{-1})) + \rho_n(b) + \rho_n(b^{-1}) - \eta I_n\right).$$

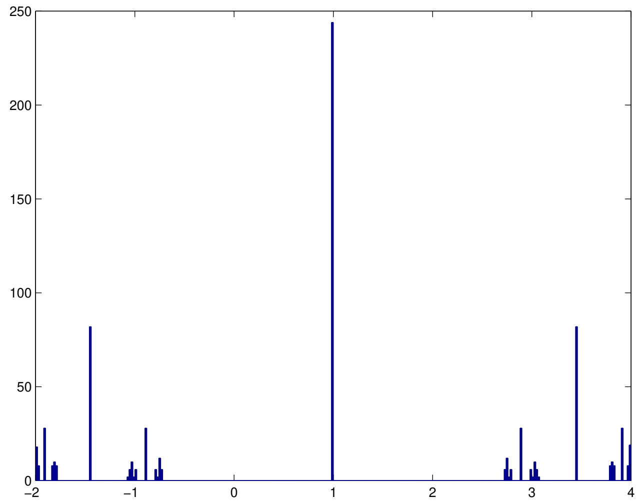

Figure 10: The histogram of the spectrum in case of the group Γ

For $\alpha = 2-\eta+\lambda$, $\beta = 2-\eta-\lambda$, $\gamma = 1+\eta+\lambda$, $\delta = 1+\eta-\lambda$ we have for $n \geq 2$

$$Q_n(\lambda,\eta) = (\alpha\beta\gamma^2)^{3^{n-2}} Q_{n-1}(F(\lambda,\eta)),$$

where

$$F(\lambda,\eta) = \left(\frac{\lambda^2(2-\eta-\lambda)}{(2-\eta+\lambda)(\eta^2-\lambda^2-\eta-2)}, \eta + 2\frac{\lambda^2(\eta^2-\lambda^2-\eta-2)}{(2-\eta+\lambda)(\eta^2-\lambda^2-2\eta-\lambda)}\right).$$

Proposition 10.3. *The map F is integrable with*

$$\begin{aligned}\psi(\lambda,\eta) &= -\frac{\eta^2-\lambda\eta-2\lambda^2-2-\eta}{\lambda},\\ \alpha(x) &= 4-2x-x^2.\end{aligned}$$

Proof. Indeed the direct computation shows

$$\begin{aligned}\alpha\psi &= -(4\lambda^4+4\lambda^3+4\eta\lambda^3-3\eta^2\lambda^2+4\lambda^2+6\eta\lambda^2-2\lambda\eta^3+4\lambda+6\lambda\eta\\ &\quad +\eta^4-2\eta^3-3\eta^2+4+4\eta)/\lambda^2 = \psi F.\end{aligned}$$

□

Let

$$H_\theta(\lambda,\eta) = \eta^2-\lambda\eta-2\lambda^2-2-\eta+\theta\lambda.$$

The above proposition is related to the fact that

$$H_\theta(F(\lambda,\eta)) = \frac{\beta}{\alpha\gamma} H_{\theta_1}(\lambda,\eta)H_{\theta_2}(\lambda,\eta)$$

where θ_1, θ_2 are preimages of θ under α.

This leads to the following factorization

$$Q_n(\lambda,\eta) = (2+2\lambda-\eta)(2-\lambda-\eta)^{3^{n-1}+1} \prod_{\substack{2\le m\le n \\ \theta\in\alpha^{-m}(-1)}} H_\theta^{3^{n-m}+1}(\lambda,\eta).$$

In the same way as it was done for the group G one can prove

Proposition 10.4. *The following relation holds for the graph associated to the group* Γ

$$\ln\zeta(t) = -\ln(1-t^2) - \frac{1}{3}\ln(1-t+3t^2)$$
$$-\sum_{j=0}^{\infty} \ln\left(1-2t+2t^2-6t^3+9t^4+\left(1\pm\underbrace{\sqrt{6\pm\sqrt{6\pm\ldots\pm\sqrt{6}}}}_{j\text{ times}}\right)t^2\right)\frac{1}{3^{j+2}}$$

for small $|t|$, *where all possible combinations of the sign* $\pm$ *appear in the above sum.*

Proof. Using the factorization for Q_n we have

$$\begin{aligned}
\zeta_n(t) &= (1-t^2)^{-3^n}\det(1-tM_n+3t^2)^{-1} \\
&= (1-t^2)^{-3^n} t^{-3^n} Q_n^{-1}\left(1, \frac{1+3t^2}{t}\right) \\
&= (1-t^2)^{-3^n} t^{-3^n}\left(4-\frac{1+3t^2}{t}\right)^{-1}\left(1-\frac{1+3t^2}{t}\right)^{-3^{n-1}-1} \\
&\quad \Pi_{2\le m\le n\ \theta\in\alpha^{-m}(-1)} H_\theta^{-3^{n-m}-1}\left(1, \frac{1+3t^2}{t}\right).
\end{aligned}$$

Thus because of the definition of the measure μ from Theorem 10.1 we get

$$\lim_{n\to\infty}\frac{1}{3^n}\ln\zeta_n(t) = -\ln(1-t^2) - \frac{1}{3}\ln(1-t+3t^2)$$
$$-\frac{1}{3}\sum_{j=1}^{\infty} \ln\left(1-2t+2t^2-6t^3+9t^4+\left(1\pm\underbrace{\sqrt{6\pm\sqrt{6\pm\ldots\pm\sqrt{6}}}}_{j\text{ times}}\right)t^2\right)\frac{1}{3^{j+1}}.$$

