Math437, Exam 1, 10/04/2016

Name: Key

SHOW ALL WORK!

Problem 1. Let $g(x) = (x+10)^{\frac{1}{3}}$.

15 **(a)** Prove that the fixed point iteration $p_{n+1} = g(p_n)$ for $n = 0, 1, \ldots$, converges to the fixed point p of $x = g(x)$ if the starting guess p_0 is in the interval $[-2, 17]$.

(A) $\frac{1}{3} \cdot 27^{-2/3} \le g'(x) = \frac{1}{3}(x+10)^{-2/3} \le \frac{1}{3} 8^{-2/3}$

$\Rightarrow \quad |g'(x)| \le \frac{1}{12}$ for $-2 \le x \le 17$

(B) If $-2 \le x \le 17$ then $g(-2) \le g(x) \le g(17)$

and $-2 < g(-2) = 2 \le g(x) \le g(17) = 3 < 17$

We used that g and g' are monotone on $[-2, 17]$

$[a, b] = [-2, 17]$

10 **(b)** Derive an error estimate for $|p_n - p|$ and find an upper bound for the number of iterations needed to achieve accuracy 10^{-4}.

$$|p_n - p| \le \left(\frac{1}{12}\right)^n \max(p_0 - a, b - p_0) \le \frac{19}{12^n}$$

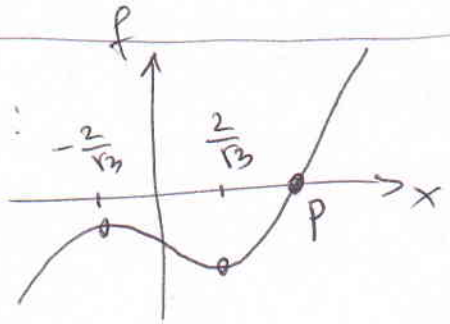
If $n \geq 5$ then $\frac{19}{12^n} < \frac{1}{10000}$

Problem 2. Let $f(x) = x^3 - 4x - 2016$. Show that f has only one real zero p, $f(p) = 0$. Write the Newton's method with a starting guess $p_0 = 20$ and show that the sequence $\{p_n\}$ satisfies $p < p_{n+1} < p_n$ for all $n = 0, 1, \ldots$.

$f'(x) = 3x^2 - 4$ $\qquad$ $f'(x) = 0 \iff x = \pm \frac{2}{\sqrt{3}}$

$f''(x) = 6x$; $\qquad$ $f\left(\pm \frac{2}{\sqrt{3}}\right) = \pm \frac{8}{3\sqrt{3}} \mp \frac{8}{\sqrt{3}} - 2016 < 0$

Hence :

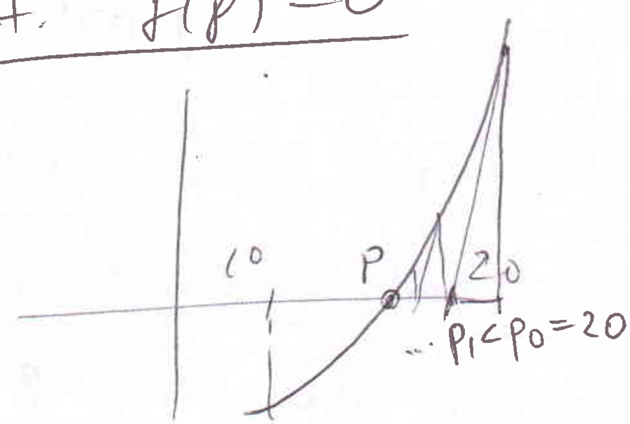


there is a unique $p > \frac{2}{\sqrt{3}}$ s.t. $f(p) = 0$

$f(10) < 0 < f(20) \Rightarrow$

$f'(x) > 0$ on $[10, 20]$

$f''(x) > 0$ on $[10, 20)$

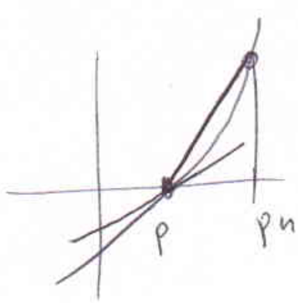


(*) f = increasing, concave up on $[10, 20]$, then

Geometry : $\quad p_0 > p \Rightarrow p < p_{n+1} < p_n$

Algebraic proof: ① $p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} < p_n$ (using (*))

② $p_{n+1} - p = p_n - p - \frac{f(p_n)}{f'(p_n)} = \frac{(p_n - p) f'(p_n) - f(p_n)}{f'(p_n)} \geq 0$



$\to$ Concave up $\Rightarrow f'(p) \leq \boxed{\frac{f(p_n) - f(p)}{p_n - p}} \leq f'(p_n)$

$\Rightarrow p_{n+1} - p > 0$

Problem 3. Find an LU factorization of $A = \begin{pmatrix} 2 & -2 & 2 & 2 \\ 3 & 3 & 9 & 1 \\ 3 & 3 & 7 & 1 \\ 3 & 3 & 7 & 9 \end{pmatrix}$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 3/2 & 1 & 0 & 0 \\ 3/2 & 1 & 1 & 0 \\ 3/2 & 1 & 1 & 1 \end{pmatrix}}_{L} \cdot \underbrace{\begin{pmatrix} 2 & -2 & 2 & 2 \\ 0 & 6 & 6 & -2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix}}_{U} = A$$

Problem 4. Let $a(x, y)$ be a nonnegative, symmetric bilinear form on $\mathbb{R}^n$. That is, for all $x, y, w \in \mathbb{R}^n$. and any $s \in \mathbb{R}$ we have:

$$a(x, x) \geq 0,\ a(x, y) = a(y, x), \text{ and } a(x + sy, w) = a(x, w) + sa(y, w).$$

Prove that for all $x, y \in \mathbb{R}^n$ we have $a(x, y) \leq \sqrt{a(x, x)}\sqrt{a(y, y)}$ even if $a(x, y)$ is degenerate, i.e., $a(x, x) = 0$ for some $x \neq 0$.

Let $\underline{f(t) = a(x+ty, x+ty) \quad \text{for } t \in \mathbb{R}}$

$f(t) \geq 0 \quad \forall t$ because $\underline{a}(\cdot,\cdot)$ is non-negative, but

$$f(t) = \underbrace{a(y,y)}_{A} t^2 + \underbrace{2a(x,y)}_{B} t + \underbrace{a(x,x)}_{C} \geq 0$$

f = quadratic polynomial in t

$$f \geq 0 \iff \left| \begin{array}{l} A, C \geq 0 \\ \mathcal{D} = B^2 - 4AC \leq 0 \end{array} \right.$$

even if A or $C = 0$

Then $\underline{\cancel{4}\, a(x,y)^2 \leq \cancel{4}\, a(x,x)\, a(y,y)}$ $\square$