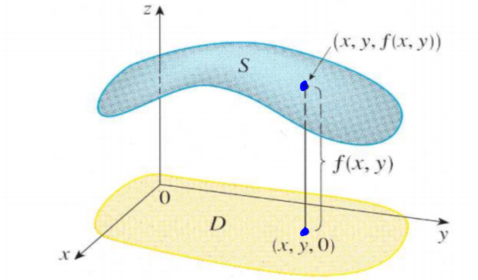
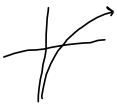



$z = f(x,y)$

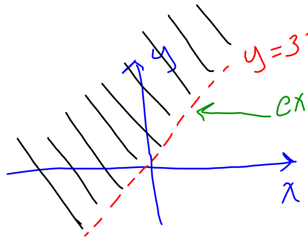
1. Find and sketch the domain of the following functions.

a.) $f(x, y) = \ln(y - 3x)$




$y - 3x > 0 \longrightarrow y > 3x$

strict inequality since $\ln(0)$ does not exist

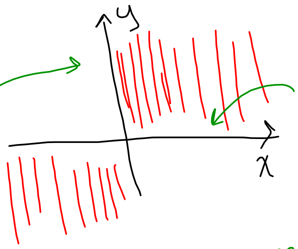



excludes the line $y=3x$

b.) $f(x, y) = \sqrt[4]{xy}$

$xy \geq 0$

includes y-axis



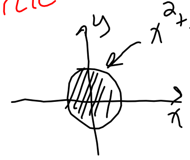

includes x-axis

c.) $f(x, y) = \dfrac{\sqrt{8 - x^2 - y^2}}{x + 2y}$

square root need $8 - x^2 - y^2 \geq 0$

$8 \geq x^2 + y^2$

inside and on the circle $x^2 + y^2 = 8$

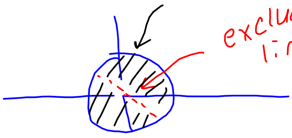



$x + 2y \neq 0$

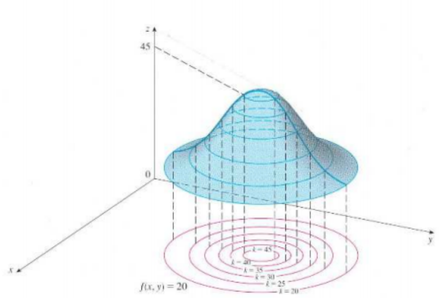
$y \neq -\frac{1}{2}x$

includes circle

exclude line $y = -\frac{1}{2}x$



**Definition**: The **level curves** of a function $f$ of two variables are the curves with equations $f(x, y) = k$, where $k$ is a constant in the range of $f$. In other words, a level curve shows where the graph of $f$ has height $k$. The level curves $f(x, y)$ are just the horizontal traces of the graph of $f$ in the plane $z = k$ projected down to the $xy$ plane. A graph of the level curves is called a **countour plot**.



## 2. Sketch several level curves for the following surfaces:

a.) $f(x, y) = 2 + 4x - y$

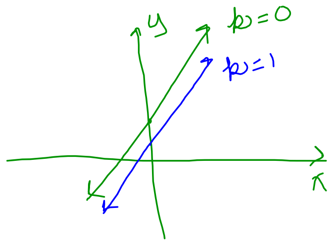
$z = 2 + 4x - y$

$\underline{z = 0}$ $\quad 0 = 2 + 4x - y$

$y = 2 + 4x$

$z = 1$ $\quad 1 = 2 + 4x - y$

$y = 4x + 1$




level curves

family of parallel lines

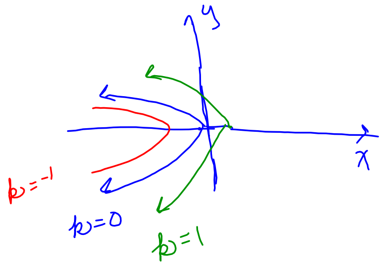
$(m = 4)$

b.) $f(x, y) = x + y^2$

$z = x + y^2$

$z = 0$ $\quad 0 = x + y^2$

$x = -y^2$



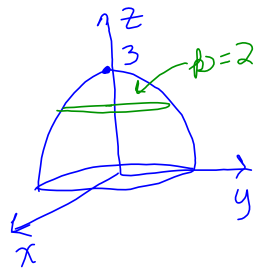

Family of parabolas

$z = 1$ $\quad 1 = x + y^2$

$x = 1 - y^2$

c.) $\boxed{f(x,y) = \sqrt{9 - x^2 - y^2}}$

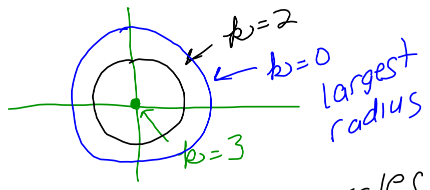
$z = \sqrt{9 - x^2 - y^2}$ $\quad 0 \le k \le 3$




