Homework 1. Due FRIDAY (only for the first time), August 23, 9:10 am.

Problem 1. *Problem 1.6 from Classical Mechanics by J. R. Tylor*
By evaluating their dot product, find the values of the scalar s for which the two vectors $\vec{b} = \hat{x} + s\hat{y}$ and $\vec{c} = \hat{x} - s\hat{y}$ are orthogonal. (Remember that two vectors are orthogonal if and only of their dot product is zero.) Explain your answers with a sketch.

Problem 2. *Problems 1.8 from Classical Mechanics by J. R. Tylor*

1. Use the definition $\vec{r} \cdot \vec{s} = r_1 s_1 + r_2 s_2 + r_3 s_3$ to prove that the scalar product is distributive; that is, that $\vec{r} \cdot (\vec{u} + \vec{v}) = \vec{r} \cdot \vec{u} + \vec{r} \cdot \vec{v}$.

2. If $\vec{r}$ and $\vec{s}$ are vectors that depend on time, prove that the product rule for differentiating products applies to $\vec{r} \cdot \vec{s}$, that is, that
$$\frac{d}{dt}(\vec{r} \cdot \vec{s}) = \vec{r} \cdot \frac{d\vec{s}}{dt} + \frac{d\vec{r}}{dt} \cdot \vec{s}.$$

Problem 3. *Problem 1.10 from Classical Mechanics by J. R. Tylor*
A particle moves in a circle (center O and radius R) with constant angular velocity ω counterclockwise. The circle lies in the $x - y$ plane and the particle lies on the x axis at time $t = 0$.

1. Show that the particle's position is given by
$$\vec{r}(t) = \hat{x} R \cos(\omega t) + \hat{y} R \sin(\omega t).$$

2. Find the particle's velocity and acceleration. What is the magnitude and the direction of the acceleration? Relate your answers to the well-known properties of uniform circular motion.

Problem 4. *Problem 1.17 from Classical Mechanics by J. R. Tylor*

1. Prove that the vector product $\vec{r} \times \vec{s}$ (as defined by the following $\vec{p} = \vec{r} \times \vec{s}$ means $p_x = r_y s_z - r_z s_y$, $p_y = r_z s_x - r_x s_z$, $p_z = r_x s_y - r_y s_x$) is distributive; that is, that $\vec{r} \times (\vec{u} + \vec{v}) = \vec{r} \times \vec{u} + \vec{r} \times \vec{v}$.

2. If $\vec{r}$ and $\vec{s}$ are vectors that depend on time, prove that the product rule for differentiating products applies to $\vec{r} \times \vec{s}$, that is, that
$$\frac{d}{dt}(\vec{r} \times \vec{s}) = \vec{r} \times \frac{d\vec{s}}{dt} + \frac{d\vec{r}}{dt} \times \vec{s}.$$

   Be careful with the order of the factors.

Problem 5. *Problem 1.19 from Classical Mechanics by J. R. Tylor*
If $\vec{r}$, $\vec{v}$, $\vec{a}$ denote the position, velocity, and acceleration of a particle, prove that

$$\frac{d}{dt}[\vec{a} \cdot (\vec{v} \times \vec{r})] = \dot{\vec{a}} \cdot (\vec{v} \times \vec{r}).$$

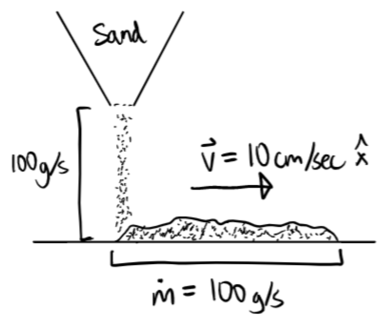
Problem 6. *Problem 1.23 from Classical Mechanics by J. R. Tylor*
The unknown vector $\vec{v}$ satisfies $\vec{b} \cdot \vec{v} = \lambda$ and $\vec{b} \times \vec{v} = \vec{c}$, where λ, $\vec{b}$, and $\vec{c}$ are fixed and known. Find $\vec{v}$ in terms of λ, $\vec{b}$, and $\vec{c}$.

Homework 2. Due August 28, at 9:10 am.

Problem 1. *Rising Snake*
A snake of length L and linear mass density ρ rises from the table. It's head is moving straight up with the constant velocity v. What force does the snake exert on the table?

Problem 2. *1983-Spring-CM-U-1.*
A ball, mass m, hangs by a massless string from the ceiling of a car in a passenger train. At time t the train has velocity $\vec{v}$ and acceleration $\vec{a}$ in the same direction. What is the angle that the string makes with the vertical? Make a sketch which clearly indicates the relative direction of deflection.

Problem 3. *1984-Fall-CM-U-1.*



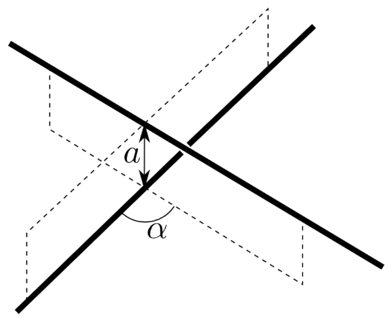
Sand drops vertically from a stationary hopper at a constant rate of 100 gram per second onto a horizontal conveyor belt moving at a constant velocity, $\vec{v}$, of 10 cm/sec.

1. What force (magnitude and direction relative to the velocity) is required to keep the belt moving at a constant speed of 10 cm/sec?
2. How much work is done by this force in 1.0 second?
3. What is the change in kinetic energy of the conveyor belt in 1.0 second due to the additional sand on it?
4. Should the answers to parts 2. and 3. be the same? Explain.

Problem 4. *1994-Fall-CM-U-1*
Two uniform very long (infinite) rods with identical linear mass density ρ do not intersect. Their directions form an angle α and their shortest separation is a.

