1. Apply $\left\langle x_i \frac{\partial H}{\partial x_j} \right\rangle = \delta_{ij} kT$.

(a) $\left\langle r \frac{\partial H}{\partial r} \right\rangle = \langle r \cdot Kr \rangle = kT \Rightarrow \boxed{\langle r^2 \rangle = \frac{kT}{K}}$

(b) $kT = \left\langle r \frac{\partial H}{\partial r} \right\rangle = \left\langle r \left(-2 \cdot \frac{p_\theta^2}{2\mu} \cdot r^{-3} - 2 \frac{p_\phi^2}{2\mu \sin^2\phi} r^{-3} + Kr \right) \right\rangle$

$$= -2 \left\langle \frac{p_\theta^2}{2\mu r^2} \right\rangle - 2 \left\langle \frac{p_\phi^2}{2\mu r^2 \sin^2\theta} \right\rangle + \langle K r^2 \rangle$$

but $kT = \left\langle p_\theta \cdot 2 \frac{p_\theta}{2\mu r^2} \right\rangle \Rightarrow \left\langle \frac{p_\theta^2}{2\mu r^2} \right\rangle = \frac{1}{2} kT$

$$kT = \left\langle p_\phi \cdot 2 \frac{p_\phi}{2\mu r^2 \sin^2\theta} \right\rangle \Rightarrow \left\langle \frac{p_\phi^2}{2\mu r^2 \sin^2\theta} \right\rangle = \frac{1}{2} kT$$

so $kT = -2 \cdot \frac{1}{2} kT - 2 \cdot \frac{1}{2} kT + \langle K r^2 \rangle$

$$\Rightarrow \langle K r^2 \rangle = 3kT \Rightarrow \boxed{\langle r^2 \rangle = \frac{3kT}{K}}$$

2. (a) $\rho(\omega)\, d\omega = 4\pi \frac{V}{h^3} \left[2 \left(\frac{\hbar\omega}{v_t} \right)^2 d\left(\frac{\hbar\omega}{v_t} \right) + \left(\frac{\hbar\omega}{v_\ell} \right)^2 d\left(\frac{\hbar\omega}{v_\ell} \right) \right]$

$$= \frac{V}{2\pi^2} \left(\frac{2}{v_t^3} + \frac{1}{v_\ell^3} \right) \omega^2 d\omega$$

$$\Rightarrow \boxed{\rho(\omega) = \frac{3V}{2\pi^2 \bar{v}^3} \omega^2} \boxed{= A\omega^2}$$

(b) $3N = \int_0^{\omega_D} d\omega\, \rho(\omega) = A \int_0^{\omega_D} d\omega\, \omega^2 = A \frac{\omega_D^3}{3}$

$$\Rightarrow \boxed{\omega_D^3 = \frac{9N}{A}} = 9N \cdot \frac{2\pi^2 \bar{v}^3}{3V} \Rightarrow \boxed{\omega_D = \left(6\pi^2 \frac{N}{V} \right)^{1/3} \bar{v}}$$

(c) $\boxed{E_0 =} \int_0^{\omega_D} d\omega\, \rho(\omega) \cdot \frac{1}{2} \hbar\omega$

$$= A \cdot \frac{1}{2} \hbar \int_0^{\omega_D} d\omega\, \omega^3$$

$$= \frac{1}{2} \hbar A \left[\frac{\omega_D^4}{4} \right] = \frac{1}{2} \hbar A \cdot \frac{1}{4} \omega_D^3 \cdot \omega_D$$

$$= \frac{1}{8} \hbar \cancel{A} \cdot \frac{9N}{\cancel{A}} \omega_D = \frac{9N}{8} \hbar\omega_D = \boxed{\frac{9}{8} N k_B \Theta_D}$$

3. (a) $N = 2\int_0^{p_F} \rho(p)\,dp = 2\frac{4\pi V}{h^3}\int_0^{p_F} p^2\,dp = \frac{8\pi V}{h^3}\frac{p_F^3}{3}$

spins

$$\Rightarrow \boxed{p_F = \left(\frac{3}{8\pi}\right)^{1/3} h\, n^{1/3}} \Rightarrow \boxed{\varepsilon_F = \left(\frac{3}{8\pi}\right)^{2/3}\frac{h^2}{2m} n^{2/3}}$$

(b) $$\boxed{\frac{E}{V} =} \frac{8\pi}{h^3}\int_0^{p_F} dp\, p^2 \cdot \frac{p^2}{2m} = \frac{8\pi}{2mh^3}\frac{p_F^5}{5} = \boxed{\frac{4\pi}{5}\frac{h^2}{m}\left(\frac{3}{8\pi}\right)^{5/3} n^{5/3}}$$

$$\boxed{= C n^{5/3}}$$

(c) $$\frac{\partial P}{\partial \rho} = \frac{2}{3}\frac{1}{m}\frac{\partial}{\partial n}\left(\frac{E}{V}\right) = \frac{2}{3}\frac{1}{m} C \cdot \frac{5}{3} n^{2/3}$$

$$= \frac{10}{9}\frac{1}{m} C n^{2/3}$$

$$\Rightarrow u^2 = \frac{10}{9}\frac{1}{m}\frac{4\pi}{5}\frac{h^2}{m}\left(\frac{3}{8\pi}\right)^{5/3} n^{2/3} = \frac{1}{3}\left(\frac{8\pi}{3}\right)\left(\frac{3}{8\pi}\right)^{5/3}\left(\frac{h}{m}\right)^2 n^{2/3}$$

$$= \frac{1}{3}\left(\frac{3}{8\pi}\right)^{2/3}\left(\frac{h}{m}\right)^2 n^{2/3}$$

$$\Rightarrow \boxed{u = \frac{1}{\sqrt{3}}\left(\frac{3}{8\pi}\right)^{1/3}\frac{h}{m} n^{1/3}}$$

