Physics 607 Final - Solution

1. (a) $F = -kT\ln Z = -NkT\ln\left(\frac{V-Nb}{\lambda_{th}^3}\right) - a\frac{N^2}{V} + kT\ln N!$

(b) $F = E - TS$ & $dE = TdS - PdV + \mu dN$

give $dF = -SdT - PdV + \mu dN$

(c) $P = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = +NkT\frac{1}{V-Nb} - \frac{aN^2}{V^2}$

$$\Rightarrow \left(P + \frac{aN^2}{V^2}\right)(V-Nb) = NkT$$

(d) $S = -\left(\frac{\partial F}{\partial T}\right)_{N,V} = +Nk\ln\left(\frac{V-Nb}{\lambda_{th}^3}\right) - 3NkT\underbrace{\frac{\partial\ln\lambda}{\partial T}}_{= -\frac{1}{2}\frac{\partial\ln T}{\partial T} = -\frac{1}{2}\frac{1}{T}} - k\ln N!$

$$= Nk\ln\left(\frac{V-Nb}{\lambda_{th}^3}\right) + \frac{3}{2}Nk - k\ln N!$$

(e) $\mu = \left(\frac{\partial F}{\partial N}\right)_{T,V} = -kT\ln\left(\frac{V-Nb}{\lambda_{th}^3}\right) - NkT\overbrace{\frac{\partial}{\partial N}\ln(V-Nb)}^{= \frac{-b}{V-Nb}} - 2a\frac{N}{V}$

Stirling's approx. $\rightarrow$ $+kT\underbrace{\frac{\partial}{\partial N}(N\ln N - N)}_{= \ln N}$

$$= -kT\ln\left(\frac{V-Nb}{\lambda_{th}^3}\right) + NkT\frac{b}{V-Nb} - 2a\frac{N}{V} + kT\ln N$$

(f) $E = kT^2\frac{\partial}{\partial T}\ln Z = kT^2\left(\frac{3N}{2T} - \frac{aN^2}{V}\cdot\frac{1}{kT^2}\right)$

$\uparrow$ as in (d) $\qquad \uparrow$ from given Z

$$= \frac{3}{2}NkT - \frac{aN^2}{V}$$

(g) $a = b = 0 \Rightarrow$

$PV = NkT$

$S = Nk\ln\left(\frac{V}{\lambda_{th}^3}\right) + \frac{3}{2}Nk - k\ln N!$

$\mu = -kT\ln(v/\lambda_{th}^3)$, $v \equiv \frac{V}{N}$

$E = \frac{3}{2}NkT$

2. (a) $E = TS - PV + \mu N$ & $dE = TdS - PdV + \mu dN$

$$\Rightarrow SdT - VdP + Nd\mu = 0$$

(b) $d\mu = 0 \Rightarrow dP = \frac{S}{V} dT \Rightarrow \boxed{\left(\frac{\partial P}{\partial T}\right)_\mu = \frac{S}{V}}$

$dT = 0 \Rightarrow dP = \frac{N}{V} d\mu \Rightarrow \boxed{\left(\frac{\partial P}{\partial \mu}\right)_T = \frac{N}{V}}$

(c) one way: $d\mu = 0 \Rightarrow \left[k \ln\left(\frac{\lambda_{th}^3}{v}\right) + kT\left(-\frac{3}{2}\right)\frac{\partial}{\partial T}\ln T - kT\frac{\partial \ln v}{\partial T}\right] dT = 0$

$$\Rightarrow \frac{kT}{v}\frac{\partial v}{\partial T} = k \ln\left(\frac{\lambda_{th}^3}{v}\right) - \frac{3}{2}k$$

Also, $Pv = kT \Rightarrow \frac{\partial P}{\partial T} = \frac{\partial}{\partial T}\left(\frac{kT}{v}\right) = \frac{k}{v} - kT\frac{1}{v^2}\frac{\partial v}{\partial T}$

$$\Rightarrow \boxed{\left(\frac{\partial P}{\partial T}\right)_\mu =} \frac{k}{v} - \frac{1}{v}\frac{kT}{v}\frac{\partial v}{\partial T} = \frac{k}{v} - \frac{1}{v}\left[k\ln\left(\frac{\lambda_{th}^3}{v}\right) - \frac{3}{2}k\right]$$

$$= \frac{5}{2}\frac{k}{v} - \frac{k}{v}\ln\left(\frac{\lambda_{th}^3}{v}\right) = \frac{k}{v}\ln\left(\frac{v}{\lambda_{th}^3}e^{5/2}\right)$$

$$= \frac{Nk}{V}\ln\left(\frac{v}{\lambda_{th}^3}e^{5/2}\right) = \boxed{\frac{S}{V}}$$

3. (a) $T = \text{const} \Rightarrow \dfrac{dP}{kT} = \dfrac{1}{\lambda_{th}^3} \underbrace{\dfrac{\partial g_{5/2}}{\partial \lambda}}_{=\frac{1}{\lambda} g_{3/2}} \dfrac{\partial \lambda}{\partial V} dV$

$$\Rightarrow \left(\frac{\partial P}{\partial V}\right)_T = \frac{kT}{\lambda_{th}^3} \frac{1}{\lambda} g_{3/2}(\lambda) \frac{\partial \lambda}{\partial V}$$