1. Find the force of attraction between them due to Newton's law of gravity.
2. Give a dimensional argument to explain why the force is independent of a.

Note: for $A^2 < 1$, $\int_{-\pi/2}^{\pi/2} \frac{d\theta}{1-A^2\sin^2\theta} = \frac{\pi}{\sqrt{1-A^2}}$


a
α

Homework 3. Due Wednesday, September 4, at 9:10 am.

Problem 1. *Height for quadratic friction.*
A body of mass m was thrown straight up with initial velocity v_0. The force of the air resistance is $-\gamma v|v|$

- How long will it take for the body to reach the top of its trajectory?
- What the answer will be if $\gamma \to 0$.
- How long it will take for the body to reach the top of its trajectory if initial velocity is infinite?
- For finite initial velocity, how far up the body will go?
- What the answer will be if $\gamma \to 0$.

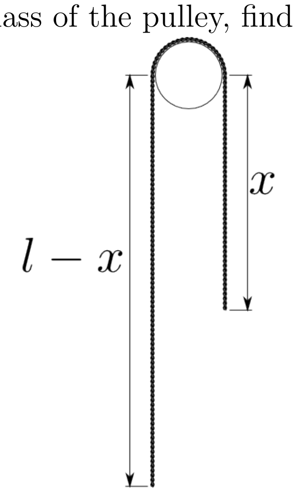
[Hint: $\int \frac{dx}{x^2+1} = \tan^{-1}(x)$, $\int \tan(x)dx = -\log(\cos(x))$, $\cos(\tan^{-1}(x)) = \frac{1}{\sqrt{1+x^2}}$.]

Problem 2. *Hole of Earth.*
Assume that the earth is a sphere, radius R and uniform mass density. Suppose a shaft were drilled all the way through the center of the earth from the north pole to the south. Suppose now a bullet of mass m is fired from the center of the earth, with velocity v_0 up the shaft. Assuming the bullet does not go beyond the earth's surface and neglecting the air resistance

- calculate in what time the bullet will return back?
- how does this time depend on v_0?
- how does this time depend on m?

Express all results using the acceleration of the free fall on the Earth's surface g.

Problem 3. *Rope*
A simple Atwood's machine consists of a heavy rope of length l and linear density ρ hung over a pulley (see Fig.). Neglecting the part of the rope in contact with the pulley, and the mass of the pulley, find $x(t)$ if the initial conditions are $x(t = 0) = l/2$ and $v(t = 0) = v_0$.



Homework 4. Due Wednesday, September 11, at 9:10 am.

Problem 1. *Problem 5.44 from Classical Mechanics by J. R.Tylor*
Another interpretation of the quality factor Q of a resonance comes from the following: Consider a motion of a driven damped oscillator (damping force $F_{dmp} = -2\beta mv$) after any transients have died out, and suppose that it is being driven close to resonance, so you can set $\omega = \omega_0$, where ω is the frequency of the force and ω_0 is the natural frequency of the oscillator.

1. Show that the oscillator's total energy (kinetic plus potential) is $E = \frac{1}{2}m\omega^2 A^2$, where A is the amplitude.

2. Show that the energy ΔE_{dis} dissipated during one cycle by the damping force F_{dmp} is $2\pi m\beta\omega A^2$. (Remember that the rate at which a force does work is Fv.)

3. Hence show that Q is 2π times the ratio $E/\Delta E_{dis}$.

Problem 2. *1985-Spring-CM-U-3.*
A damped one-dimensional linear oscillator is subjected to a periodic driving force described by a function $F(t)$. The equation of motion of the oscillator is given by

$$m\ddot{x} + b\dot{x} + kx = F(t),$$

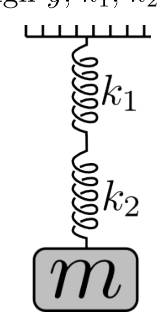
where $F(t)$ is given by

$$F(t) = F_0\left(1 + \sin(\omega t)\right).$$

The driving force frequency is $\omega = \omega_0$ and the damping by $b/2m = \omega_0$, where $\omega_0^2 = k/m$. At $t = 0$ the mass is at rest at the equilibrium position, so that the initial conditions are given by $x(0) = 0$, and $\dot{x}(0) = 0$. Find the solution $x(t)$ for the position of the oscillator vs. time.

Problem 3. *1986-Spring-CM-U-1.*
A mass m hangs vertically with the force of gravity on it. It is supported in equilibrium by two different springs of spring constants k_1 and k_2 respectively. The springs are to be considered ideal and massless.

Using your own notations (clearly defined) for any coordinates and other physical quantities you need develop in logical steps an expression for the net force on the mass if it is displaced vertically downward a distance y from its equilibrium position. (Clarity and explicit expression of your physical reasoning will be important in the evaluation of your solution to this problem. Express your result through y, k_1, k_2.)



Problem 4. **Extra for Honors**

Plot coordinate vs. time for an overdamped oscillator with initial conditions $x(t=0)=0$, $v(t=0)=v_0$.

- What is the slope of the graph at $t=0$?
- What is the dependence of coordinate on time at $t\to\infty$?
- Find the general solution for the damped oscillator in the case $\gamma=\omega_0$. Hint: The solution must still have two independent constants.

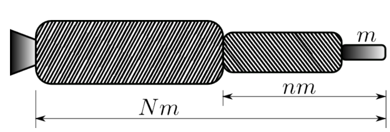
Homework 5. Due Wednesday, September 18, at 9:10 am.

Problem 1. *Rocket velocity and acceleration*
In gravity-free space a rocket of total initial mass M_0 and initial velocity $v_0 = 0$ at time $t = 0$ starts to burn fuel with the constant rate α. The thrusters stop when the total mass of the rocket is M_f. The burnt fuel is ejected at a speed (relative to the rocket) V.

