1. The quark-gluon plasma is believed to have existed in the early universe about a microsecond after the Big Bang, and experiments indicate that it can also be produced in collisions of nuclei moving near the speed of light. In the simplest description, it is composed of spin 1/2 quarks and antiquarks – which come in 2 flavors (up and down) and 3 colors – plus 8 kinds of spin 1 gluons.

The gluons are massless and the masses of these light quarks can be neglected. At these high temperatures, it is assumed that there are equal numbers of quarks and antiquarks. Just as for photons, the particle numbers are not fixed.

One then has an ideal gas of 2 x 2 x 3 = 12 kinds of fermions, each with 2 polarizations, plus 8 kinds of bosons, each again with 2 polarizations, at a temperature T and in a volume V.

(a) (5) Obtain the density of states $\rho\left(\varepsilon\right)$, for one set of massless particles with 2 polarizations in 3 dimensions. Recall that $\varepsilon = cp$, where ε is the one-particle energy and p is the magnitude of the 3-momentum, and that

$$\rho\left(p\right) dp = \frac{4\pi p^2 dp}{h^3/V}$$

in an obvious notation.

(b) (5) Show that the energy density of quarks and antiquarks (the fermions) is given by

$$\frac{E_{fermions}}{V} = C_f T^n$$

where you will obtain the constants C_f and n.

(c) (5) Show that the energy density of gluons (the bosons) is given by

$$\frac{E_{bosons}}{V} = C_b \, T^{\,n}$$

where you will obtain the constant C_b.

(d) (5) Recall that we proved

$$P = \frac{1}{3}\frac{N}{V}\left\langle pu \right\rangle$$

for an ideal quantum gas, where u is the particle velocity. Use this result to obtain the total pressure P of the quark-gluon plasma as a function of temperature.

(e) (5) Use the Gibbs-Duhem relation to obtain a relation between S and $(dP/dT)_V$. (Justify each step.) Then calculate the entropy density S/V of the quark-gluon plasma as a function of temperature.

2. A classical nonrelativistic ideal gas of N distinguishable particles with mass m is placed in a 3-dimensional potential

$$U(r) = \frac{|\overrightarrow{r}|^2}{2R^2} = \frac{x^2 + y^2 + z^2}{2V^{2/3}} \quad , \quad V \equiv R^3 \ .$$

(a) (7) Calculate the canonical partition function Z, obtaining an answer in terms of V, m, $\hbar$, and kT.

(b) (7) Calculate the Helmholtz free energy F. Then, using $dF = -SdT - PdV + \mu dN$, obtain the equation of state which relates P, V, N, and T.

(c) (7) Calculate the entropy S, the energy E, and the heat capacity at constant volume $C_{V.}$

(d) (4) Now use the equipartition theorem to calculate the energy E. Does your answer agree with that from part (c)?

3. (a) (12) In the canonical ensemble, the probability that the system is in a state r, with an energy E_r, is

$$p_r = \frac{e^{-\beta E_r}}{Z}$$

where Z is the partition function. The variance in the energy is

$$\sigma_E^2 = \left\langle (E - \langle E \rangle)^2 \right\rangle$$

where $\langle \cdots \rangle$ denotes an average.

Show that

$$\sigma_E^2 \propto C_V$$

where you will determine the proportionality factor, **being clear in every step**.

(b) (13) In the grand canonical ensemble, the probability that the system is in a state N, r, with N particles and an energy E_{Nr}, is

$$p_{Nr} = \frac{e^{-\beta E_{Nr}} e^{\gamma N}}{\mathcal{Z}} \quad , \quad \gamma = \beta \mu$$

where $\mathcal{Z}$ is the grand partition function. The variance in the number of particles is

$$\sigma_N^2 = \left\langle (N - \langle N \rangle)^2 \right\rangle \ .$$

Using the Helmholtz free energy, with some clever thermodynamic manipulations, one can show that

$$\left(\frac{\partial \mu}{\partial \langle N \rangle} \right)_{V,T} = -\frac{V^2}{N^2} \left(\frac{\partial P}{\partial V} \right)_{N,T}$$

but you can just assume this result here, in obtaining the result below.

Show that

$$\sigma_N^2 \propto \kappa_T$$

where you will determine the proportionality factor (again being clear in every step).

4. (25) The potential energy of a classical 1-dimensional anharmonic oscillator is

$$V\left(x\right) = cx^2 - gx^3 - fx^4$$

where c, g, and f are positive constants, with g and f assumed to be very small (and the obvious instability assumed to be relieved by still higher-order terms). Show that the leading contribution to the heat capacity from the anharmonic (g and f) terms is

$$\frac{3}{2}k^2\left(\frac{f}{c^2} + \frac{5}{4}\frac{g^2}{c^3}\right)T\ ,$$

exhibiting each step clearly.