$k=3 \qquad 3 = \sqrt{9 - x^2 - y^2}$

$9 = 9 - x^2 - y^2$

$0 = x^2 + y^2$




Family of circles and a point

$k = 2 \qquad 2 = \sqrt{9 - x^2 - y^2}$

$4 = 9 - x^2 - y^2$

$x^2 + y^2 = 5$

$k = 0 \qquad 0 = \sqrt{9 - x^2 - y^2}$

$x^2 + y^2 = 9$

d.) $f(x, y) = \sqrt{x^2 - y^2}$

$z = \sqrt{x^2-y^2}$

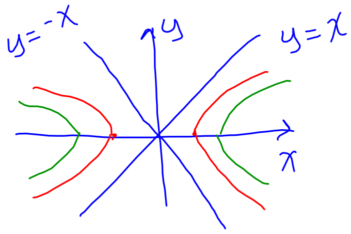
only choose non negative values of $k$.

$k = 0 \longrightarrow 0 = \sqrt{x^2-y^2}$

$0 = x^2 - y^2$

$y^2 = x^2$

$y = \pm x$

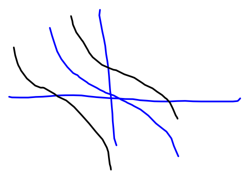



Family of hyperbolas

$k \neq 0 \quad k = 1 \quad \boxed{1 = x^2 - y^2}$

one more $f(x,y) = x^3 + y$

$z = x^3 + y$



3. Describe the level surfaces of $f(x, y, z) = x + y + z$.

$w = x + y + z$

level surfaces of form $k = x + y + z$

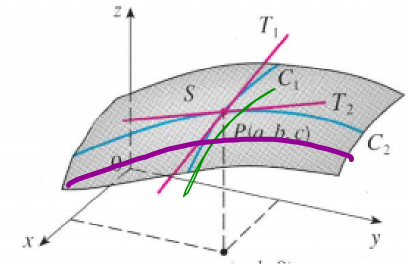
$k$ constant family of parallel planes

4. Describe the level surfaces of $f(x, y, z) = x^2 + y^2 + z^2$.

$w = x^2 + y^2 + z^2$

for every positive $k$, $k = x^2 + y^2 + z^2$ is a sphere

Family of spheres centered at origin




Fix y cut surface with A plane parallel to the xz plane

① differentiate $z = f(x,y)$ with respect to x, holding y constant denoted by $f_x(x,y)$ = slope of $C_1$

② differentiate $z = f(x,y)$ with respect to y, holding x constant denoted by $f_y(x,y)$ slope of $C_2$

5. Find $f_x(-1, 2)$ and $f_y(-1, 2)$ for $f(x, y) = x^3 - y^4 - 6x^2y^3$

- $f_x(x,y) = 3x^2 - 0 - 12xy^3$

  $f_x(-1,2) = 3 + 12(8)$

- $f_y(x,y) = 0 - 4y^3 - 6x^2 \cdot 3y^2$

  $f_y(-1,2) = -4(8) - 6(12)$

6. Find $f_x(x, y)$ and $f_y(x, y)$ for $f(x, y) = x^2 e^{\cos(2x^4y^2)}$

$f_x(x,y)$ requires product rule

$$f_x(x,y) = 2x e^{\cos(2x^4y^2)} + x^2\left(e^{\cos(2x^4y^2)}\left(-\sin(2x^4y^2)8x^3y^2\right)\right)$$

$$f_y(x,y) = x^2\left(e^{\cos(2x^4y^2)}\left(-\sin(2x^4y^2)\right)(4x^4y)\right)$$

7. Find all higher order partial derivatives for $f(x, y) = \ln(2x + 3y)$

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$= \frac{\partial}{\partial x}\left(\frac{2}{2x+3y}\right)$$

$$= \frac{\partial}{\partial x}\, 2(2x+3y)^{-1}$$

$$= -2(2x+3y)^{-2}\cdot 2$$

$$f_{xx} = \frac{-4}{(2x+3y)^2}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{2}{2x+3y}\right)$$

$$= \frac{-6}{(2x+3y)^2} = f_{yx}$$

$f_{xy} = f_{yx}$ always!