- Sketch the dependence of the velocity on time. Clearly indicate the initial value, the slope of the graph at the initial point, the asymptotic value at large times.
- Sketch the dependence of the traveled distance on time. Clearly indicate the initial value, the slope of the graph at the initial point, the asymptotic dependence at large times.
- Sketch the dependence of the acceleration on time. Clearly indicate the initial value, the slope of the graph at the initial point, the asymptotic value at large times.

Problem 2. *1994-Fall-CM-U-2*
Suppose that the payload (e.g., a space capsule) has mass m and is mounted on a two-stage rocket (see figure below). The total mass — both rockets fully fueled, plus the payload — is Nm. The mass of the second-stage plus the payload, after first-stage burnout and separation, is nm. In each stage the ratio of burnout mass (casing) to initial mass (casing plus fuel) is r, and the exhaust speed is v_0.

1. Find the velocity v_1 gained from first-stage burn starting from rest (and ignoring gravity). Express your answer in terms of v_0, N, n, and r.
2. Obtain a corresponding expression for the additional velocity, v_2 gained from the second-stage burn.
3. Adding v_1 and v_2, you have the payload velocity v in terms of N, n, v_0, and r. Taking N, v_0, and r as constants, find the value for n for which v is a maximum.
4. Show that the condition for v to be a maximum corresponds to having equal gains of velocity in the two stages. Find the maximum value of v, and verify that it makes sense for the limiting cases described by $r = 0$ and $r = 1$.
5. You need to build a system to obtain a payload velocity of 10km/sec, using rockets for which $v_0 = 2.5$km/sec and $r = 0.1$. Can you do it with a two-stage rocket?
6. Find an expression for the payload velocity of a single-stage rocket with the same values of r and v_0. Can you reach a payload velocity of 10km/s with a single-stage rocket by taking the same conditions as in the previous point?



Problem 3. *Hovering rocket*
How a rocket should burn its fuel (what is $m(t)$) in order to hover in the Earth gravitational field close to the Earth surface.

Homework 6. Due Wednesday, September 25, at 9:10 am.

Problem 1. *Tensor of inertia.*
In the frame where the tensor of inertia is diagonal calculate the FULL TENSOR for:

1. A uniform disc of mass m and radius R.
2. A uniform solid (not hollow) sphere of mass m and radius R.
3. A thin stick of length L and mass m, if the origin is at one end of the stick.
4. A thin stick of length L and mass m, if the origin is at the center of the stick.

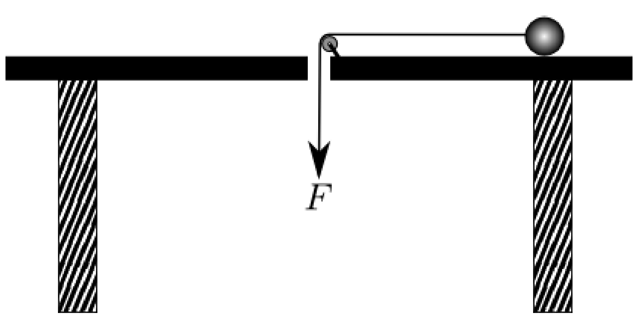
You have to show how you compute them. Just copying the answers from Wikipedia is not enough.

Problem 2. *1989-Spring-CM-U-2.*
A platform is free to rotate in the horizontal plane about a frictionless, vertical axle. About this axle the platform has a moment of inertia I_p. An object is placed on a platform a distance R from the center of the axle. The mass of the object is m and it is very small in size. The coefficient of friction between the object and the platform is μ. If at $t = 0$ a torque of constant magnitude τ_0 about the axle is applied to the platform when will the object start to slip?

Problem 3. *A ball on a string.*
A ball is on a frictionless table. A string is connected to the ball as shown in the figure. The ball is started in a circle of radius R with angular velocity ω_0. The force exerted on a string is then increases so that the distance between the hole and the ball decreases and is a given function of time $r(t)$. Assuming the string stays straight and that it only exerts a force parallel to its length.

1. Find the velocity of the ball as a function of time.
2. What is polar ϕ component of the force must be in order for this motion to be possible?

   [**Hint:** This question means that as from the previous question you know the full motion of the ball, you need to find the ϕ component of the acceleration and from it find the ϕ component of the force.]



Extra for Honors

Problem 4. *1990-Spring-CM-U-2*

A uniform rod, of mass m and length $2l$, is freely rotated at one end and is initially supported at the other end in a horizontal position. A particle of mass αm is placed on the rod at the midpoint. The coefficient of static friction between the particle and the rod is μ. If the support is suddenly removed:

1. Calculate by what factor the force on the hinge is instantaneously reduced?
2. The particle begins to slide when the falling rod makes an angle θ with the horizontal. Calculate this angle.

   [**Hint:** To do this part, you need both components of the force equation on the mass αm; plus the torque equation and the equation for the conservation of the energy of the whole system.]

Homework 7. Due Wednesday, October 2, at 9:10 am.

Problem 1. *Conservative forces*

- Which of the two $2D$ forces $\vec{F}^A(x,y)$ or $\vec{F}^B(x,y)$ given below is conservative? The x and y components of the force $\vec{F}^A(x,y)$ are given by

$$F_x^A(x,y) = -6x - y\cos(xy) - 12xy,$$
$$F_y^A(x,y) = -5 - x\cos(xy) - 7x^2$$

  and the x and y components of the force $\vec{F}^B(x,y)$ are given by

$$F_x^B(x,y) = -6x - y\cos(xy) - 12xy,$$
$$F_y^B(x,y) = -5 - x\cos(xy) - 6x^2.$$

- Write the potential $U(x,y)$ for the conservative one.

