Physics 607 Final Exam

Please be well-organized, and show all significant steps clearly in all problems.

You are graded on your work, so please do not just write down answers with no explanation!

Do all your work on the blank sheets provided, writing your name clearly. (You may keep this exam.)

The variables have their usual meanings: $E =$ energy, $S =$ entropy, $V =$ volume, $N =$ number of particles, $T =$ temperature, $P =$ pressure, $\mu =$ chemical potential, $B =$ applied magnetic field, $C_V =$ heat capacity at constant volume, $C_P =$ heat capacity at constant pressure, $F =$ Helmholtz free energy, $G =$ Gibbs free energy, $k =$ Boltzmann constant, $h =$ Planck constant. Also, $\langle \cdots \rangle$ represents an average.

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \quad , \quad \int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \frac{1}{2}\sqrt{\pi} \quad , \quad \int_0^{\infty} x^n e^{-x^2} dx = \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad , \quad \Gamma(1) = 1 \quad , \quad \Gamma(z+1) = z\Gamma(z) \quad , \quad \ln N! \approx N \ln N - N \quad , \quad e^x = 1 + x + \frac{x^2}{2} + \ldots$$

$$E = TS - PV + \mu N \quad , \quad E = kT^2 \frac{\partial}{\partial T} \ln Z \quad , \quad F = -kT \ln Z \quad , \quad \left\langle x_i \frac{\partial H}{\partial x_j} \right\rangle = \delta_{ij} kT \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad , \quad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt \quad \Rightarrow \quad \int_0^{\infty} x^n e^{-ax} dx = \frac{\Gamma(n+1)}{a^{n+1}} = \frac{n!}{a^{n+1}} \quad \text{(if n is an integer)}$$

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1. (a) (4) Consider a paramagnetic system of distinguishable atoms in a magnetic field $\vec{B}$. Each atom has spin ½ and a magnetic dipole moment whose component parallel to $\vec{B}$ is $\pm m_B$. Its energy is then $\mp m_B B$ (with the dipole interaction between spins neglected here).

Obtain the partition function z for a single atom. Then obtain the partition function Z_N for the system of N atoms.

(b) (4) Now let us switch to a more general system, for which a given state *j* has a magnetic dipole moment whose component parallel to $\vec{B}$ is M_j. The energy for this state is then $-M_j B$.

Show that the average value of the total magnetic dipole moment of the system is given by

$$\langle M \rangle = kT\left(\frac{\partial \ln Z}{\partial B}\right)_T$$

(at temperature T, with N implicitly fixed), where Z is the partition function of this system.

(c) (4) Using the definition of the Helmholtz free energy, $F = E - TS$, and the fundamental expression for *dE,* obtain the expression for dF in terms of dT, dV, and dN.

(d) (4) Using the results of parts (b) and (c), obtain the (Maxwell) relation between $\left(\frac{\partial \langle M \rangle}{\partial T}\right)_B$ and $\left(\frac{\partial S}{\partial B}\right)_T$ (with N and V implicitly fixed).

(e) (4) Using your answer to Part (d), give a clear and convincing explanation of how $\left(\frac{\partial \langle M \rangle}{\partial T}\right)_B$ behaves as $T \to 0$.

(f) (4) Give a different (clear and convincing) physical explanation of why $\langle M \rangle$ should show this behavior in a paramagnetic system.

2. Recall that the classical partition function for N indistinguishable particles has the form

$$Z = \frac{1}{N!}\int \frac{dp\, dq}{h^{3N}}\, e^{-\beta H(p,q)} \quad .$$

(a) (10) Consider an ultrarelativistic ideal gas of particles (with no internal structure), so that the one-particle energy and momentum are related through $\varepsilon = pc$.

Calculate Z for this system in terms of N, V, and $\frac{kT}{hc}$.

(b) (5) Using the result of part (a), calculate the energy E as a function of the temperature T.

(c) (5) Using the result of part (a), and the result for dF in part (c) of problem 1 (i.e., the expression for dF in terms of dT , dV , and dN , where F is the Helmholtz free energy), calculate the pressure P, and then obtain the equation of state relating P and V to N and T.

(d) (5) Using the above results, obtain the relation between the pressure P and the energy density $\frac{E}{V}$.

3. (22) Suppose that a D-dimensional system has bosonic excitations for which the dispersion relation is

$$\varepsilon = a\,p^n$$

where p is the momentum and a and n are constants. Assume a maximum energy $\varepsilon_{\max}$, just as in the Debye model for phonons.

Recall that the density of states in momentum space is given by

$$\rho(p)dp = A\,p^{D-1}dp \quad , \quad A = \frac{2\pi^{D/2}}{\Gamma(D/2)} \cdot \frac{1}{h^D/V}$$

where V is the D-dimensional volume in which the excitations are confined.

The number of excitations is not fixed (just as for phonons or photons), so the average number in a given mode with energy ε is given by the Bose-Einstein distribution function with $\mu = 0$.

For low temperatures, $\frac{\varepsilon_{\max}}{kT} >> 1$, show that

$$C_V = \text{ constant } \times T^{\alpha}$$

where C_V is the heat capacity at constant volume and you will determine α, in terms of D and n, as well as the constant prefactor, in terms of the various constants and a definite integral (which you need not evaluate).

