Physics 607 - Exam 2 Solution

1. (a) $z_{rot} = \sum_{j=0}^{\infty} (2j+1) e^{-j(j+1)\hbar^2/2IkT} \underbrace{\Delta j}_{=1}$

$$\boxed{\approx} \int_0^\infty d(j^2+j)\, e^{-(j^2+j)\hbar^2/2IkT}$$

$$= \frac{2IkT}{\hbar^2} \int_0^\infty du\, e^{-u} = \frac{2IkT}{\hbar^2} \underbrace{\left[\frac{e^{-u}}{-1}\right]_0^\infty}_{=1} = \boxed{\frac{2IkT}{\hbar^2}}$$

(b) $\boxed{\mathcal{E}_{rot} =} kT^2 \frac{\partial}{\partial T} \ln z = kT^2 \frac{\partial}{\partial T} (\ln T + \text{const}) = kT^2 \cdot \frac{1}{T} = \boxed{kT}$

(c) Since $H_{vib} = \frac{1}{2} K u^2$, and $\underline{u}$ does not occur elsewhere in $\underline{H}$, the integrals for the other degrees of freedom cancel in numerator and denominator, so

$$\boxed{\langle u^2 \rangle =} \frac{\int_{-\infty}^{\infty} du\, u^2 e^{-Ku^2/2kT}}{\int_{-\infty}^{\infty} du\, e^{-Ku^2/2kT}}$$

$$= \frac{(2kT/K)^{3/2} \int_{-\infty}^{\infty} dx\, x^2 e^{-x^2}}{(2kT/K)^{1/2} \int_{-\infty}^{\infty} dx\, e^{-x^2}}$$

$$= \frac{2kT}{K} \frac{\sqrt{\pi}/2}{\sqrt{\pi}}$$

$$= \boxed{\frac{kT}{K}}$$

(d) equipartition theorem: $\left\langle u \frac{\partial H}{\partial u} \right\rangle = kT$

with $\left\langle u \frac{\partial H}{\partial u} \right\rangle = \left\langle u \cdot \frac{1}{2} \cdot 2Ku \right\rangle = K \langle u^2 \rangle$

so $\boxed{\langle u^2 \rangle = \frac{kT}{K}}$

2. (a) The equilibrium state is found by minimizing the free energy:

$$0 = \frac{\partial F}{\partial \psi} = a(T-T_c)\cdot 2\psi + \frac{1}{2} b \cdot 4\psi^3$$

$$\Rightarrow \psi = 0 \quad \text{or} \quad \psi^2 = -\frac{a(T-T_c)}{b},$$ with second solution allowed only for $T < T_c$

$$\Rightarrow F = 0 \quad \text{or} \quad \boxed{F =} -\frac{a^2}{b}(T-T_c)^2 + \frac{1}{2}\frac{a^2}{b}(T-T_c)^2$$

$T > T_c$

$$= \boxed{-\frac{1}{2}\frac{a^2}{b}(T-T_c)^2}$$ lower for $T < T_c$

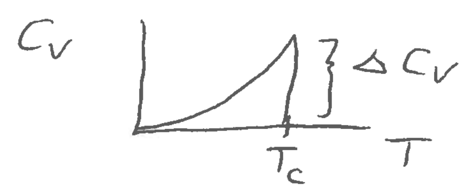
Then for $T > T_c$, $\psi = 0$; and for $T < T_c$, $\psi = \sqrt{\frac{a}{b}(T_c - T)}$, so there is a phase transition as described.