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{3}{2x+3y}\right)$$

$$= \frac{-9}{(2x+3y)^2}$$

8. Find the equation of the tangent plane to the surface $z = x^3 - 3y^2$ at the point $(-1, 1, -4)$

The equation of the tangent plane at $(x_0, y_0, z_0)$ is $\boxed{z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$

$x_0 = -1$, $y_0 = 1$, $z_0 = -4$

$f_x(x,y) = 3x^2$ $\boxed{f_x(-1,1) = 3}$

$f_y(x,y) = -6y$ $\boxed{f_y(-1,1) = -6}$

$z + 4 = f_x(-1,1)(x+1) + f_y(-1,1)(y-1)$

$\boxed{z + 4 = 3(x+1) - 6(y-1)}$

9. Find the equation of the tangent plane to the surface $z = e^{x-y}$ at the point $(2, 2, 1)$. What is the equation of the normal line to this tangent plane the point $(2, 2, 1)$?

$x_0 = 2$
$y_0 = 2$
$z_0 = 1$

$$\boxed{z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

$$z - 1 = f_x(2,2)(x-2) + f_y(2,2)(y-2)$$

$f(x, y) = z = e^{x-y}$

$f_x(x, y) = e^{x-y} \longrightarrow f_x(2,2) = 1$

$f_y(x, y) = -e^{x-y} \longrightarrow f_y(2,2) = -1$

$\boxed{z - 1 = x - 2 - (y-2)} \rightarrow z = x - y + 1$

$\vec{n} = \langle 1, -1, -1 \rangle$

$0 = x - y - z + 1$

normal line is perpendicular to the tangent plane.

line in space $\vec{r}_0 + t\vec{v}$

$r_0 = (2, 2, 1)$

$v = \langle 1, -1, -1 \rangle$

$\langle 2, 2, 1 \rangle + t \langle 1, -1, -1 \rangle$

$$\boxed{\begin{aligned} x &= 2 + t \\ y &= 2 - t \\ z &= 1 - t \end{aligned}}$$

10. Find the differential of $z = e^{-2x}\sin(\pi y)$.

Recall: $y = f(x)$ differential

$\boxed{dy = f'(x)\,dx}$

$z = f(x,y)$

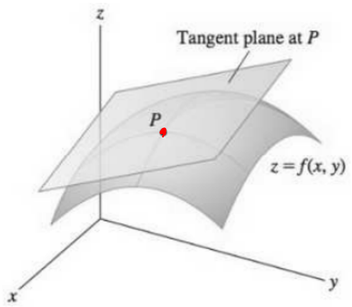
the differential $dz = f_x(x,y)\,dx + f_y(x,y)\,dy$

$z = e^{-2x}\sin(\pi y)$

$dz = -2e^{-2x}\sin(\pi y)\,dx + e^{-2x}(\cos(\pi y)\pi)\,dy$

11. Use differentials to approximate $f(1.02, 0.97)$ for $f(x,y) = 1 - xy\cos(\pi y)$

$f(1,1) = 1 - \cos(\pi) = 2$




tangent plane approximates a surface at its point of tangency.

① Find the tangent plane to $f(x,y) = 1 - xy\cos(\pi y)$ at $(1, 1, 2)$.

$x_0 = 1,\ y_0 = 1,\ z_0 = 2$

$$z - z_0 = \underbrace{f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)}_{dz}$$

$z - 2 = x - 1 + y - 1$

$\boxed{z = x + y}$ tangent plane

$f(x,y) = 1 - xy\cos(\pi y)$

$f_x(x,y) = -y\cos(\pi y)$, $f_x(1,1) = 1$

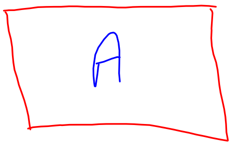
$f_y(x,y) = -x\cos(\pi y) - xy(-\sin(\pi y)\pi)$

$f_y(1,1) = -1\cos(\pi) = 1$

use tangent line to approximate $f(1.02, .97)$

$f(1.02, .97) \approx 1.02 + .97$
$= 1.99$

12. The length and width of a rectangle are measured as 30 cm and 24 cm, respectively, with an error in measurement of 0.1 cm in both. Use differentials to approximate the maximum error in the calculated area of the rectangle.




$w \approx 24$

$|dw| < .1$

$\ell \approx 30$

$|d\ell| < .1$

Approximate $\Delta A$ by using $\boxed{dA}$

$\boxed{A = \ell w}$ max value of

$\Delta A = A(24.1, 30.1) - A(24, 30)$

$$A = \ell w$$

$$dA = (A_\ell)\, d\ell + (A_w)\, dw$$

$$= w\, d\ell + \ell\, dw$$

max approximation of $dA = (24)(.1) + (30)(.1)$

$$= 2.4 + 3$$

$$= 5.4 \text{ cm}^2$$