(d) $$\boxed{u_F =} \frac{p_F}{m} = \boxed{\left(\frac{3}{8\pi}\right)^{1/3}\frac{h}{m} n^{1/3}} \Rightarrow \boxed{u = \frac{1}{\sqrt{3}} u_F}$$

(e) $dE = T\,dS - P\,dV + \mu\,dN$ & $dE = T\,dS + S\,dT - P\,dV - V\,dP + \mu\,dN + N\,d\mu$

$$\Rightarrow \boxed{S\,dT - V\,dP + N\,d\mu = 0} \text{ or } \boxed{s\,dT - v\,dP + d\mu = 0}$$

(f) T fixed $\Rightarrow dP - n\,d\mu = 0 \Rightarrow \frac{\partial P}{\partial n} - n\frac{\partial \mu}{\partial n} = 0$ or $\boxed{\left(\frac{\partial P}{\partial n}\right)_T = n\left(\frac{\partial \mu}{\partial n}\right)_T}$

or $\boxed{\left(\frac{\partial P}{\partial \rho}\right)_T = \frac{n}{m}\left(\frac{\partial \mu}{\partial n}\right)_T}$

(g) Then at $T=0$, from part (a)

$$u^2 = \frac{n}{m}\frac{\partial}{\partial n}\left[\left(\frac{3}{8\pi}\right)^{2/3}\frac{h^2}{2m} n^{2/3}\right]$$

$$= \frac{n}{m}\left(\frac{3}{8\pi}\right)^{2/3}\frac{h^2}{2m}\cdot\frac{2}{3} n^{-1/3}$$

$$= \frac{1}{3}\left(\frac{3}{8\pi}\right)^{2/3}\frac{h^2}{m^2} n^{2/3}$$

$$\Rightarrow \boxed{u = \frac{1}{\sqrt{3}} u_F} \quad \text{again}$$

4. (a) $\boxed{N - N_0 =} \int_0^\infty dp\, \rho(p) \langle n(\varepsilon(p)) \rangle = \boxed{A \int_0^\infty dp\, p^{D-1} \frac{\lambda}{e^{\varepsilon(p)/kT} - \lambda}}$

(b) $\varepsilon_0 = 0 \Rightarrow N_0 = \frac{\lambda}{1-\lambda} \to \infty \Rightarrow \lambda \to 1 \Rightarrow e^{\mu/kT} \to 1 \Rightarrow \boxed{\mu \to 0}$

[$\mu \to 0$ as $N \to \infty$ or $V \to \infty$ with $\frac{N}{V}$ fixed]

(c) $\boxed{N - N_0 = A \underbrace{\int_0^\infty dp\, p^{D-1} \frac{1}{e^{\varepsilon(p)/kT} - 1}}_{\equiv I}}$

(d) $I = \int_0^\infty dp\, p^{D-1} \frac{1}{e^{ap^s/kT} - 1}$

$u \equiv \frac{ap^s}{kT} \Rightarrow p = \left(\frac{kT}{a} u\right)^{1/s}$ and $dp = \left(\frac{kT}{a}\right)^{1/s} u^{1/s - 1}\, du$

$\Rightarrow I = \int_0^\infty \left(\frac{kT}{a}\right)^{1/s} u^{1/s-1} du \cdot \left(\frac{kT}{a}\right)^{\frac{D-1}{s}} u^{\frac{D-1}{s}} \frac{1}{e^u - 1}$

$= B\, T^{1/s}\, T^{\frac{D-1}{s}}$, $\boxed{B \equiv \left(\frac{k}{a}\right)^{D/s} \int_0^\infty du\, u^{\frac{D}{s}-1} \frac{1}{e^u - 1}}$ [a constant]

$= B\, T^{D/s}$

$\Rightarrow \boxed{N - N_0 = C\, T^{D/s}, \quad C \equiv A \cdot B}$

(e) If B is finite, then $C T^{D/s}$ will become less than N for $T < T_c$, $N = C T_c^{D/s}$ or $\boxed{T_c = \left(\frac{N}{C}\right)^{s/D}}$ forcing N_0 to be finite (i.e., macroscopic).

B will be finite if the integral converges at the lower limit, with $u^{\frac{D}{s}-1} \frac{1}{e^u - 1} \to u^{\frac{D}{s}-1} \frac{1}{1+u-1} = u^{\frac{D}{s}-2}$ as $u \to 0$

and $\int du\, u^{\frac{D}{s}-2} = \frac{u^{\frac{D}{s}-1}}{\frac{D}{s}-1}$, so we need $\frac{D}{s} > 1$ or $\boxed{D > s}$.

(f) $\boxed{\frac{N_0}{N} =} 1 - \frac{C T^{D/s}}{N} = 1 - \frac{T^{D/s}}{T_c^{D/s}} = \boxed{1 - \left(\frac{T}{T_c}\right)^{D/s}}$

If $D = 3$ & $s = 2$, $\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2}$.