$\square$

Let us check that the zeta functions from Corollary 10.2 and Proposition 10.4 coincide. This follows from the following

$$\ln\left(1-t\left(1+\underbrace{\sqrt{6\pm\sqrt{6\pm\ldots\pm\sqrt{6}}}}_{j\text{ times}}\right)+3t^2\right)\frac{1}{3^{j+1}}$$

$$+\quad\ln\left(1-t\left(1-\underbrace{\sqrt{6\pm\sqrt{6\pm\ldots\pm\sqrt{6}}}}_{j\text{ times}}\right)+3t^2\right)\frac{1}{3^{j+1}}$$

$$=\quad\frac{1}{3^{j+1}}\ln\left(1-2t+2t^2-6t^3+9t^4+\left(1\pm\underbrace{\sqrt{6\pm\sqrt{6\pm\ldots\pm\sqrt{6}}}}_{j-1\text{ times}}\right)t^2\right).$$

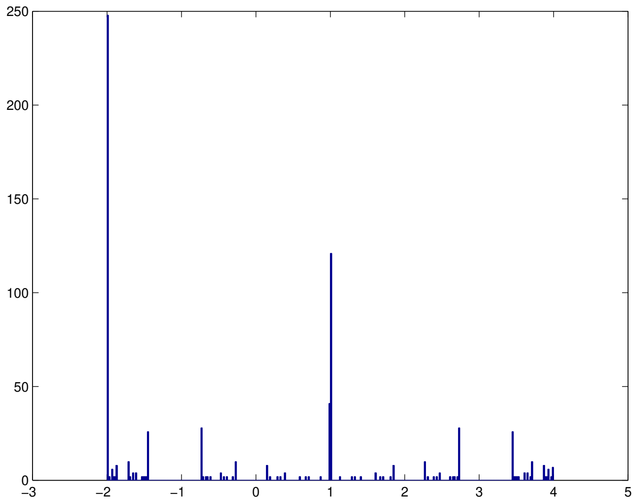

Figure 11: The histogram of the spectrum in case of the group $\overline{\Gamma}$

The computation of the spectrum and spectral measures for the graphs $\overline{\mathcal{S}}$ and $\overline{\overline{\mathcal{S}}}$ is similar to the computation for the graph $\mathcal{S}$.

Theorem 10.5 ([1]). *The spectra of the graphs $\overline{\mathcal{S}}$ and $\overline{\overline{\mathcal{S}}}$ coincide and are the closure of the set of points*

$$4,-2,1,1\pm\frac{1}{\sqrt{2}}\sqrt{9\pm\underbrace{\sqrt{45\pm4\sqrt{45\pm\ldots\pm4\sqrt{45\pm4\cdot3}}}}_{n\text{ times}}}$$

for $n=0,1,\ldots$.

The above set is a union of a Cantor set of null Lebesgue measure and a countable collection of isolated points supporting the KNS spectral measure μ which is discrete and which has value 0, $\frac{1}{3}$, $\frac{2}{9}$ for the first three points and $\frac{2}{3^{n+2}}$ at the point whose definition involves n radicals.

Corollary 10.6. *The Ihara zeta functions for the above graph is equal to*

$$\begin{aligned}\zeta_{\overline{S}}(t) &= (1-t^2)^{-1}(1+16t+7t^2)^{\frac{1}{3}}(1-8t+7t^2)^{\frac{2}{3}}\\ &\prod_{j=0}^{\infty}\left(1-t\left(1\pm\frac{1}{\sqrt{2}}\sqrt{9\pm\sqrt{45\pm 4\sqrt{45\pm\ldots\pm 4\cdot 3}}}\right)+4t^2\right)^{\frac{2}{3^j}}.\end{aligned}$$

For these groups one has the following relation for $n \geq 2$

$$Q_n(\lambda,\eta) = (\gamma\delta)^{2\cdot 3^{n-2}}(\alpha\beta)^{3^{n-2}}Q_{n-1}(F(\lambda,\eta))$$

where $\alpha = 2-\eta+\lambda$, $\beta = 2-\eta-\lambda$, $\gamma = 1+\eta+\lambda$, $\delta = 1+\eta-\lambda$ and

$$F(\lambda,\eta) = \left(-2\frac{\lambda^2}{(2-\eta+\lambda)(1+\eta-\lambda)}, \eta+2\frac{\lambda^2(\eta-\lambda-1)}{(2-\eta+\lambda)(1+\eta-\lambda)}\right).$$

Proposition 10.7. *The map F is integrable with*

$$\begin{aligned}\psi(\lambda,\eta) &= \frac{\eta^2-\lambda\eta-2\lambda^2-2-\eta}{-\lambda},\\ \alpha(x) &= 6+\frac{1}{2}x-\frac{1}{2}x^2.\end{aligned}$$

Proof. Indeed the direct computation shows

$$\begin{aligned}\alpha\psi &= 6-\frac{1}{2}(\eta^2-\lambda\eta-2\lambda^2-2-\eta)/\lambda-\frac{1}{2}(\eta^2-\lambda\eta-2\lambda^2-2-\eta)^2/\lambda^2\\ &= \psi F.\end{aligned}$$

□

11. Zeta function related to the lamplighter group

The difference between the example considered here and previous cases is that although the Markov operator has pure point spectrum, the spectrum considered just as a set is the interval $[-1,1]$ and is not totally disconnected as in the previous section.