Problem 2. *Free fall*

Derive the formula $x(t) = x_0 + v_0 t + \frac{at^2}{2}$ for the $1D$ motion with constant acceleration a starting from the energy conservation law with potential energy $U(x) = -xma$, deriving the integral representation of the solution, and computing the last integral (see Lecture 14).

Problem 3. *Not just a harmonic oscillator*

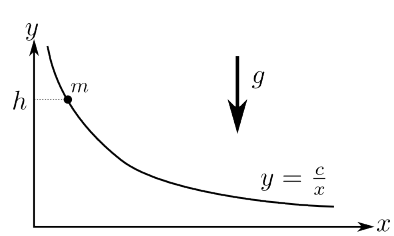
A particle of mass m is moving in $1D$ in a potential $U(x) = kx^4$, where $k > 0$. The total energy of the particle $E > 0$.

- Find the period T of motion of the particle. (You do not need to take the last integral. Just write it.)
- How the period T depends on the total energy E?
- How the period of motion T depends on the total energy E if the potential energy is given by $U(x) = kx^2$?
- What will happen in both cases if $k < 0$?

Problem 4. *1994-Spring-CM-U-3*

A wire has the shape of a hyperbola, $y = c/x$, $c > 0$. A small bead of mass m can slide without friction on the wire. The bead starts at rest from a height h as shown in the figure

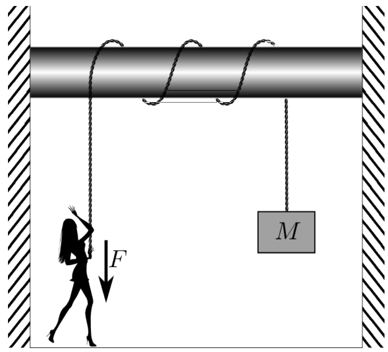
1. Find the velocity vector $\vec{v}$ of the bead as a function of x.
2. Find the force $\vec{F}$ that the bead exerts on the wire as a function of x.



Extra for Honors

Problem 5. *1994-Spring-CM-U-4*

A person wants to hold up a large object of mass M by exerting a force F on a massless rope. The rope is wrapped around a fixed pole of radius R. The coefficient of friction between the rope and the pole is μ. If the rope makes $n+1/2$ turns around the pole, what is the maximum weight the person can support?

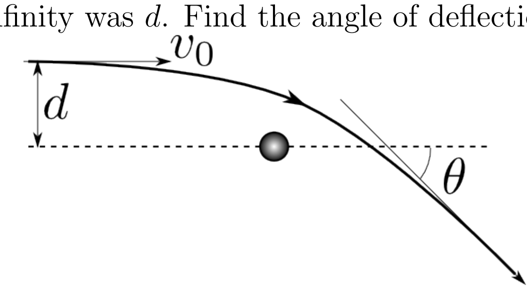


Homework 8. Due Wednesday, October 9, at 9:10 am.

Problem 1. *Kepler orbits.*
The shape of the Kepler's orbits (bounded and unbounded) is defined by its eccentricity ϵ. Find the shape (its equation in the Cartesian coordinates) for

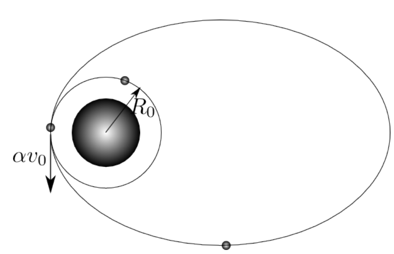
1. $\epsilon = 0$.
2. $0 < \epsilon < 1$.
3. $\epsilon = 1$. What is the energy E of the body?
4. $\epsilon > 1$.

Problem 2. *Scattering.*
A comet is approaching the sun. Its velocity at infinity was v_0. The impact parameter at infinity was d. Find the angle of deflection θ.



Problem 3. *1990-Fall-CM-U-2.jpg*
An artificial Earth satellite is initially orbiting the earth in a circular orbit of radius R_0. At a certain instant, its rocket fires for a negligibly short period of time. This firing accelerates the satellite in the direction of motion and increases its velocity by a factor of α. The change of satellite mass due to the burning of fuel can be considered negligible.

1. Let E_0 and E denote the total energy of the satellite before and after the firing of the rocket. Find E solely in terms of E_0 and α.
2. For $\alpha > \alpha_{\text{es}}$ the satellite will escape from earth. What is α_{es}?
3. For $\alpha < \alpha_{\text{es}}$ the orbit will be elliptical. For this case, find the maximum distance between the satellite and the center of the earth, R_{max}, in terms of R_0 and α.



Homework 9. Due Wednesday, October 16, at 9:10 am.

Problem 1. *1994-Fall-CM-U-4*
The most efficient way to transfer a spacecraft from an initial circular orbit at R_1 to a larger circular orbit at R_2 is to insert it into an intermediate elliptical orbit with radius R_1 at perigee and R_2 at apogee. The following equation relates the semi-major axis a, the total energy of the system E and the potential energy $U(r) = -GMm/r \equiv -k/r$ for an elliptical orbit of the spacecraft of mass m about the earth of mass M:

$$R_1 + R_2 = 2a = \frac{k}{|E|}.$$

1. Derive the relation between the velocity v and the radius R for a circular orbit.
2. Determine the velocity increase required to inject the spacecraft into the elliptical orbit as specified by R_1 and R_2. Let v_1 be the velocity in the initial circular orbit and v_p be the velocity at perigee after the first boost so $\Delta v = v_p - v_1$.
3. Determine the velocity increment required to insert the spacecraft into the second circular orbit when it reaches apogee at $r = R_2$. In this case let v_2 be the velocity in the final orbit and v_a be the velocity at apogee so $\Delta v = v_2 - v_a$.

