Physics 607 Exam 1 – Solution

1. (a) $\boxed{z =} \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})\hbar\omega/kT} = \boxed{\dfrac{e^{-\frac{1}{2}\hbar\omega/kT}}{1-e^{-\hbar\omega/kT}}}$ $\quad [e^{-\hbar\omega/kT} < 1]$, $Z = z^N$

(b) $\boxed{\dfrac{E}{N} =} kT^2 \dfrac{\partial}{\partial T} \ln z = \boxed{\dfrac{1}{2}\hbar\omega + \dfrac{\hbar\omega}{e^{\hbar\omega/kT}-1}}$ after the algebra

(d) $\dfrac{F}{N} = -kT \ln z$, $\boxed{P =} - \dfrac{\partial F}{\partial V} = \boxed{0}$ [if ω is not a function of V]

$\boxed{C_p =} \dfrac{\partial(E+PV)}{\partial T} = \dfrac{\partial E}{\partial T} = \boxed{C_V}$

(c) $C_V = \dfrac{\partial E}{\partial T} = k\left(\dfrac{\hbar\omega}{kT}\right)^2 \dfrac{e^{\hbar\omega/kT}}{(e^{\hbar\omega/kT}-1)^2}$ after algebra

(e) $\boxed{\dfrac{E}{N} =} \dfrac{1}{2}\hbar\omega + \hbar\omega\left[\left(1+\dfrac{\hbar\omega}{kT}+\dfrac{1}{2}\left(\dfrac{\hbar\omega}{kT}\right)^2+\dots\right)-1\right]^{-1}$

$= \dfrac{1}{2}\hbar\omega + \hbar\omega \cdot \dfrac{kT}{\hbar\omega}\left[1+\dfrac{1}{2}\left(\dfrac{\hbar\omega}{kT}\right)^2+\dots\right]^{-1}$

$= \cancel{\dfrac{1}{2}\hbar\omega} + kT\left[1 \cancel{- \dfrac{1}{2}\left(\dfrac{\hbar\omega}{kT}\right)} + O\left(\dfrac{\hbar\omega}{kT}\right)^2\right]$

$= \boxed{kT\left[1+O\left(\left(\dfrac{\hbar\omega}{kT}\right)^2\right)\right]}$

2. (a) $dS = \left(\dfrac{\partial S}{\partial V}\right)_T dV + \left(\dfrac{\partial S}{\partial T}\right)_V dT = \left(\dfrac{\partial S}{\partial V}\right)_T dV + \dfrac{C_V}{T}\,dT$

(b) $dF = -S\,dT - P\,dV \Rightarrow \dfrac{\partial F}{\partial T} = -S$, $\dfrac{\partial F}{\partial V} = -P$

so $\boxed{\left(\dfrac{\partial S}{\partial V}\right)_T =} -\dfrac{\partial^2 F}{\partial V \partial T} = -\dfrac{\partial^2 F}{\partial T \partial V} = \boxed{\left(\dfrac{\partial P}{\partial T}\right)_V}$

(c) $dE = T\,dS - P\,dV = T\left[\left(\dfrac{\partial P}{\partial T}\right)_V dV + \dfrac{C_V}{T}\,dT\right] - P\,dV$

$= \left[T\left(\dfrac{\partial P}{\partial T}\right)_V - P\right] dV + C_V\,dT$

(d) $\left(\dfrac{\partial E}{\partial V}\right)_T = T\left(\dfrac{\partial P}{\partial T}\right)_V - P$

$P + \dfrac{a}{v^2} = \dfrac{NkT}{V-Nb} = \dfrac{kT}{v-b} \Rightarrow \left(\dfrac{\partial P}{\partial T}\right)_V = \dfrac{k}{v-b}$

$\therefore \boxed{\left(\dfrac{\partial E}{\partial V}\right)_T =} \dfrac{kT}{v-b} - \left(\dfrac{kT}{v-b} - \dfrac{a}{v^2}\right) = \boxed{\dfrac{a}{v^2}}$

[Then $E = E_0(T) - a\dfrac{N}{v}$, $v = \dfrac{V}{N}$.]

Note: this is a skeleton solution – not a model!

3. In (a) & (b), $1 \ln(1) = 0$ and $n \ln n \to 0$ as $n \to 0$

(a) $\langle n_k \rangle$ and $1 - \langle n_k \rangle = 0$ or 1 in ground state, so $S \to 0$.

(b) $\langle n_k \rangle = 0$ for excited states, $\langle n_k \rangle = N$ for lowest state

$$
\begin{aligned}
N \ln N - (1+N)\ln(1+N) &= N \ln N - (1+N)\ln\left(N\left(1+\tfrac{1}{N}\right)\right) \\
&= N \ln N - (1+N)\ln N - (1+N)\left(\tfrac{1}{N} + O\left(\tfrac{1}{N^2}\right)\right) \\
&= -\ln N + 1 + O\left(\tfrac{1}{N}\right)
\end{aligned}
$$

$$\Rightarrow \frac{S}{N} = -\frac{\ln N}{N} + O\left(\frac{1}{N}\right) \to 0 \quad \text{as} \quad N \to \infty$$