Let $L = (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z} = (\oplus_{\mathbb{Z}}\mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}$ be the lamplighter group [14]. It is a metabelian, non-finitely presented group of exponential growth generated by the elements a, v where v is a generator of the group $\mathbb{Z}$ and

$$a = (\ldots 010\ldots) \in \oplus_{\mathbb{Z}}\mathbb{Z}/2\mathbb{Z}$$

where 1 corresponds to the zero position. This group has a presentation

$$L = \langle a, v | a^2 = [a^{v^n}, a] = 1, n = 1, 2, \ldots \rangle,$$

where $x^y = y^{-1}xy$ denotes the conjugation.

Consider the system o generators v, w where $w = av^{-1}$. Then L is isomorphic to the group $G(A) = \langle \mathcal{A}_v, \mathcal{A}_w \rangle$ generated by the automaton A from Figure 12 as was shown in [14] and the isomorphism $L \simeq G(A)$ holds via the map $v \to \mathcal{A}_v$, $w \to \mathcal{A}_w$.

Let $P = St_L(\xi)$, $\xi \in \partial T$ where T is the binary tree. Then P is either cyclic of trivial (for the set of points of ∂T of the full measure) ([14] Proposition 5). Let $\mathcal{S}$ be the Cayley graph $\mathcal{S}(L, \{v, w\})$.

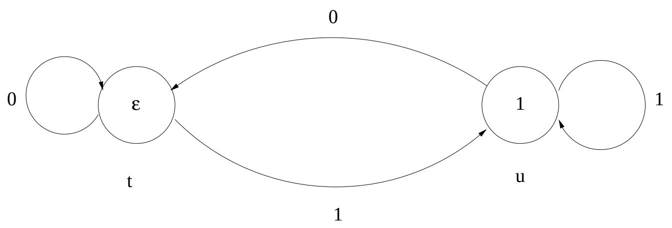

Figure 12: The automaton A generating the lamplighter group

Theorem 11.1 ([14]). *The Markov operator on the graph $\mathcal{S}$ has a pure point spectrum concentrated on the points $\cos \frac{p}{q}\pi$, $q = 2, 3, \ldots$, $p \in \mathbb{Z}$. The KNS spectral measure is discrete, concentrated on the above points with values*

$$\mu\left(\cos \frac{p}{q}\pi\right) = \frac{1}{2^q - 1} \quad \text{if} \quad (p, q) = 1.$$

Corollary 11.2. *For the lamplighter group the zeta function is equal to*

$$\zeta_{\mathcal{S}}(t) = (1 - t^2)^{-1} \prod_{q=2}^{\infty} \prod_{(p,q)=1} \left(1 - t4\cos \frac{p}{q}\pi + 3t^2\right)^{\frac{1}{2^q - 1}}.$$

Consider the operator

$$M(\lambda, \eta) = \rho(v) + \rho(v^{-1}) + \rho(w) + \rho(w^{-1}) - \lambda I - \eta\rho(a)$$

in $l^2(L/P)$ and

$$M_n(\lambda, \eta) = \rho_n(v) + \rho_n(v^{-1}) + \rho_n(w) + \rho_n(w^{-1}) - \lambda I_n - \eta\rho_n(a)$$

in $l^2(L/P_n)$. The Markov operator on $\mathcal{S}(G/P)$ and $\mathcal{S}(G/P_n)$ corresponds to the above operators when $\eta = 0$. Let $Q_n = \det M_n$. Then

$$\begin{aligned} Q_0 &= 4 - \lambda - \eta \\ Q_1 &= (\eta - \lambda)(4 - \lambda - \eta) \end{aligned}$$

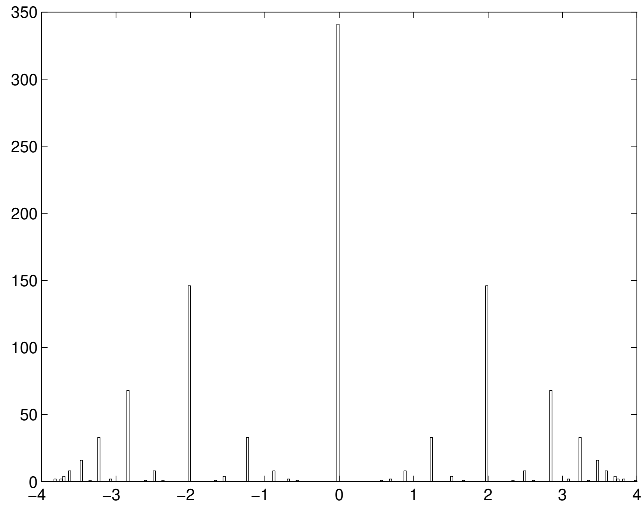

Figure 13: The histogram of the spectrum of M_n for the lamplighter group for $n = 10$

and for $n \geq 2$ the following recursion holds

$$Q_n(\lambda,\eta) = (\eta - \lambda)^{2^{n-1}} Q_{n-1}(F(\lambda,\eta)),$$

where

$$F(\lambda,\eta) = \left(\frac{\lambda^2 - \eta^2 - 2}{\lambda - \eta}, \frac{2}{\lambda - \eta}\right).$$

The map F admits the following inverse

$$F^{-1}(\lambda,\eta) = \left(\frac{\lambda}{2} + \frac{1}{\eta} + \frac{\eta}{2}, \frac{\lambda}{2} - \frac{1}{\eta} - \frac{\eta}{2}\right),$$

which is defined for $\eta \neq 0$.

