Physics 607 Final Exam

Please be well-organized, and show all significant steps clearly in all problems.

You are graded on your work, which must be clear and legible!

An answer, even if correct, will receive zero credit unless it is obtained via the work shown.

The variables have their usual meanings: E = energy, S = entropy, V = volume, N = number of particles, T = temperature, P = pressure, μ = chemical potential, B = applied magnetic field, C_V = heat capacity at constant volume, C_P = heat capacity at constant pressure, F = Helmholtz free energy, G = Gibbs free energy, k = Boltzmann constant, h = Planck constant, c = speed of light. Also, $\langle \cdots \rangle$ represents an average.

In working these problems, you may assume (when appropriate) the following:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \qquad \int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$$

$$\int_0^\infty e^{-x} x^{z-1} dx = \Gamma(z) \quad , \quad \Gamma(\nu+1) = \nu\Gamma(\nu) \quad , \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\langle E \rangle = kT^2 \frac{\partial}{\partial T} \ln Z \qquad C_V = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_V \qquad Z = \sum_r e^{-\beta E_r} \qquad \mathbb{Z} = \sum_{Nr} e^{-\beta E_{Nr}} e^{-\gamma N}$$

$$F = -kT \ln Z \qquad \Omega = -kT \ln \mathbb{Z} \qquad Z = \frac{1}{N!}\int \frac{dq\, dp}{h^{dN}} e^{-H(p,q)/kT} \qquad \left\langle x_i \frac{\partial H}{\partial x_j} \right\rangle = \delta_{ij} kT$$

$$Z = \frac{1}{N!}\left(\frac{V}{\lambda_{th}^3}\right)^N \qquad \lambda_{th} = \frac{h}{(2\pi m kT)^{1/2}} \qquad \ln N! \approx N \ln N - N \qquad C_V = k\frac{x^2 e^x}{(e^x - 1)^2} \;,\; x = \frac{\hbar\omega}{kT}$$

$$\int_0^\infty \frac{x^n}{e^x+1} dx = \left(1 - \frac{1}{2^n}\right)\int_0^\infty \frac{x^n}{e^x - 1} dx \qquad \int_0^\infty \frac{x^2}{e^x-1} dx = 2\varsigma(3) \approx 2.404 \qquad \int_0^\infty \frac{x^3}{e^x-1} dx = 6\varsigma(4) = \frac{\pi^4}{15}$$

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1 \quad , \quad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - .. \quad , \quad 1 + x^2 + x^3 + \ldots = \frac{1}{1-x}$$

$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P \;,\; \kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T \;,\; \beta = \frac{1}{kT} \;,\; \lambda = e^{\beta\mu} \;,\; \Omega = -kT \ln \mathbb{Z} = \mp kT \sum_k \ln\left(1 \pm \lambda e^{-\beta\varepsilon_k}\right)$$

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1. The following partition function can be derived for a dense gas:

$$Z(T,V,N) = \frac{1}{N!}\left(\frac{V-Nb}{\lambda_{th}^{3}}\right)^{N} e^{aN^2/VkT} \quad , \quad \lambda_{th} = \frac{h}{(2\pi mkT)^{1/2}}$$

where a and b are constants and m is the mass of a particle.

(a) (4) Obtain the Helmholtz free energy $F = E - TS$.

(b) (4) Using the expression for dE in terms of dS, dV, and dN, obtain the expression for dF in terms of dT, dV, and dN.

(c) (4) Using the results of (a) and (b), calculate the pressure P, and then write down the equation of state.

Also calculate the following:

(d) (4) entropy S.

(e) (4) chemical potential μ.

(f) (4) energy E

(g) (1) What do you get for the equation of state, S, μ, and E when $a = b = 0$?

2. (a) (5) Using Euler's theorem, obtain a simple relation involving SdT, VdP, and $Nd\mu$.

(b) (10) Use this relation to obtain a simple relation between $\left(\frac{\partial P}{\partial T}\right)_\mu$ and S, and between $\left(\frac{\partial P}{\partial \mu}\right)_T$ and N.