(b) $$\underline{C_V} = -T\left(\frac{\partial^2 F}{\partial T^2}\right) = -T\left(-\frac{1}{2}\frac{a^2}{b}\right)\frac{\partial}{\partial T}\left(2(T-T_c)\right)$$

$$= \underline{\frac{a^2}{b} T \quad \text{for } T < T_c}$$

$\underline{C_V = 0 \text{ for } T > T_c}$

$$\therefore \boxed{\Delta C_V = \frac{a^2}{b} T_c}$$



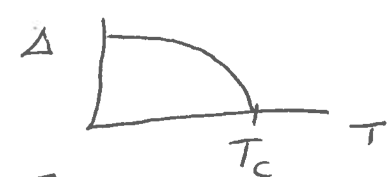
(c) $C_V = 0$ for $T > T_c$

$$C_V = 2k \frac{1}{kT_c}\rho(\varepsilon_F)\cdot\frac{1}{2}\frac{1}{kT_c}\left(\frac{1}{\partial\beta/\partial T}\cdot\frac{\partial \Delta^2}{\partial T}\right)_{T_c} \quad \text{for } T < T_c$$

with $$\frac{\partial \beta}{\partial T} = \frac{\partial}{\partial T}\left(\frac{1}{kT}\right) = -\frac{1}{kT^2}$$

$$\Rightarrow C_V = -\rho(\varepsilon_F)\left(\frac{\partial \Delta^2}{\partial T}\right)_{T_c} \quad \text{for } T < T_c$$

$$\therefore \boxed{\Delta C_V = \rho(\varepsilon_F)\left(-\frac{\partial \Delta^2}{\partial T}\right)_{T_c}}$$



(d) & (e) $$\ln 2 = (2-\alpha)\Big[\ln \underbrace{\frac{\partial}{\partial K}\left(\frac{3}{8}\ln\cosh(4K)\right)}_{\text{"}\ \frac{3}{8}\frac{\sinh(4K)}{\cosh(4K)}\cdot 4 \ = \ \frac{3}{2}\tanh((4)(0.507)) \text{ at } K = K_c}\Big]_{K=K_c}, \quad K_c = 0.507$$

$$\Rightarrow \boxed{\alpha} = 2 - \frac{\ln 2}{\ln\left(\left(\frac{3}{2}\right)(0.966)\right)} = 2 - \frac{0.693}{0.371} \boxed{\approx 0.13}$$

3. (a) "divide by dn" with $dT = 0$: $\frac{1}{n}\left(\frac{\partial P}{\partial n}\right)_T = \left(\frac{\partial \mu}{\partial n}\right)_T$

(b) $$\boxed{u_F^2 =} \frac{1}{m}\left(\frac{\partial P}{\partial n}\right)_T = \frac{n}{m}\left(\frac{\partial \mu}{\partial n}\right)_T = \boxed{\frac{n}{m}\left(\frac{\partial \mathcal{E}_F}{\partial n}\right)_T \text{ at } T=0}$$

(c) $$N = \int_0^{\mathcal{E}_F} d\mathcal{E}\, \rho(\mathcal{E}) = \int_0^{\mathcal{E}_F} d\mathcal{E}\, A V \mathcal{E}^{1/2} = AV \left[\frac{\mathcal{E}^{3/2}}{3/2}\right]_0^{\mathcal{E}_F} = AV \cdot \frac{2}{3}\mathcal{E}_F^{3/2}$$

$$\Rightarrow \frac{N}{V} = \frac{2}{3} A \mathcal{E}_F^{3/2} \Rightarrow \boxed{\mathcal{E}_F = \left(\frac{3n}{2A}\right)^{2/3}}$$

(d) $$\frac{\partial \mathcal{E}_F}{\partial n} = \left(\frac{3}{2A}\right)^{2/3} \cdot \frac{2}{3} n^{-\frac{1}{3}} = \left(\frac{3n}{2A}\right)^{2/3} \cdot \frac{2}{3n} = \frac{2\mathcal{E}_F}{3n}$$

$$\Rightarrow u_F^2 = \frac{\cancel{n}}{m} \cdot \frac{2\mathcal{E}_F}{3\cancel{n}} = \frac{\cancel{2}}{3}\frac{1}{\cancel{m}} \cdot \frac{1}{\cancel{2}}\cancel{m} v_F^2 = \frac{1}{3} v_F^2$$

$$\Rightarrow \boxed{u_F = \frac{1}{\sqrt{3}} v_F}$$

$\Gamma\left(\frac{1}{2}\right) = \pi^{1/2}$, $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\pi^{1/2}$, $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\pi^{1/2}$, $\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$