Proposition 11.3. *The map F is integrable with $\psi(\lambda,\eta) = \lambda + \eta$ and $\alpha(x) = x$.*

Proof. The direct computation shows

$$\psi F = \lambda + \eta = \alpha\psi.$$

□

We see that in case of the lamplighter group the dynamics of F is very simple. It preserves the lines l_c given by the equation $\lambda + \eta = c$. Depending on c, the dynamics of F on l_c is either a south-north dynamics or F is conjugated to a rotation of the circle. Indeed we have

Proposition 11.4. *On the line l_c parametrized by the parameter λ, the map F is the Möbius map*

$$\lambda \to \frac{c\lambda - \left(\frac{c^2}{2} + 1\right)}{\lambda - \frac{c}{2}}$$

and the corresponding matrix

$$C = \begin{pmatrix} c & -\frac{c^2}{2} - 1 \\ 1 & -\frac{c}{2} \end{pmatrix} \in SL(2, \mathbb{R})$$

is elliptic if $|c| < 4$, parabolic if $|c| = 4$ and hyperbolic if $|c| > 4$.

Proof. The second statement follows from $tr(c) = \frac{c}{2}$ and the well known fact that a Möbius map c is elliptic if and only if $|tr(c)| < 2$. The corresponding regions are shown in Figure 14. $\square$

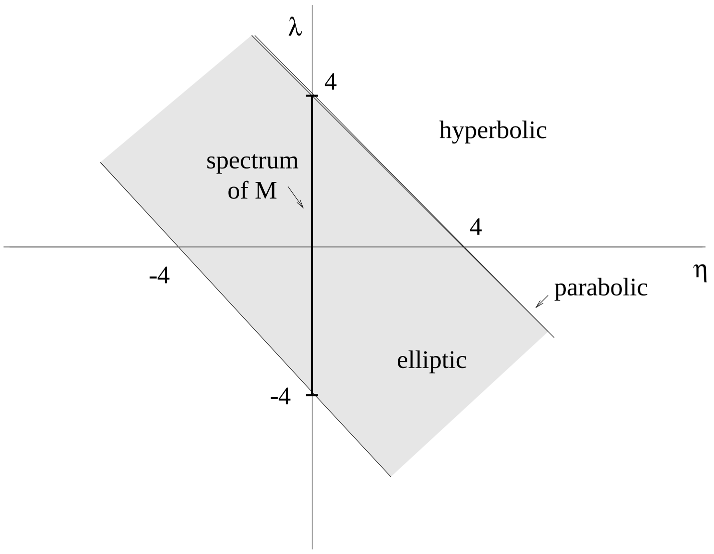

Figure 14: The dynamical system related to the automaton generating the lamplighter group

The map F has two fixed points

$$\lambda = \frac{3c \pm \sqrt{c^2 - 4}}{4}$$

on the line l_c in case it is hyperbolic.

The computation done in [14] shows that for $\lambda + \eta = 4 \cos z$

$$Q_n(\lambda,\eta) = (4-4\cos z)2^n \frac{1}{(\sin z)^{2^{n-1}}} \Pi_{k=2}^n (\sin kz)^{2^{n-k}} \sin(z(n+1)). \qquad (11.12)$$

The factor $\sin kz$ is related to Chebyshev polynomial T_k by the relation

$$\sin kz = \sqrt{1 - T_k^2\left(\frac{\lambda+\eta}{4}\right)}.$$

In contract to previous examples, in decomposition of Q_n on factors consists not of functions depending on the parameter θ which runs the preimages of the map α but the preimages by the Chebyshev polynomials. The family $\{T_k\}_{k=1}^{\infty}$ is well known in dynamics. It consists of commuting transformations preserving the measure

$$\frac{dx}{\pi\sqrt{1-x^2}}$$

on $[-1,1]$ and T_k is conjugated to $z \to z^k$.

It is not clear how this measure can be used in our situation, but the alternative presentation for the Ihara zeta function can be easily obtained from the relation (11.12). Namely as

$$\begin{aligned} \zeta_n(t) &= (1-t^2)^{-2^n} \det(1 - tM_n + 3t^2)^{-1} \\ &= (1-t^2)^{-2^n} t^{-2^n} Q_n^{-1}\left(\frac{1+3t^2}{-t}, 0\right), \end{aligned}$$

and using the factorization (11.12) we get

$$\begin{aligned} \ln\zeta(t) &= -\frac{1}{4}\ln(1-t^2) - \frac{1}{4}\ln\left(1 - T_1^2\left(\frac{1+3t^2}{-4t}\right)\right) \\ &\quad -\sum_{k=2}^{\infty} \frac{1}{2^{k+1}} \ln\left(1 - T_k^2\left(\frac{1+3t^2}{-4t}\right)\right) \end{aligned}$$

and $|T_k(x)| < \sqrt{\frac{\pi}{2}}\frac{1}{2^k}$ if $x \in [-1,1]$.

By the results from Section 5 the above series converges for $0 < |t| < \frac{1}{3}$ as then $|\frac{1+3t^2}{-4t}| < 1$.

12. An example of iterated monodromy group

Let H be a group generated by the automaton B from Figure 15 on three states. As the state c corresponds to identity map, the group H is indeed 2-generated by the elements a and b corresponding to the states of the automaton. The group H embeds into $H \wr S_2$ via the map $a \to (1,b)$ and $b \to (1,a)\varepsilon$.