(c) (10) For an ideal gas, with $PV = NkT$ and $E = \frac{3}{2}NkT$, one obtains

$$\mu = kT\ln\left(\frac{\lambda_{th}^{\ 3}}{v}\right) \quad , \quad S = Nk\ln\left(\frac{v}{\lambda_{th}^{\ 3}}e^{5/2}\right) \quad , \quad v = \frac{V}{N} \quad , \quad \lambda_{th} = \frac{h}{\left(2\pi mkT\right)^{1/2}} \ .$$

Calculate $\left(\frac{\partial P}{\partial T}\right)_\mu$ and see whether the result for this quantity in part (b) holds.

Please show all steps clearly.

3. For an ideal gas of bosons, you are given the results

$$\frac{P}{kT} = \frac{1}{\lambda_{th}^3} g_{5/2}(\lambda) \quad , \quad \frac{N - N_0}{V} = \frac{1}{\lambda_{th}^3} g_{3/2}(\lambda)$$

$$Pv^{5/3} = \text{constant} \quad , \quad vT^{3/2} = \text{constant} \quad , \quad \frac{P}{T^{5/2}} = \text{constant} \quad \text{for an adiabatic process}$$

with $N_0 = 0$ here and $v = 1/n$, $n = N/V$.

(a) (10) Show that the isothermal compressibility κ_T is given by

$$\kappa_T = \frac{1}{nkT} \frac{g_{n_T}(\lambda)}{g_{n'_T}(\lambda)}$$

where you will determine the constants n_T and n'_T , which are not integers, while doing the calculation.

(b) (10) Show that the adiabatic compressibility κ_S is given by

$$\kappa_S = \frac{3}{5} \frac{1}{nkT} \frac{g_{n_S}(\lambda)}{g_{n'_S}(\lambda)}$$

where you will determine the constants n_S and n'_S , which again are not integers, while doing the calculation.

(c) (10) One can show that $\lambda \to 0$ as $n \to 0$ and that consequently one can write $\lambda = a_1 n + a_2 n^2 + \ldots$.

Use this Taylor series expansion and the equations given at the top of this problem to obtain

$$\frac{P}{kT} = n + a\lambda_{th}^3 n^2 + \ldots$$

where you will obtain the (positive or negative) constant a .

4. (25) For the Rayleigh-Bénard instability, we obtained the following eigenvalue equation:

$$\left(\frac{d^2}{dZ^2}-\overline{\alpha}^2\right)^3 V(Z)=-R\overline{\alpha}^2 V(Z)$$

where

$$Z=\frac{z}{d} \quad , \quad \overline{\alpha}=k\,d \quad , \quad R=\frac{g\,a\,\alpha\,d^4\,\rho_0^2\,c_V}{\eta K} \quad , \quad v_z=V(Z)e^{i\left(k_x x+k_y y\right)}e^{\omega t}$$

with $\omega \to 0+$ in the initial time-dependent solution, so that one has a solution that initially grows exponentially in time and then settles down to a nontrivial steady state.

(Here x,y,z,t are the coordinates, d is the spacing between the hot plate on the bottom and the cold plate on top, $k=\left(k_x^2+k_y^2\right)^{1/2}$, v_z is the fluid velocity in the z direction (with the z axis perpendicular to the plane of each flat plate), and the parameters in the Rayleigh number R are the acceleration of gravity, temperature gradient, thermal expansion coefficient, d, unperturbed density, heat capacity per volume, shear viscosity, and thermal conductivity.

Now assume smooth boundaries at both plates, but also assume that the experiment is performed with a very thin sheet midway between the plates, which is permeable to heat etc., but forces v_z to be zero, so that the boundary conditions are now

$$V(Z)=0 \quad \text{for} \quad Z=\frac{1}{2} \quad \text{as well as} \quad V(Z)=0 \quad \text{for} \quad Z=0 \quad \text{and} \quad Z=1.$$

Calculate R_c, the smallest value of the Rayleigh number R for which there will be a nontrivial (rolling) solution for the fluid velocity.

(You can leave your answer in terms of well-defined constants like π.)

5. (6 extra credit) Specify whether each of the following **can** or **cannot** be negative in a stable thermodynamic system. In each case very briefly state a reason for your answer.

(i) temperature

(ii) heat capacity at constant volume

(iii) heat capacity at constant pressure

(iv) pressure

(v) isothermal compressibility

(vi) isentropic compressibility

6. (4 extra credit) In the context of the above question 5 (on thermodynamic stability), discuss the temperature and heat capacity of a black hole (with specific points of course).

Note that the maximum credit on this exam is intentionally 105 + 10.