Physics 607 Exam 1

Please be well-organized, and show all significant steps clearly in all problems.

You are graded on your work, so please do not just write down answers with no explanation!

Do all your work on the blank sheets provided, writing your name clearly. (You may keep this exam.)

The variables have their usual meanings: $E =$ energy, $S =$ entropy, $V =$ volume, $N =$ number of particles, $T =$ temperature, $P =$ pressure, $\mu =$ chemical potential, $B =$ applied magnetic field, $C_V =$ heat capacity at constant volume, $C_P =$ heat capacity at constant pressure, $F =$ Helmholtz free energy, $G =$ Gibbs free energy, $k =$ Boltzmann constant, $h =$ Planck constant. Also, $\langle \cdots \rangle$ represents an average.

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \quad , \quad \int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \frac{1}{2}\sqrt{\pi} \quad , \quad \int_0^{\infty} x^n e^{-x^2} dx = \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad , \quad \Gamma(1) = 1 \quad , \quad \Gamma(z+1) = z\Gamma(z) \quad , \quad \ln N! \approx N \ln N - N \quad , \quad e^x = 1 + x + \frac{x^2}{2} + \ldots$$

$$E = TS - PV + \mu N \quad , \quad E = kT^2 \frac{\partial}{\partial T} \ln Z \quad , \quad F = -kT \ln Z \quad , \quad \left\langle x_i \frac{\partial H}{\partial x_j} \right\rangle = \delta_{ij} kT \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad , \quad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

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1. In the Einstein model, a solid consists of N atoms, each vibrating with an angular frequency ω. They are regarded as distinguishable, so that the partition function is $Z = z^N$, where z is given by the sum over the states n of a single oscillator with energies $\varepsilon_n = \left(n + \frac{1}{2}\right)\hbar\omega$.

(a) (5) Obtain z in a simple form, using $1 + x + x^2 + ... = (1-x)^{-1}$, and then the partition function Z. (Is the function you are summing within the radius of convergence?)

(b) (5) Calculate the thermodynamic energy E in a simple form.

(c) (5) Calculate the specific heat at constant volume, C_V, in a simple form.

(d) (5) Assuming that ω does not depend on the volume V, calculate the pressure and the heat capacity at constant pressure, C_P.

(e) (5) Show that $E = NkT + O\left(\left(\frac{\hbar\omega}{kT}\right)^2\right)$ for $kT >> \hbar\omega$ (with no first-order correction in $\hbar\omega / kT$).

2. (a) (6) With V and T treated as the independent variables (and N held fixed) show that

$$dS = \left(\frac{\partial S}{\partial V}\right)_T dV + \frac{C_V}{T} dT \ .$$

(b) (6) Use the expression for dF (where $F = E - TS$) to obtain the Maxwell relation involving

$$\left(\frac{\partial S}{\partial V}\right)_T \text{ and } \left(\frac{\partial P}{\partial T}\right)_V \ .$$

(c) (6) Using the fundamental expression for dE in terms of dS and dV, show that

$$dE = \left[T\left(\frac{\partial P}{\partial T}\right)_V - P\right] dV + C_V dT \ .$$

(d) (7) For a van der Waals gas, with the equation of state $\left(P + \frac{a}{v^2}\right)(V - Nb) = NkT$, $v = \frac{V}{N}$, obtain $\left(\frac{\partial E}{\partial V}\right)_T$ in terms of a and V.

3. Suppose that the entropy of an ideal quantum gas is given by

$$S = -\sum_k \left[\langle n_k \rangle \ln \langle n_k \rangle \pm \left(1 \mp \langle n_k \rangle\right) \ln\left(1 \mp \langle n_k \rangle\right) \right]$$

where $\langle n_k \rangle$ is the average number of particles in state k. Here the upper sign holds for fermions and the lower sign for bosons.

Also suppose that the particles in all the other states act as a reservoir for the specific state k, so that $\sum_k \langle n_k \rangle$ and $\sum_k \langle n_k \rangle \varepsilon_k$ are both fixed constants, which can be treated with Lagrange multipliers γ and β respectively.