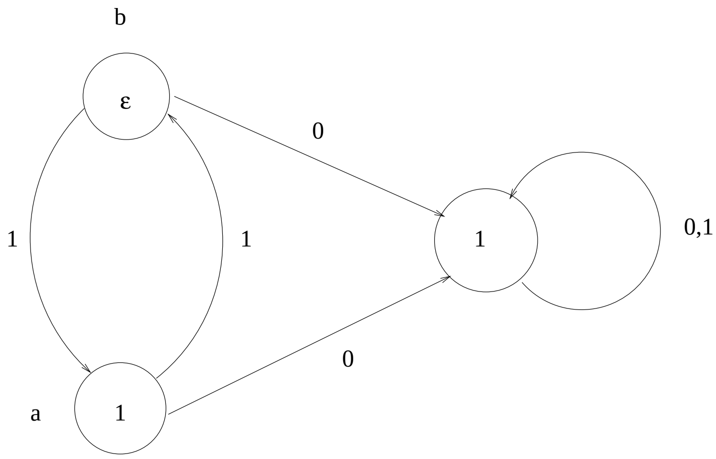

Figure 15: The automaton B

This group was studied in [15] and [16] where many interesting facts about algebraic and algorithmic properties of it were proven. In [2] it was shown that H is the iterated monodromy group of the map $z \to z^2 - 1$ and that the geometry of the Schreier graphs of this group is closely related to geometry of the Julia set of the map. An open question about H related to some famous problems is whether this group is amenable or not [16].

Now we are going to restate some results from [16] about spectral properties of H and to discuss them in connection with the subject of this paper.

We consider the quasi-regular representations ρ and ρ_n of H in the spaces $l^2(H/P)$ and $l^2(H/P_n)$. Let us consider the operators

$$M(\lambda, \eta) = \rho(a) + \rho(a^{-1}) + \lambda(\rho(b) + \rho(b^{-1})) - \eta I$$

and

$$M_n(\lambda, \eta) = \rho_n(a) + \rho_n(a^{-1}) + \lambda(\rho_n(b) + \rho_n(b^{-1})) - \eta I_n.$$

The operators M_n are given by $2^n \times 2^n$ matrices. Let $Q_n = \det M_n$. For instance

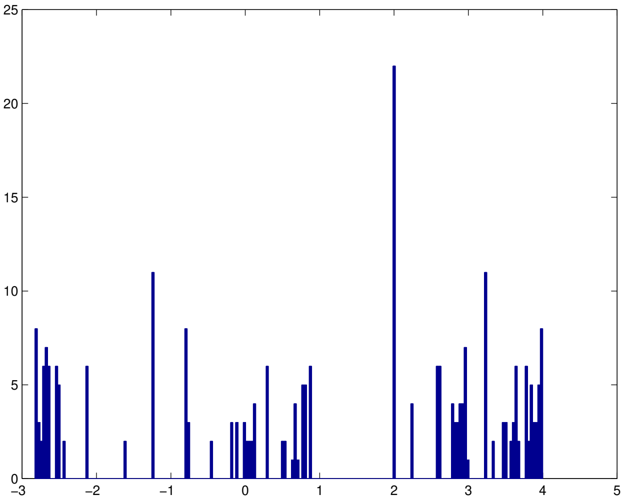

Figure 16: The histogram of the spectrum in case of the group H

$$
\begin{aligned}
Q_1(\lambda,\eta) &= 2\eta+2-\lambda,\\
Q_2(\lambda,\eta) &= -(2\eta-2+\lambda)(2\eta+2-\lambda),\\
Q_3(\lambda,\eta) &= (2\eta+2-\lambda)(2\eta-2+\lambda)(4\eta^2-\lambda^2+4)\\
Q_4(\lambda,\eta) &= -(-2+\lambda)(2\eta+2-\lambda)(2\eta-2+\lambda)\\
&\quad (2\lambda^2-8-\lambda^3+4\lambda+4\eta^2\lambda)(4\eta^2-\lambda^2+4)\\
Q_5(\lambda,\eta) &= (-2+\lambda)^2(2\eta+2-\lambda)(2\eta-2+\lambda)(4\eta^2+2\lambda-\lambda^2)\\
&\quad (4\eta^2+4-\lambda^2)(4\eta^2\lambda-8+4\lambda+2\lambda^2-\lambda^3)\\
&\quad (16\eta^4\lambda-8\eta^2\lambda^3+16\eta^2\lambda^2+8\eta^2\lambda-16\eta^2\\
&\quad +\lambda^5-4\lambda^4+16\lambda^2-16\lambda)\\
Q_6(\lambda,\eta) &= -(-2+\lambda)^3(2\eta-2+\lambda)(2\eta+2-\lambda)\\
&\quad (2\lambda^2-8-\lambda^3+4\lambda+4\eta^2\lambda)(4\eta^2-\lambda^2+4)\\
&\quad (16\eta^4\lambda-16\eta^2+8\eta^2\lambda-16\lambda+16\eta^2\lambda^2+16\lambda^2-8\eta^2\lambda^3\\
&\quad -4\lambda^4+\lambda^5)(8\eta^4-16+12\eta^2\lambda+16\lambda-6\eta^2\lambda^2-4\lambda^3+\lambda^4)\\
&\quad (-8\lambda^8+16\lambda^7-384\eta^2\lambda-160\eta^6\lambda^3-320\eta^2\lambda^2+72\eta^4\lambda^5\\
&\quad -14\eta^2\lambda^7-512\lambda^2+256\eta^2+256\lambda-608\eta^4\lambda+608\eta^4\lambda^2\\
&\quad +128\lambda^4+256\lambda^3+128\eta^8\lambda+736\eta^2\lambda^3+320\eta^6\lambda^2+96\eta^6\lambda\\
&\quad -272\eta^2\lambda^4-160\lambda^5+136\eta^4\lambda^3-288\eta^4\lambda^4+32\lambda^6+\lambda^9\\
&\quad -104\eta^2\lambda^5+84\eta^2\lambda^6-192\eta^6)(4\eta^2+2\lambda-\lambda^2)^2
\end{aligned}
$$