Please give complete arguments and be clear in each step.

4. Let us estimate the limiting mass for a compact object (white dwarf or neutron star) using rough approximations, including the assumption that it is entirely supported by the degeneracy pressure of the relevant fermions (electrons or neutrons), that the density is uniform, and that these fermions are highly relativistic. You may assume that the central gravitational pressure is given by

$$P_{grav} = \frac{\alpha}{4\pi}\frac{GM^2}{R^4} \quad , \quad V = \frac{4}{3}\pi R^3 \quad , \quad M = Nm'$$

where α is a dimensionless constant of order unity and $G = 6.67 \times 10^{-11}$ N m^2 / kg^2 is the gravitational constant. Also, N is the number of relevant fermions in the compact object, which has mass M, radius R, and volume V.

We distinguish between the actual mass m of a relevant fermion (9.11×10^{-31} kg for an electron and 1.67×10^{-27} kg for a neutron) and the gravitational mass m' per fermion ($2 \times 1.67 \times 10^{-27}$ kg for an electron in a white dwarf composed of ^{4}He, ^{12}C, ^{16}O, etc. and 1.67×10^{-27} kg for a neutron star).

The relativistic expression for the energy ε of a particle is

$$\varepsilon^2 = p^2c^2 + m^2c^4 \qquad \text{with} \qquad c = 3.00 \times 10^8 \text{ m/s}.$$

In the following, treat the N fermions as simply a quantum ideal gas confined to a volume V. According to a general argument for a quantum ideal gas

$$P = \frac{1}{3}n\langle pu \rangle \quad , \quad n \equiv \frac{N}{V} \quad , \quad u = \frac{\partial \varepsilon}{\partial p}$$

where P is the pressure due to the gas and u is the particle velocity.

In the following, assume that $pc \gg mc^2$, but keep the leading 2 terms in the energy and pressure, as indicated below.

(a) (3) Obtain ε in the form $\qquad \varepsilon \approx pc + \dfrac{ac}{p} \qquad$ (where you will determine the constant a).

(b) (3) Obtain the pressure P in the form $\qquad P = \dfrac{1}{3}nc\left\langle p + \dfrac{b}{p} \right\rangle \qquad$ (where you will determine the constant b).

(c) (3) Obtain the density of states in momentum space, in the form $A p^2$ (where you will determine the constant A).

(d) (4) Calculate the Fermi momentum p_F as a function of $n = N/V$.

(e) (4) Calculate the degeneracy pressure P as a function of n.

(f) (4) Set P equal to P_{grav} and show that there is a limiting mass M_0 requiring $M \le M_0$. Express M_0 in terms of the various constants.

(g) (4) In the case of a neutron star, assuming the value $\alpha = 3/2$, calculate (or carefully estimate) the value of M_0 in solar masses $M_\odot$, with $M_\odot = 1.99 \times 10^{30}$ kg. Show all the steps in SI units, but you may use the fact that $(\hbar c / G)^{1/2}$ = Planck mass $= 2.18 \times 10^{-8}$ kg .

(h) (4) For a neutron star with mass $M_0/2$, estimate the radius R in kilometers. (It is possible that you may want to use the fact that Compton wavelength of neutron $\equiv \dfrac{h}{m_n c} = 1.32 \times 10^{-15}$ m .)

5. **For extra credit:** Let us obtain the BCS scaling prediction

$$\frac{2\Delta(0)}{kT_c} = 3.53$$

starting with the BCS gap equation

$$1 = N(0)V_0 \int_0^{\hbar\omega_D} d\varepsilon \, \frac{\tanh\left(\dfrac{E}{2kT}\right)}{E} \qquad , \qquad E = \left(\varepsilon^2 + \Delta(T)^2\right)^{1/2} \ .$$

Here $\Delta(0)$ is the gap in the electronic spectrum at $T = 0$, which can be measured in microwave absorption or tunneling experiments. T_c is the transition temperature, below which the resistivity falls to zero. Also, $N(0)$ is the electronic density of states (for one spin), V_0 is the effective phonon-mediated attraction, and $\hbar\omega_D$ is the maximum (Debye) phonon energy. Using $\tanh(\infty) = 1$, you will need the integrals

$$\int_0^a \frac{\tanh x}{x} dx \approx \ln(2.268\, a) \qquad \text{for } a \gg 1$$

$$\int_0^b \frac{1}{\left(1+x^2\right)^{1/2}} dx = \sinh^{-1} b \qquad \text{with } \sinh y \approx \frac{1}{2} e^y \ \text{ for } y \gg 1 .$$

Assume weak coupling,

$$N(0)V_0 \ll 1 \ ,$$

which according to the results below also implies that

$$\frac{kT_c}{\hbar\omega_D} \ll 1 \qquad \text{and} \qquad \frac{\Delta(0)}{\hbar\omega_D} \ll 1 \ .$$

(a) (5) By using the gap equation at $T = T_c$, show that $kT_c = \text{prefactor} \times e^{-1/N(0)V_0}$, while at the same time determining the prefactor.

(b) (5) By using the gap equation at $T = 0$, show that $\Delta(0) =$ different prefactor $\times\, e^{-1/N(0)V_0}$, while at the same time determining this different prefactor.

Happy Holidays, Merry Christmas, and best wishes for the New Year!