Problem 2. *2000-Spring-CM-U-1*
A populated spherical planet, diameter a, is protected from incoming missiles by a repulsive force field described by the potential energy function:

$$V(r) = ka(a+r)e^{-r/a}, \qquad r > a/2.$$

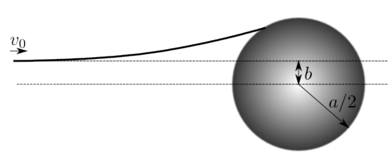
Here $k > 0$ and r is the distance of the missile from the center of the planet. Neglect all other forces on the missile.

The initial speed of a missile of mass m relative to the planet is v_0 when it is a long way away, and the missile is aimed in such a way that the closest it would approach the center of the planet, if it were not deflected at all by the force field or contact with the surface, would be at an impact parameter b (see the diagram). The missile will not harm the planet if it does not come into contact with its surface. Therefore, we wish to explore, as a function of v_0 the range of values of b:

$$0 \le b \le B$$

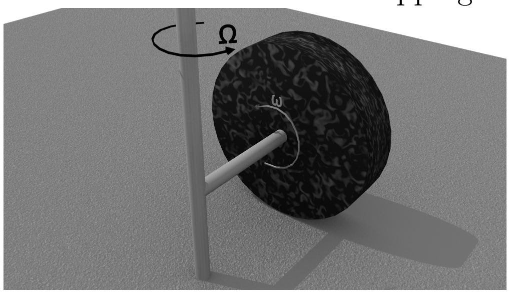
such that the missile will hit the planet.

1. If v_0 is less than a certain critical velocity, v_c, the missile will not be able to reach the planet at all, even if $b = 0$. Determine v_c.
2. For missiles with velocity greater than v_c find B as a function of v_0. Write this function of v_0 in terms of v_c and a.



Problem 3. **Extra for Honors**

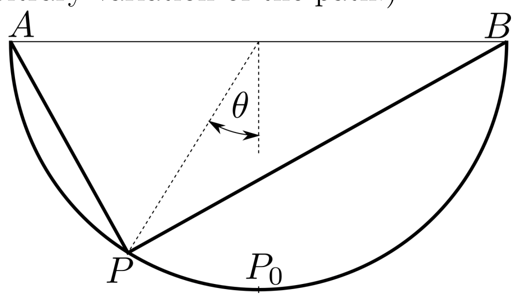
In a disk mill a massive cylinder of radius R and mass M can rotate around its geometrical axis. After the initial push it freely rotates with angular velocity Ω around vertical axis and is rolling on the horizontal plate, as shown on the figure. What force the disk applies to the plate? The disk rolls without slipping.



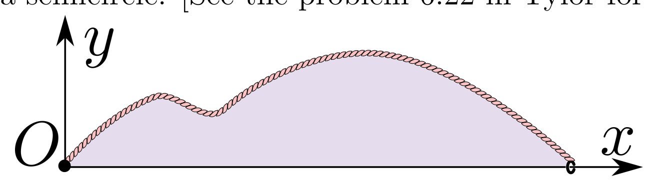
Homework 10. Due FRIDAY, October 25, at 9:10 am.

Problem 1. *Problem 6.2 from Taylor*
Find the shortest path from point (R, z_1, ϕ_1) to point (R, z_2, ϕ_2) on a cylinder of radius R in cylindrical coordinates.

Problem 2. *Problem 6.5 from Classical Mechanics by J. R.Tylor*
Consider a concave, hemispherical mirror with points A and B at opposite ends of a diameter. Consider a ray of light traveling in a vacuum from A to B with one reflection at a point P, in the same vertical plane as A and B. According to the law of reflection, the actual path goes via point P_0 at the bottom of the hemisphere ($\theta = 0$). Find the time of travel along the path APB as a function of θ and show that it is a *maximum* at $P = P_0$! (the correct statement of the Fermat's principle is that the time of travel for light must be *stationary* for arbitrary variation of the path.)



Problem 3. *Problem 6.22 from Classical Mechanics by J. R.Tylor*
You are given a pink string of fixed length L with one end fastened at the origin O, and you are to place the string in the xy plane with its other end on the x axis in such a way as to enclose the maximum area between the string and the x axis. Show, that the required shape is a semicircle. [See the problem 6.22 in Tylor for hints.]



Problem 4. *Problem 6.24 from Classical Mechanics by J. R.Tylor*
Consider a medium in which the refractive index is inversely proportional to r^2; that is, $n = a^2/r^2$, where r is the distance from the origin, and a is a known constant. Use Fermat's principle to find the path of a ray of light traveling in a plane containing the origin. [See the problem 6.24 in Tylor for hints.]

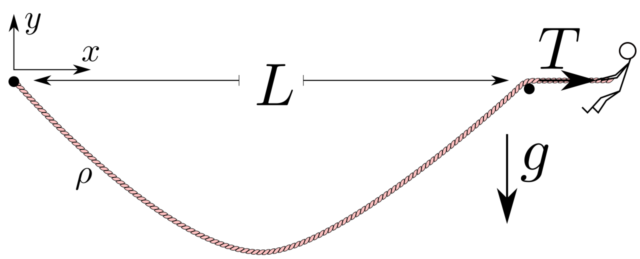
Problem 5. **Extra for Honors** *(and fun for the rest of us.)*
A pink rope of linear mass density ρ is connected to a peg on the left and goes over a peg on the right. The pegs are on the same height distance L apart from each other. The force T is applied to the right end of the rope. There is no friction. y is the vertical coordinate pointing up, and x the is horizontal coordinate.

- Write down the functional of potential energy of the string vs. the shape of the string

$y(x)$. Specify the boundary conditions for the function $y(x)$.

[**Hint:** You have two contributions to the potential energy. The first one is the gravitational potential energy of the rope of shape $y(x)$, and the second one is the work one needs to do against the force T in order to be able to give the rope the shape $y(x)$.]