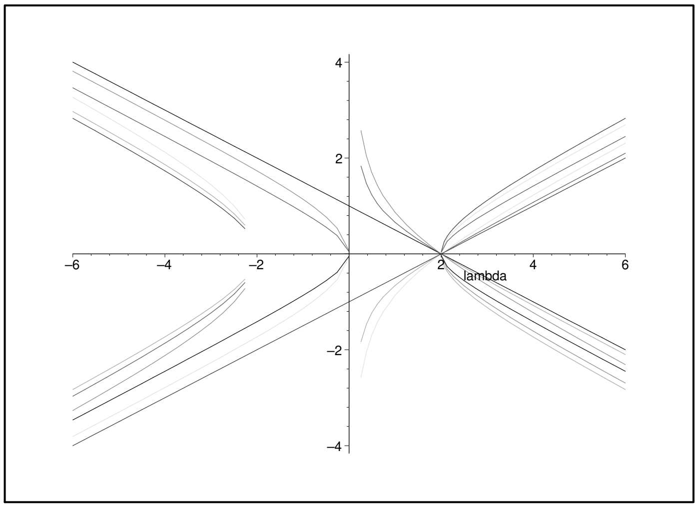

Figure 17: Zeros of $Q_n(\lambda, \eta)$

and we see that the degrees of factors grow. We have

Theorem 12.1 ([16]). *a) If* $n \geq 1$ *then*

$$Q_{n+1}(\lambda, \eta) = \lambda^{2^{n+1}} Q_n(F(\lambda, \eta))$$

where

$$F : \begin{cases} \lambda & \to & -2 - \frac{\lambda(2-\lambda)}{\eta^2} \\ \eta & \to & \frac{\lambda-2}{\eta^2} \end{cases} .$$

b) The spectrum Σ *of* $M(\lambda, \eta)$*, i.e. the set of pairs* (λ, η) *(including multiplicities) for which the operator* $M(\lambda, \eta)$ *is not invertible, is invariant with respect to the map* $F : \mathbb{R}^2 \to \mathbb{R}^2$*, i.e.* $F^{-1}(\Sigma) = \Sigma$.

The spectrum Σ_n of M_n is given in Figure 16 while the set of accumulation points of the set $\cup_{n=1}^{\infty} F^{-n}(x_0, y_o)$ is shown on Figure 18, suggesting that the map F should have an attractor topology which is very different from the topology of the attractor shown on Figure 5.

We see that the situation for this example is more complicated than in the previous examples. The degree of the curves into which splits Σ_n is growing while in the previous examples it was constant and equal to 2. At the moment we are

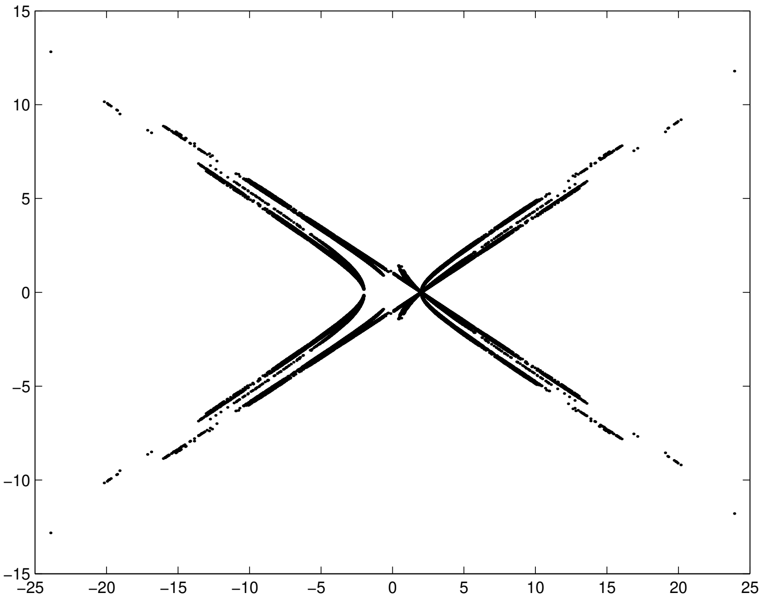

Figure 18: The attractor associated to the automaton from Figure 15

not able to solve the problem of complete factorization of $Q_n(\lambda, \eta)$ and we do not know if the map F is integrable in the sense of (8.10). Looking at the picture of the histogram of the distribution of the spectrum of the operators

$$M_n(1, \eta) = \rho_n(a + a^{-1} + b + b^{-1})$$

we can guess that the spectrum of the Markov operator

$$M(1, 0) = \rho(a + a^{-1} + b + b^{-1})$$

is a totally disconnected set but this is only a conjecture.

13. Related topics and concluding remarks

There are three ways of averaging in case of action of a group. First of all one can use Følner averaging which works for amenable groups and was used in dynamical systems (for instance in ergodic theorems) and in statistical physics (when defining averaged statistical sum) [5].

The second way is to use the von Neumann trace (this was used in [29]).

The third way (which works for residually finite groups) is to use approximation by finite quotients (see [12] for the definition of the averaged free energy on infinite residually finite groups and [7, 30] for computation of L^2 Betti numbers).

The second approach was used in this paper to define Ihara zeta function for residually finite graphs. One may ask if the first method works for amenable graphs. Our answer is yes at least in the situation of Cayley graphs. Here are our arguments.

Let G be an amenable group generated by a finite set S. Thus there exists a sequence of finite sets $A_n \subset G$, such that for any $r \geq 1$

$$\lim_{n\to\infty} \frac{|\partial_r A_n|}{|A_n|} = 0, \tag{13.13}$$

where $\partial_r A_n$ denotes the elements in $G \setminus A_n$ which are at distance at most r from A_n (by a distance we mean the distance on the Cayley graphs (G, S)).