(a) (5) Demonstrate with clear arguments that the entropy per particle goes to zero for fermions in the ground state.

(b) (5) Demonstrate with clear arguments that the entropy per particle goes to zero for bosons in the ground state. [Show that $S/N \to 0$ as $N \to \infty$ in the ground state.]

(c) (5) By maximizing the entropy, subject to the constraints above, obtain the equilibrium value of $\langle n_k \rangle$ for fermions in terms of β, ε_k, and γ.

(d) (5) By maximizing the entropy, subject to the constraints above, obtain the equilibrium value of $\langle n_k \rangle$ for bosons in terms of β, ε_k, and γ.

Do these results make sense?

4. Consider the average of the quantity $\frac{dG}{dt}$ over a time τ, with $\tau \to \infty$ and

$$G \equiv \sum_i q_i\, p_i \quad .$$

The q_i are a set of coordinates, and the p_i are the corresponding momenta. We assume a classical system in which the magnitude of G is bounded.

(a) (5) Show that

$$\sum_i \langle u_i p_i \rangle = -\sum_i \langle q_i F_i \rangle$$

where u_i is the velocity and F_i is the force corresponding to the coordinate q_i. (Here, and in all cases where you are given the answer, your arguments must be clear, complete, and convincing.)

Use this result in obtaining the results of parts (b) and (c) below.

(b) (5) For a planet revolving around a star (subject to only the Newtonian gravitational force of the star), obtain the relation between the average kinetic energy $\langle K_{\text{planet}} \rangle$ and the average potential energy $\langle U_{\text{planet}} \rangle$. The motion is nonrelativistic but the orbit can be highly elliptical.

(c) (5) Consider a star revolving at a distance r around the center of a dark matter halo with a spherically symmetric matter density $\rho(r)$. The matter enclosed in the spherical volume of radius r is

$$M(r) = \int_0^r dr' 4\pi r'^2 \rho(r') \ .$$

The observed average velocity of the stars around a given halo is found to be equal to a constant v_0 with respect to r, out to large distances from the center. Determine $M(r)$ as a function of r (involving v_0). In this part you can assume circular orbits for simplicity.

(d) (5) Now let us switch to a different application of the virial theorem: Consider N particles that are confined to a volume V by a pressure P. We first wish to obtain

$$-\sum_i \langle q_i F_i \rangle = -\sum_i \langle q_i F_i' \rangle + 3PV \qquad (1)$$

where the forces F_i' are due to interactions among the particles (and $\langle \cdots \rangle$ represents an average over particles as well as a time average).

Let $\vec{F}_{\text{walls}}$ be the total force on the particles due to their collisions with the walls enclosing the volume. Start by assuming that

$$N \langle \vec{r} \cdot \vec{F}_{\text{walls}} \rangle = -P \int_S \vec{r} \cdot d\vec{S}$$

where S is the surface enclosing V and $\vec{r}$ is the position vector of a particle. Show that this relation leads to the equation labeled (1) above (giving a clear mathematical argument).

[next page for parts (e) and (f)]

(e) (5) Now suppose that

$$F_i' = -\frac{\partial U}{\partial q_i}$$

and that the potential energy U is a homogeneous function of order n of all the particle coordinates:

$$U(\lambda q_1, \lambda q_2, \ldots) = \lambda^n U(q_1, q_2, \ldots) \ .$$

For nonrelativistic particles, show that

$$\langle K \rangle \ = \ \text{constant} \times \langle U \rangle \ + \ \text{different constant} \times PV$$

while at the same time obtaining the constants. Here $\langle K \rangle$ is the mean value of the total kinetic energy and $\langle U \rangle$ is the mean value of the total potential energy.

(f) (5) Finally, consider an ideal gas (with $F_i' = 0$). Using the equipartition theorem, plus Hamilton's equations, obtain the equation of state relating P to N, V, and T.