Also, $N\left(-\dfrac{1}{V^2}\right) = \dfrac{1}{\lambda_{th}^3} \underbrace{\dfrac{\partial g_{3/2}}{\partial \lambda}}_{=\frac{1}{\lambda} g_{1/2}} \dfrac{\partial \lambda}{\partial V} \Rightarrow \dfrac{\partial \lambda}{\partial V} = -\dfrac{N}{V^2} \lambda_{th}^3 \dfrac{\lambda}{g_{1/2}(\lambda)}$

so $\left(\dfrac{\partial P}{\partial V}\right)_T = \dfrac{kT}{\lambda_{th}^3} \dfrac{1}{\lambda} g_{3/2} \left(-\dfrac{N}{V^2} \lambda_{th}^3\right) \dfrac{\lambda}{g_{1/2}} = -\dfrac{N}{V^2} kT \dfrac{g_{3/2}}{g_{1/2}}$

and $\boxed{\kappa_T =} -\dfrac{1}{V}\left(-\dfrac{V^2}{N} \dfrac{1}{kT} \dfrac{g_{1/2}}{g_{3/2}}\right) = \boxed{\dfrac{1}{nkT} \dfrac{g_{1/2}(\lambda)}{g_{3/2}(\lambda)}}$

(b) $P v^{5/3} = \text{const} \Rightarrow dP\, v^{5/3} + \dfrac{5}{3} P v^{2/3} dv = 0$

$$\Rightarrow \frac{dP}{dv} = -\frac{5}{3} \frac{P}{v}$$

so $\kappa_S = -\dfrac{1}{v}\left(-\dfrac{3}{5} \dfrac{v}{P}\right) = \dfrac{3}{5} \dfrac{1}{P}$

and $P = \dfrac{kT}{\lambda_{th}^3} g_{5/2}$ with $\dfrac{N}{V} = \dfrac{1}{\lambda_{th}^3} g_{3/2} \Rightarrow \dfrac{P}{nkT} = \dfrac{g_{5/2}}{g_{3/2}}$

which gives $\boxed{\kappa_S = \dfrac{3}{5} \dfrac{1}{nkT} \dfrac{g_{3/2}(\lambda)}{g_{5/2}(\lambda)}}$

(c) keep the first 2 terms in $g_n(\lambda) = \sum_{\ell=1}^{\infty} \dfrac{\lambda^\ell}{\ell^n}$

for, first, $n = \frac{3}{2}$ to get $\lambda = n\lambda_{th}^3 - \dfrac{1}{2^{3/2}} n^2 \lambda_{th}^6 + \ldots$

and then $\boxed{\dfrac{P}{kT}} = \dfrac{1}{\lambda_{th}^3}\left[\left(n\lambda_{th}^3 - \dfrac{1}{2^{3/2}} n^2 \lambda_{th}^6 + \ldots\right) + \dfrac{1}{2^{5/2}} n^2 \lambda_{th}^6 + \ldots\right]$

$$= \boxed{n - \frac{\lambda_{th}^3}{2^{5/2}} n^2 + \ldots}$$

4. With $\alpha \to \kappa$ for convenience :

$V(z) = V_0 \sin(k_z z)$ to satisfy equation & $V(0) = V(\tfrac{1}{2}) = 0$

$$\Rightarrow \tfrac{1}{2} k_z = n\pi \Rightarrow k_z = 2\pi n$$

so $$\left(\frac{d^2}{dz^2} - \kappa^2\right)^3 V(z) = \left(-(2\pi n)^2 - \kappa^2\right)^3 V(z) = -R\kappa^2 V(z)$$

and $$R = \frac{\left((2\pi n)^2 + \kappa^2\right)^3}{\kappa^2}$$

For smallest R, $n = 1$ and

$$0 = \frac{dR}{d\kappa^2} = -\frac{\left((2\pi)^2 + \kappa^2\right)^3}{(\kappa^2)^2} + \frac{3\left((2\pi)^2 + \kappa^2\right)^2}{\kappa^2}$$

Which $\Rightarrow$ $-\left((2\pi)^2 + \kappa^2\right) + 3\kappa^2 = 0$

or $\kappa^2 = 2\pi^2$.

Then $$\boxed{R =} \frac{\left((2\pi)^2 + 2\pi^2\right)^3}{2\pi^2} = \frac{6^3\pi^6}{2\pi^2} = 3\cdot 36\pi^4 = \boxed{108\pi^4}$$

5. Reasoning & examples omitted here, but should be given — there are various possibilities

(i) $T < 0$ possible
(ii) $C_V < 0$ not (for stable system)
(iii) $C_p < 0$ not
(iv) $P < 0$ possible
(v) κ_T not
(vi) κ_S not

6. Adding heat increases $E = Mc^2$, but larger mass $\Rightarrow$ lower temperature, so $C < 0$ & thermodynamically unstable