(e) To first order, $\alpha \equiv \frac{\pi^2}{6}\frac{1}{\xi^2}$

$$\boxed{\gamma} = \frac{5}{3}\, \frac{\xi^{5/2} \cdot \xi^{1/2}}{(\xi^{3/2})^2}\, \frac{\left(\Gamma\left(\frac{5}{2}\right)\right)^2}{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{3}{2}\right)} \left[1 + \frac{5}{2}\cdot\frac{3}{2}\alpha\right]\left[1 + \frac{1}{2}\left(-\frac{1}{2}\right)\alpha\right]\left[1 + \frac{3}{2}\cdot\frac{1}{2}\alpha\right]^{-2}$$

$$= \frac{5}{3}\, \frac{\left(\frac{3}{4}\pi^{1/2}\right)^2}{\left(\frac{15}{8}\pi^{1/2}\right)\left(\frac{1}{2}\pi^{1/2}\right)} \left[1 + \frac{15}{4}\alpha\right]\left[1 - \frac{1}{4}\alpha\right]\left[1 - 2\cdot\frac{3}{4}\alpha + \cdots\right]$$

$$= \frac{5}{3}\cdot\frac{3}{5}\left[1 + \frac{15}{4}\alpha - \frac{1}{4}\alpha - \frac{6}{4}\alpha\right] + O(\alpha^2)$$

$$= 1 + 2\cdot\frac{\pi^2}{6}\frac{1}{(\mu/kT)^2} + O(\alpha^2)$$

$$\boxed{\approx 1 + \frac{\pi^2}{3}\left(\frac{kT}{\mathcal{E}_F}\right)^2}$$

so $\kappa_T \to \kappa_S$ as $T \to 0$ $\quad [\mu(T=0) = \mathcal{E}_F]$

(f) For $T < T_c$, $\lambda = 1$, which means $\mu = 0$.

[This is in the thermodynamic limit, N & $V \to \infty$.]

Then $\boxed{u_B =} \left(\frac{n}{m}\left(\frac{\partial \mu}{\partial n}\right)_T\right)^{1/2} = \boxed{0}$.

4. (a) $\rho(p) = A\, p^{D-1}$

$$N = \underbrace{\int_0^\infty dp\, \rho(p)}_{\text{number of states in } dp \text{ at } p} \underbrace{\frac{1}{e^{cp/kT} - 1}}_{\text{number of bosons in each state (Bose-Einstein distribution function)}} + \underbrace{N_0}_{\text{number of bosons with } p=0}$$

Bose-Einstein condensation is forced if the largest possible value of the above integral at a given low T is less than N, forcing N_0 to be nonzero.

$$N - N_0 = \int_0^\infty dp \cdot A\, p^{D-1} \frac{1}{e^{cp/kT} - 1}$$

$$= A \cdot \left(\frac{kT}{c}\right)^D \int_0^\infty du\, u^{D-1} \frac{1}{e^u - 1} \qquad u \equiv \frac{cp}{kT}$$

If the integral is convergent, then the right-hand side $\to 0$ as $T \to 0$, and $N_0 \to N$.

If the integral is divergent, this mechanism does not prevail. Instead, we need a more careful treatment, but the point of this problem is to determine the largest value of D for which the usual mechanism does not force condensation.

Near the lower limit $u = 0$, we have $e^u - 1 \approx u$, and

$$\int du\, u^{D-1} \frac{1}{e^u - 1} \approx \int du\, u^{D-2} = \frac{u^{D-1}}{D-1} \quad \text{for } D \neq 1$$

so the integral is divergent for $D < 1$.

If $D = 1$, $\int du \frac{1}{e^u - 1} \approx \int \frac{du}{u} = \ln u$

which again diverges as $u \to 0$.

So $\boxed{D = 1}$ is the answer.

(b) For the nonrelativistic case, we learned that the corresponding answer is $\boxed{D = 2}$.