Our idea is to define the zeta function for the graph using relation (4.5). Therefore we need to show that for any r the limit

$$c_r(A_n)/|A_n|$$

exists when n tends to infinity and A_n is considered as a subgraph of $S(G, S)$.

Proposition 13.1. *Let G be an amenable group and let $A_n \subset G$ be a Følner sequence. Then the limit*

$$c_r(A_n)/|A_n|$$

exists.

Proof. For any m, n we have

$$c_r(A_n)/|A_n \cup \partial_r A_n| \leq c_r(A_m \cup \partial_r A_m)/|A_m| \leq (c_r(A_m) + |S|^r|\partial_r A_m|)\,/|A_m|.$$

This implies that when $n, m \to \infty$

$$|c_r(A_n)/|A_n| - c_r(A_m)/|A_m||$$

tends to zero. Thus the sequence $c_r(A_n)/|A_n|$ indeed converges to some limit. □

We can use the relation (4.6) to define $\widetilde{\zeta}_X(t)$.

Our definition of the Ihara function of an infinite graph X depends on the approximating sequence $\{X_n\}$ of finite graphs, but we do not have an example of two approximating sequences $\{X_n\}$ and $\{Y_n\}$ of the same graph, which lead to different zeta functions. In particular this question is open for Schreier graphs related to automata groups.

It is not clear how our definition of the Ihara zeta function of Cayley graphs of amenable groups agrees with other definitions. For simple examples like $\mathbb{Z}^d$ this can be checked by a direct computation.

It is also an open problem for which finite automata one can associate a multidimensional map F relating the n-th and $n+1$-th determinants and whether such a map is integrable. New examples in this direction would be welcome. Such examples could provide new examples of spectra of Schreier graphs and Cayley graphs related to automata groups.

In all examples from this paper with an integrable map F, the spectrum of the Markov operator is an intersection of the line $\{\lambda = const\}$ with this part of $Im(F)$ which corresponds to the chaotic region of behavior of the map α (this is the case for the examples from Sections 10 and 11 in which $\alpha = z^2 - 1, 4 - z - z^2, 6 + z - z^2$) or the chaotic behavior of the map F on the integral lines (which is the case for the lamplighter group, when F on l_c acts as a rotation of a circle). It would be interesting to understand better this phenomenon with irrational angle for most values of $|c| < 4$.

In analogy with the finite case, one can define an infinite k-regular Ramanujan graph by the condition $spM \subset [-\frac{2\sqrt{k-1}}{k}, \frac{2\sqrt{k-1}}{k}]$. For a residually finite graph this corresponds to the situation when all zeros are on $Re(z) = \frac{1}{2}$. The characterization of such graphs in terms of the cogrowth follows from the formula for the spectral radius found in [9] (namely as graphs with cogrowth at most $\sqrt{2m-1}$). Approximating an infinite Ramanujan graph X (which is not necessary a tree) by finite ones could possibly produce new examples of finite Ramanujan graphs or at least of expanders.

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References

[1] L. Bartholdi, R. I. Grigorchuk, *On the spectrum of Hecke type operators related to some fractal groups*, Tr. Mat. Inst. Steklova 231 (2000), 5–45.

[2] L. Bartholdi, R. I. Grigorchuk, V. Nekrashevich *From fractal groups to fractal sets*, Proceedings of the Graz conference on Fractals (to appear).

[3] H. Bass, *The Ihara-Selberg zeta function of a tree lattice*, Internat. J. Math. 3 (1992), no. 6, 717–797.

[4] D. M. Cvetković, M. Doob, H. Sachs, *Spectra of graphs, Theory and applications*, Third edition, Johann Ambrosius Barth, Heidelberg, 1995.

[5] J. Dodziuk, V. Mathai, *Approximating L^2 invariants of amenable covering spaces: a combinatorial approach*, J. Funct. Anal. 154 (1998), no. 2, 359–378.

[6] A. E. Eremenko, *Some functional equations connected with the iteration of rational functions*, Algebra i Analiz 1 (1989), no. 4, 102–116; translation in Leningrad Math. J. 1 (1990), no. 4, 905–919.

[7] M. Farber, *Geometry of growth: approximation theorems for L^2 invariants*, Math. Ann. 311 (1998), no. 2, 335–375.

[8] F. Gecseg, I. Peak, *Algebraic theory of automata*, 1972, Disc. Math. Hung., Budapest.

[9] R. I. Grigorchuk, *Symmetrical random walks on discrete groups*, Multicomponent random systems, pp. 285–325, Adv. Probab. Related Topics, 6, Dekker, New York, 1980.

[10] R. I. Grigorchuk, *On Burnside's problem on periodic groups*, Funktsional. Anal. i Prilozhen. 14 (1980), no. 1, 53-54.

[11] R. I. Grigorchuk, *On the Milnor problem of group growth*, Dokl. Akad. Nauk SSSR 271 (1983), no. 1, 30–33.

[12] R. I. Grigorchuk, A. M. Stepin, *Gibbs states on countable groups*, Teor. Veroyatnost. i Primenen. 29 (1984), no. 2, 351–354.

[13] R. I. Grigorchuk, A. Żuk, *On the asymptotic spectrum of random walks on infinite families of graphs*, Random walks and discrete potential theory (Cortona, 1997), 188–204, Sympos. Math., XXXIX, Cambridge Univ. Press, Cambridge, 1999.

[14] R. I. Grigorchuk, A. Żuk, *The lamplighter group as a group generated by a 2-state automaton and its spectrum*, Geometriae Dedicata, 87, p. 209–244, 2001.