(c) & (d) We have essentially

$$\frac{\partial}{\partial n}\left[-n \ln n \mp (1 \mp n)\ln(1 \mp n) - \gamma n - \beta n \varepsilon\right] = 0$$

$$
\begin{aligned}
\Rightarrow 0 &= -\ln n - n \cdot \tfrac{1}{n} \mp (\mp 1)\ln(1 \mp n) \mp (1 \mp n)\tfrac{1}{1 \mp n}(\mp 1) - \gamma - \beta\varepsilon \\
&= -\ln n \cancel{- 1} + \ln(1 \mp n) \cancel{+ 1} - \gamma - \beta\varepsilon \\
&= \ln\left(\frac{1 \mp n}{n}\right) - \gamma - \beta\varepsilon
\end{aligned}
$$

$$\Rightarrow \ln(n^{-1} \mp 1) = \gamma + \beta\varepsilon$$

$$\Rightarrow n^{-1} \mp 1 = e^{\gamma + \beta\varepsilon}$$

$$\Rightarrow n^{-1} = e^{\beta\varepsilon + \gamma} \pm 1$$

$$\Rightarrow \boxed{n = \frac{1}{e^{\beta\varepsilon + \gamma} \pm 1}}$$

If we let $\gamma = -\beta\mu$, we have the form

$$\boxed{n_k = \frac{1}{e^{\beta(\varepsilon_k - \mu)} \pm 1}}$$

of the Fermi-Dirac and Bose-Einstein distribution functions.

4. (a) $\left\langle \frac{dG}{dt} \right\rangle = \frac{1}{\tau}\int_0^\tau dt \frac{dG}{dt} = \frac{1}{\tau}\left(G(\tau) - G(0)\right) \to 0$ as $\tau \to \infty$

$$\frac{dG}{dt} = \frac{d}{dt}\sum_i q_i p_i = \sum_i \dot{q}_i p_i + \sum_i q_i \dot{p}_i$$

$\therefore \sum_i \langle \dot{q}_i p_i \rangle + \sum_i \langle q_i \dot{p}_i \rangle = 0$ or $\boxed{\sum_i \langle u_i p_i \rangle = -\sum_i \langle q_i F_i \rangle}$

since $u_i = \dot{q}_i$ and $\dot{p}_i = F_i$

(b) $\sum_i \langle u_i p_i \rangle = \sum_i \langle v_i \cdot m v_i \rangle = 2\langle \frac{1}{2} m v^2 \rangle = 2\langle K_{planet} \rangle$

$$\sum_i \langle q_i F_i \rangle = \langle \vec{r} \cdot \vec{F} \rangle = \langle r\left(-G\frac{mM}{r^2}\right)\rangle = \langle U_{planet} \rangle$$

$$\therefore \boxed{\langle U_{planet} \rangle = -2\langle K_{planet} \rangle}$$

(c) $2\langle \frac{1}{2}\cancel{m} v^2 \rangle = G\left\langle \frac{\cancel{m}M}{r} \right\rangle \Rightarrow v_0^2 = \frac{GM(r)}{r}$

$$\Rightarrow \boxed{M(r) = \frac{v_0^2}{G} r}$$

(d) $\int_S \vec{r} \cdot d\vec{S} = \int_V \vec{\nabla} \cdot \vec{r}\, d^3r = \int_V \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) d^3r = 3\int d^3r = 3V$

$\Rightarrow N\langle \vec{r} \cdot \vec{F}_{walls} \rangle = -P\,3V$ and this is $\sum_i \langle q_i F_i^{walls} \rangle$

so $\boxed{-\sum_i \langle q_i F_i \rangle = -\sum_i \langle q_i F_i' \rangle + 3PV}$

(e) $$\frac{\partial U(\lambda q_1, \lambda q_2, \ldots)}{\partial \lambda} = \frac{\partial U(\lambda q_1, \ldots)}{\partial(\lambda q_1)} \frac{\partial(\lambda q_1)}{\partial \lambda} + \ldots$$

$$= q_1 \frac{\partial U(\lambda q_1, \ldots)}{\partial(\lambda q_1)} + \ldots$$

and this also $= \frac{\partial}{\partial \lambda}\left(\lambda^n U(q_1, \ldots)\right) = n\lambda^{n-1} U(q_1, \ldots)$

Set $\lambda = 1$: $q_1 \frac{\partial U(q_1, \ldots)}{\partial q_1} + q_2 \frac{\partial U(q_2, \ldots)}{\partial q_2} + \ldots = n\,U(q_1, q_2, \ldots)$

Then $\sum_i \langle q_i F_i' \rangle = -n\langle U \rangle$, with $\langle K \rangle = \frac{1}{2} m \langle \sum_i v_i^2 \rangle$;

$\Rightarrow 2\langle K \rangle = +n\langle U \rangle + PV \Rightarrow \boxed{\langle K \rangle = \frac{n}{2}\langle U \rangle + \frac{3}{2}PV}$

[$P = 0$, $n = -1 \Rightarrow$ result like (b); $\langle U \rangle = 0 \Rightarrow \langle K \rangle = \frac{3}{2}PV$]

(f) $-\sum_i \langle q_i F_i \rangle = -\sum_i \langle q_i \dot{p}_i \rangle = +\sum_i \left\langle q_i \frac{\partial H}{\partial q_i} \right\rangle = 3NkT$

$\Rightarrow$ from (d), $3NkT = 3PV \Rightarrow \boxed{PV = NkT}$