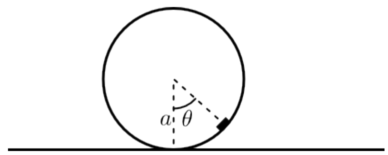
- Write down the equation which gives the shape of minimal energy for the string.
- Find the solution of the equation which satisfies the boundary conditions. (Do not try to solve the transcendental equation for one of the constants. Just write it down.)
- In the case $T \gg \rho g L$, the shape is approximately given by $y \approx -\frac{\alpha}{2}x(L-x)$. Find α.



Homework 11. Due October 30, at 9:10 am.

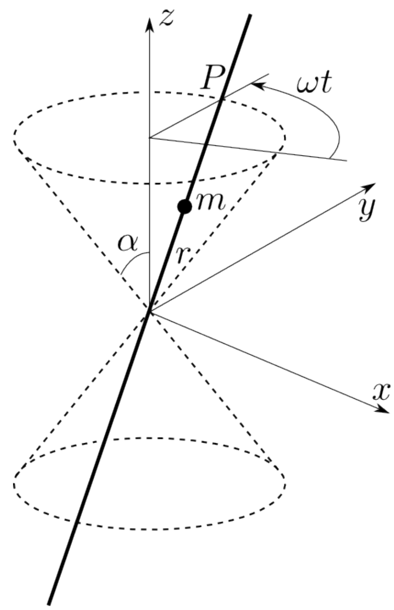
Problem 1. *1991-Fall-CM-U-3.*
A small block of mass m is attached near the outer rim of a solid disk of radius a which also has mass m. The disk rolls without slipping on a horizontal straight line.

1. Find the equation of motion for the angle $\theta(t)$ (measured with respect to the vertical as shown) for all θ.
2. Find the system's small amplitude oscillation frequency about its stable equilibrium position.

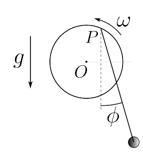


Problem 2. *1999-Spring-CM-U-2*
An infinitely long straight frictionless wire, which passes through the origin of the coordinate system, is held at a constant angle α with respect to the (vertical) z axis. The wire rotates about the z axis at constant angular velocity ω, so that it describes the surface of a pair of right circular cones, centered on the vertical axis, with their common vertices at the origin. Hence, an arbitrary point P, fixed on the wire, describes a horizontal circle as the wire rotates. A bead of mass m is free to slide along this wire under the influence of gravity and without friction. Let r be the distance of the bead from the apex of the cone, positive if above and negative if below the vertex.

1. Write the Lagrangian for this system in terms of r, α, ω, m, and g.
2. From the Lagrangian, obtain the differential equation of motion for the bead.
3. Solve the equation of motion subject to the initial conditions that $r = r_0$ and $dr/dt = 0$ at $t = 0$. Find the condition on r_0 which determines whether the bead will rise or fall on the wire.
4. Use your solution to the equation of motion to find $r(t)$ in the limit that ω goes to zero and show that it is consistent with simple kinematics.



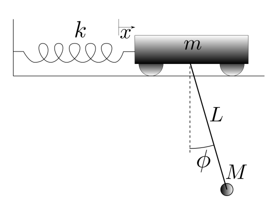
Problem 3. *Problem 7.29 from Classical Mechanics by J. R. Tylor*
A simple pendulum (mass m, length l) whose point of support P is attached to the edge of a vertical wheel (center O, radius R) that is forced to rotate at a fixed angular velocity ω. At $t = 0$, the point P is level with O on the right. Write down the Lagrangian and find the equation of motion for the angle ϕ. Check that your answer makes sense in the special case that $\omega = 0$. [See the problem 7.29 in Tylor for hints.]



Problem 4. *Problem 7.31 from Classical Mechanics by J. R. Tylor*
A simple pendulum (mass M, length L) is suspended from a cart (mass m) that can oscillate on the end of the spring of force constant k.

1. Write the Lagrangian in terms of the two generalized coordinates x and ϕ, where x is the extension of the spring from the its equilibrium length. Find the two Lagrange equations. (Warning: They're pretty ugly!)

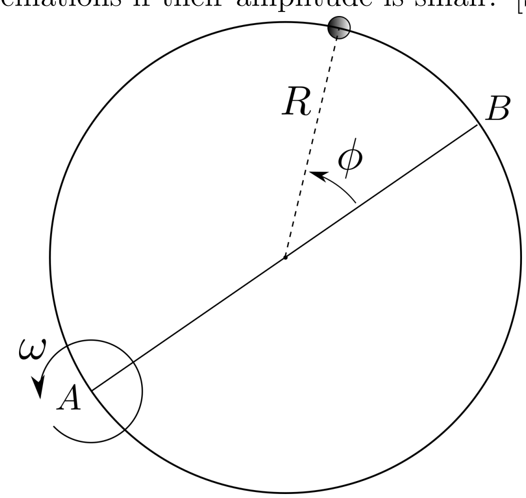
2. Simplify the equations of motion for the case that both x and ϕ are small. (They're still pretty ugly, and still are coupled.)



Homework 12. Due Wednesday November 6, at 9:10 am.

Problem 1. *Problem 7.33 from Classical Mechanics by J. R. Tylor*
A bar of soap (mass m) is at rest on a frictionless plate that rests on a horizontal table. At time $t = 0$, I start raising one edge of the plate so that the plate pivots about the opposite edge with constant angular velocity ω, and the soap starts to slide toward the downhill edge. Show that the equation of motion for the soap has the form $\ddot{x} - \omega^2 x = -g\sin(\omega t)$, where x is the soap's distance from the downhill edge. Solve this for $x(t)$, given the bar start from rest at $x(0) = x_0$. [See the problem 7.33 in Tylor for hints.]