[15] R. I. Grigorchuk, A. Żuk, On a torsion-free weakly branch group defined by a three state automaton, *Internat. J. Algebra Comput.*, Vol. 12, No. 1 (2002), 1-24.

[16] R. I. Grigorchuk, A. Żuk, *On the spectrum of a torsion-free weakly branch group defined by a three state automaton*, in "Computational and Statistical Group Theory" (Ed. by R. Gilman, A. Myasnikov, V. Shpilrain), Contemp. Math., Amer. Math. Soc., vol. 298, 2002, 57–82.

[17] R. I. Grigorchuk, A. Żuk, *Self-similar C^* algebras and spectra of Markov elements*, (in preparation).

[18] R. I. Grigorchuk, P. Linnell, T. Schick, A. Żuk, *On a question of Atiyah*, C. R. Acad. Sci. Paris, t. 331, Série I, p. 663–668, 2000.

[19] R. I. Grigorchuk, V. V. Nekrashevych, V. I. Sushchansky, *Automata, dynamical systems and groups*,in Proceedings of the Steklov Institute of Mathematics, vol. 231, p. 134–214, 2000.

[20] R. I. Grigorchuk, J. S. Wilson,*The conjugacy problem for certain branch groups*, Tr. Mat. Inst. Steklova 231 (2000), Din. Sist., Avtom. i Beskon. Gruppy, 215–230;

[21] M. Hall, *Coset representation in free groups*, Trans. Amer. Math. Soc. 67 (1949) 421-432.

[22] K. Hashimoto, *Zeta functions of finite graphs and representations of p-adic groups*, Avanced Studies in Pure Mathematics 15 91989), in: Automorphisc Forms and Geometry of Arithmetic Varieties, 211-280.

[23] Y. Ihara, *On discrete subgroups of the two by two projective linear group over p-adic fields*, J. Math. Soc. of Japan 18 (1966), 219-235.

[24] H. Kesten, *Symmetric random walks on groups*, Trans. Amer. Math. Soc., Vol. 92, 1959, 336-354.

[25] M. Kotani, T. Sunada, *Zeta Functions of Finite Graphs*, J. Math. Sci. Univ. Tokyo 7 (2000), 7–25.

[26] W. Li, *Geometry, graph theory and number theory*, Algebra and geometry (Taipei, 1995), 83–102, Lect. Algebra Geom., 2, Internat. Press, Cambridge, MA, 1998.

[27] A. Lubotzky, *Discrete Groups, Expanding Graphs and Invariant Measures*, Birkhäuser, 1994.

[28] M. Lyubich, *The quadratic family as a qualitatively solvable model of chaos*, Notices Amer. Math. Soc. 47 (2000), no. 9, 1042–1052.

[29] F. Lund, M. Rasetti, T. Regge, *The Ising model and dimer model on the Lobachevskii plane*, Theoretical and Mathematical Physics, 1977, vol. 33, n. 2, 246–271.

[30] W. Lück, *Approximating L^2-invariants by their finite-dimensional analogues*, Geom. Funct. Anal. 4 (1994), no. 4, 455–481.

[31] W. Lück, *L^2-Invariants: Theory and Applications to Geometry and K-theory*, Mathematics Monograph, Springer 2002.

[32] W. S. Massey, *A basic course in algebraic topology*, Graduate Texts in Mathematics, 127 Springer-Verlag, New York, 1991.

[33] B. Mohar, W. Woess, *A survey on spectra of infinite graphs*, Bull. London Math. Soc. 21 (1989), no. 3, 209–234.

[34] H. Nagoshi, *On arithmetic infinite graphs*, Proc. Japan Acad. Ser. A Math. Sci. 76 (2000), no. 2, 22–25.

[35] S. Northshield, *Two proofs of Ihara's theorem*, Emerging applications of number theory (Minneapolis, MN, 1996), 469–478, IMA Vol. Math. Appl., 109, Springer, New York, 1999.

[36] S. Northshield, *A note on the zeta function of a graph*, J. Combin. Theory Ser. B 74 (1998), no. 2, 408–410.

[37] P. Sarnak, *Some applications of modular forms*, Cambridge Tracts in Mathematics, 99, Cambridge University Press, Cambridge, 1990.

[38] J.-P. Serre, *Répartition asymptotique des valeurs propres de l'opérateur de Hecke* T_p, J. Amer. Math. Soc. 10 (1997), no. 1, 75–102.

[39] H. Stark, *Multipath zeta functions of graphs*, Emerging applications of number theory (Minneapolis, MN, 1996), 601–615, IMA Vol. Math. Appl., 109, Springer, New York, 1999.

[40] H. Stark, A. Terras, *Zeta functions of finite graphs and coverings*, II. Adv. Math. 154 (2000), no. 1, 132–195.

[41] T. Sunada, *Fundamental groups and Laplacians*, in: Lecture Notes in Mathematics 1339, (T. Sunada editor), Springer Verlag, (1987), 248-277.

[42] A. P. Veselov, *Integrable mappings*, Uspekhi Mat. Nauk 46 (1991), no. 5(281), 3–45, 190; translation in Russian Math. Surveys 46 (1991), no. 5, 1–51

[43] W. Woess, *Random walks on infinite graphs and groups*, Cambridge Tracts in Mathematics, 138. Cambridge University Press, Cambridge, 2000.

[44] *From number theory to physics*, Papers from the Meeting on Number Theory and Physics held in Les Houches, March 7–16, 1989, Edited by M. Waldschmidt, P. Moussa, J. M. Luck and C. Itzykson. Springer-Verlag, Berlin, 1992.

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