Problem 2. *Problem 7.35 from Classical Mechanics by J. R. Tylor*
The figure shows top view of a smooth horizontal wire hoop that is forced to rotate at a fixed angular velocity ω about a vertical axis through the point A. A bead of mass m is threaded on the hoop and is free to move around it, with its position specified by the angle ϕ that it makes at the center with the diameter AB. Find the Lagrangian for this system using ϕ as your generalized coordinate. Use the Lagrangian equations of motion to show that the bead oscillates about point B exactly like a simple pendulum. What is the frequency of these oscillations if their amplitude is small? [See the problem 7.35 in Tylor for hints.]



Problem 3. *Problem 7.48 from Classical Mechanics by J. R. Tylor*
Let $F(q_1, \dots, q_n)$ be any function of the generalized coordinates $(q_1, \dots, q_n)$ of a system with Lagrangian $L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t)$. Prove directly, by writing the equations of motion that the two Lagrangians L and $L' = L + dF/dt$ give exactly the same equations of motion.

Problem 4. *Problem 7.49 from Classical Mechanics by J. R. Tylor*
Consider a particle of mass m and charge q moving in a uniform constant magnetic field $\vec{B}$ in the z direction.

1. Prove, that $\vec{B}$ can be written as $\vec{B} = \nabla \times \vec{A}$ with $\vec{A} = \frac{1}{2}\vec{B} \times \vec{r}$. Prove equivalently, that in cylindrical coordinates, $\vec{A} = \frac{1}{2}B\rho\hat{\phi}$.

2. Write the Lagrangian in sylindrical polar coordinates and find the three corresponding Lagrange equations.

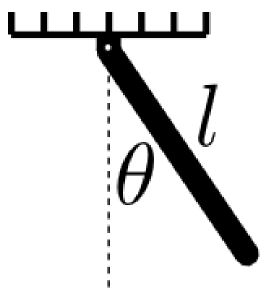
3. Describe (shortly!) those solutions of the Lagrange equations in which ρ is constant.

[See the problem 7.49 in Tylor for hints.]

Homework 13. Due Wednesday, November 13, at 9:10 am.

Problem 1. *1986-Spring-CM-U-2.*
A "physical pendulum" is constructed by hanging a thin uniform rod of length l and mass m from the ceiling as shown in the figure. The hinge at ceiling is frictionless and constrains the rod to swing in a plane. The angle θ is measured from the vertical.

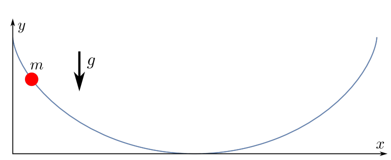
1. Find the Lagrangian for the system.
2. Use Euler-Lagrange differential equation(s) to find the equation(s) of motion for the system. (BUT DON'T SOLVE).
3. Find the approximate solution of the Euler-Lagrange differential equation(s) for the case in which the maximum value of θ is small.
4. Find the Hamiltonian $H(p, q)$ for the system.
5. Use the canonical equations of Hamilton to find the equations of motion for the system and solve for the case of small maximum angle θ. Compare your results with b. and c.



Problem 2. *1985-Spring-CM-G-5*
A bead slides without friction on a wire in the shape of a cycloid:

$$x = a(\theta - \sin\theta)$$
$$y = a(1 + \cos\theta)$$

1. Write down the Hamiltonian of the system.
2. Derive Hamiltonian's equations of motion.



Problem 3. *Problem 7.37 from Classical Mechanics by J. R.Tylor*
Two equal masses $m_1 = m_3 = m$, are joined be a massless string of length L, that passes through a hole in a frictionless horizontal table. The first mass slides on the table with the second hangs below the table and moves up and down in a vertical line.

1. Assuming that the string remains taut, write down the Lagrangian for this system in terms of polar coordinates (r, ϕ) of the mass on the table.

2. Find the two Lagrangian equations of motion and interpret the ϕ equation in terms of the angular momentum l of the first mass.

3. Express $\dot{\phi}$ in terms of l and eliminate $\dot{\phi}$ from the r equation. Now use the r equation to find the value $r = r_0$ at which the first mass can move in a circular path. Interpret your answer in Newtonian terms.

4. Suppose the first mass is moving in a circular path and is given a small radial nudge. Write $r(t) = r_0 + \epsilon(t)$, and rewrite the r equation in terms of $\epsilon(t)$ dropping all powers of $\epsilon(t)$ higher than linear. Show, that the circular path is stable and that $r(t)$ oscillates about r_0.

5. Find the frequency of these oscillations.

Problem 4. *Problem 7.41 from Classical Mechanics by J. R.Tylor*
Consider a bead of mass m sliding without friction on a wire that is bent in the shape of a parabola $z = k\rho^2$, which is spun with constant angular velocity ω about its vertical axis z.

1. Use cylindrical polar coordinates and write down the Lagrangian in terms of ρ as the generalized coordinate.

2. Find the equation of motion of the bead and determine whether there are positions of equilibrium, that is, values of ρ at which the bead can remain fixed, without sliding up or down the spinning wire.

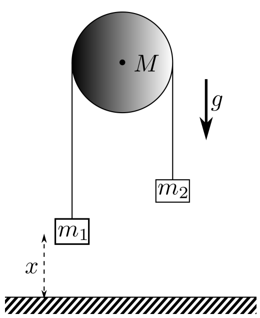
3. Find out the conditions at which the equilibrium points you found are stable.

Homework 14. Due Wednesday, November 20, at 9:10 am.

Problem 1. *Problem 13.2 from Classical Mechanics by J. R. Tylor*
Consider a mass m constrained to move in a vertical line under the influence of gravity. Using the coordinate x measured vertically down from a convenient origin O, write down the Lagrangian L and find the generalized momentum $p = \partial L/\partial \dot{x}$. Find the Hamiltonian H as a function of x and p, and write down the Hamiltonian's equations of motion.

Problem 2. *Problem 13.3 from Classical Mechanics by J. R. Tylor*
Consider the Atwood machine, but suppose that the pulley is a uniform disc of mass M. There is no slipping between the disc and the rope. Using x as you generalized coordinate,

1. Write down the Lagrangian.
2. Write down the generalized momentum p, and the Hamiltonian H.
3. Find Hamilton's equations and use them to find the acceleration $\ddot{x}$.



Problem 3. *Problem 13.5 from Classical Mechanics by J. R. Tylor*
A bead of mass m is threaded on a frictionless wire that is bent into a helix with cylindrical polar coordinates (ρ, ϕ, z) satisfying $z = c\phi$ and $\rho = R$, with c and R constants. The z axis points vertically up and gravity vertically down. Using ϕ as your generalized coordinate,

1. Write down the kinetic and potential energies.
2. Write down the Hamiltonian H.
3. Write down the Hamiltonian's equations and solve for $\ddot{\phi}$ and hence $\ddot{z}$.
4. Explain your result in terms of Newtonian dynamics.
5. Consider a special case $R = 0$.

Problem 4. *Problem 7.34 from Classical Mechanics by J. R.Tylor*
Consider the well-known problem of a cart of mass m moving along the x axis attached to a spring (force constant k), whose other end is held fixed. If we ignore the mass of the spring, then we know that the cart executes simple harmonic motion with angular frequency $\omega = \sqrt{k/m}$.

1. Assuming, that the spring has mass M, that it is uniform and stretches uniformly, show that its kinetic energy is $\frac{1}{6}M\dot{x}^2$. (As usual x is extension of the spring from its equilibrium length.)

2. Write down the Lagrangian for the system of cart plus spring. Write down the Lagrange equation of motion. Find the frequency of the small oscillations.

3. Write down the Hamiltonian.

4. Write down the Hamiltonian equations of motion.

5. Solve the equations and find out the frequency of the oscillations.

[See the problem 7.34 in Tylor for hints.]

Problem 5.
Consider a $1D$ particle with the Hamiltonian $H(p) = \sqrt{m_0^2 c^4 + c^2 p^2}$, where p is the momentum, m_0 is some constant with the units of mass, and c is some constant with the units of velocity.

1. Show that the momentum is conserved.

2. Find how the velocity v depends on the momentum. What is the largest possible velocity? How does velocity depend on momentum in case $m_0 = 0$?

3. Express momentum as a function of velocity v (and constants m_0 and c). Introduce a function $m(v)$ (function of velocity, not momentum) such, that $p = m(v)v$.

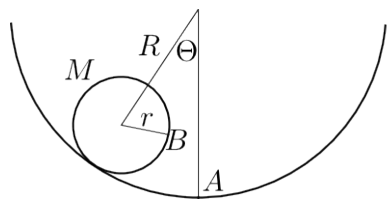
4. Express energy E in terms of the velocity (it is no longer Hamiltonian, as Hamiltonian must be a function of momentum). Use the function m to simplify your answer.

5. Find the Lagrangian for this particle. Write down the action.

Homework 15. The Last One! Due MONDAY, December 2, at 9:10 am.

Problem 1. *1983-Fall-CM-U-3.*
A hollow thin walled cylinder of radius r and mass M is constrained to roll without slipping inside a cylindrical surface with radius $R + r$ (see diagram). The point B coincides with the point A when the cylinder has its minimum potential energy.

1. What is the frequency of small oscillations around the equilibrium position?
2. What would the frequency of small oscillations be if the contact between the surfaces is frictionless?



Problem 2. *1991-Spring-CM-U-2.*
A circular platform of mass M and radius R is free to rotate about a vertical axis through its center. A man of mass M is originally standing right at the edge of the platform at the end of a line painted along a diameter of the platform. The platform and man are set spinning with an angular velocity ω_0. At $t = 0$ the man begins to walk toward the center of the platform along the line so that his distance from the center is $R - v_0 t$. If the man slips off the line when he is at $R/2$, what must be the coefficient of friction between the man and the platform?

Problem 3. *Problem 7.38 from Classical Mechanics by J. R.Tylor*
A particle is confined to move on the surface of a circular cone with its axis on the vertical z axis, vertex at the origin (pointing down), and half angel α.

1. Write down the Lagrangian L in terms of spherical polar coordinates r and ϕ.
2. Find the two equations of motion. Interpret the ϕ equation in terms of the angular momentum l_z, and use it to eliminate $\dot{\phi}$ from the r equation in favor of the constant l_z.
3. Show that your r equation make sense in the case that $l_z = 0$.
4. Find the value r_0 of r at which the particle can remain in a horizontal circular path.
5. Suppose the particle is given a small radial kick, so that $r(t) = r_0 + \epsilon(t)$, where $\epsilon(t)$ is small. Use the r equation to decide whether the circular path is stable. If so, which what frequency does r oscillate about r_0?

Problem 4. **Extra for Honors**

To solve this problem you need to study the Lecture #37 in the lecture notes.

A parallel plate capacitor is charged so that the electric field inside is $\mathcal{E}_0$. A ball of mass m and charge q is bouncing vertically off the lower plate without loss of energy and charge transfer. Its initial energy E_0 is such, that it never reaches the top plate. Neglect the gravity.

- What is the height x_{E_0} (measured from the lower plate) of the ball's trajectory initially as a function of its energy, mass, and charge?
- Write the Lagrangian for the ball's motion using its height as the coordinate x.
- Write the Hamiltonian for the ball's motion.
- Write the adiabatic invariant for the ball's motion in terms of energy, electric field, mass and charge of the ball.
- The electric field in the capacitor is increased very slowly to the value $\mathcal{E} = 8\mathcal{E}_0$. What is the energy E of the ball now in terms of the initial energy E_0?
- What is the height of the ball's trajectory x_E in terms of the initial height x_{E_0}?