1. (a) $z = e^{-(-m_B B)/kT} + e^{-m_B B/kT} = e^{m_B B/kT} + e^{-m_B B/kT}$

$Z = z^N$

(b) $\langle M \rangle = \sum_j M_j \underbrace{\frac{e^{-(-M_j B/kT)}}{Z}}_{= \mathcal{P}_j} = \frac{1}{Z} \sum_j M_j e^{M_j B/kT}$

$$Z = \sum_j e^{M_j B/kT} \Rightarrow kT \frac{\partial \ln Z}{\partial B} = kT \frac{1}{Z} \sum_j \frac{M_j}{kT} e^{M_j B/kT} = \langle M \rangle$$

(c) $\boxed{dF =} (TdS - PdV + \mu dN) - (TdS - SdT) = \boxed{-SdT - PdV + \mu dN}$

(d) $F = -kT \ln Z \Rightarrow \langle M \rangle = -\left(\frac{\partial F}{\partial B}\right)_T$ with $-S = \left(\frac{\partial F}{\partial T}\right)_B$

$$\therefore \boxed{\left(\frac{\partial \langle M \rangle}{\partial T}\right)_B} = -\frac{\partial^2 F}{\partial T \, \partial B} = -\frac{\partial^2 F}{\partial B \, \partial T} = \boxed{\left(\frac{\partial S}{\partial B}\right)_T}$$

(e) 3rd law $\Rightarrow S \to 0$ as $T \to 0$

For any X, $\left(\frac{\partial S}{\partial X}\right)_T \to 0$ as $T \to 0$

since $\frac{\Delta S}{\Delta X} = \frac{S_2 - S_1}{\Delta X}$ with S_2 & $S_1 \to 0$ as $T \to 0$

Then $\left(\frac{\partial S}{\partial B}\right)_T \to 0$ as $T \to 0$, and

$$\boxed{\left(\frac{\partial \langle M \rangle}{\partial T}\right)_B \to 0 \text{ as } T \to 0}.$$

(f) For any fixed $\vec{B}$, as $T \to 0$ all the dipole moments are aligned with $\vec{B}$, so $\langle M \rangle$ approaches a constant and it is plausible that $\frac{\partial \langle M \rangle}{\partial T} \to 0$.

2. (a) $$\boxed{Z =} \frac{1}{N!}\frac{1}{h^{3N}} \underbrace{\int d^{3N}x}_{= V^N} \Big(\underbrace{\int_0^\infty 4\pi p^2 dp \cdot e^{-pc/kT}}_{= 4\pi \frac{2!}{(c/kT)^3}} \Big)^N$$

$$= \boxed{\frac{1}{N!}\left[8\pi V\left(\frac{kT}{hc}\right)^3\right]^N}$$

(b) $$\boxed{E =} kT^2 \frac{\partial}{\partial T}\left(\ln T^{3N} + \ln(\text{function of } V \text{ and } N)\right)$$

$$= kT^2 \cdot 3N\frac{1}{T} = \boxed{3NkT}$$

(c) $$dF = -SdT - PdV + \mu dN$$

$$\Rightarrow \quad P = -\frac{\partial F}{\partial V} = -\frac{\partial}{\partial V}(-kT \ln Z)$$

$$= kT\frac{\partial}{\partial V}\left(\ln V^N + \ln(\text{function of } T \,\&\, N)\right)$$

$$= kT \cdot \frac{N}{V}$$

$$\Rightarrow \boxed{PV = NkT}$$

(d) $$\boxed{P = \frac{NkT}{V} = \frac{1}{3}\frac{E}{V}}$$

3. $E = \int_0^{\varepsilon_{max}} \rho(\varepsilon)\, d\varepsilon \cdot \frac{\varepsilon}{e^{(\varepsilon-\mu)/kT} - 1}$, $\mu = 0$,

with $\rho(\varepsilon)\, d\varepsilon = \rho(p)\, dp = A p^{D-1} dp$ & $\varepsilon = a p^n$

This can be done with $\int d\varepsilon$ or $\int dp$,

and with E calculated first or

C_V calculated directly.

One way, with $\int dp$ and C_V calculated directly:

$$C_V = \frac{\partial E}{\partial T} = \frac{\partial}{\partial T} \int_0^{p_{max}} \rho(p)\, dp \cdot \frac{\varepsilon(p)}{e^{\varepsilon(p)/kT} - 1}$$

$$\frac{\partial}{\partial T} \frac{\varepsilon}{e^{\varepsilon/kT} - 1} = - \frac{\varepsilon}{(e^{\varepsilon/kT} - 1)^2}\, e^{\varepsilon/kT} \left(\frac{\varepsilon}{k}\right)\left(-\frac{1}{T^2}\right)$$

$$= k \frac{x^2 e^x}{(e^x - 1)^2} \quad , \quad x \equiv \frac{\varepsilon}{kT}$$

Then $C_V = \int_0^{p_{max}} A p^{D-1} dp \cdot k \frac{x^2 e^x}{(e^x - 1)^2}$.

$$p = \left(\frac{\varepsilon}{a}\right)^{1/n} \Rightarrow p^{D-1} = \left(\frac{\varepsilon}{a}\right)^{\frac{D-1}{n}} \quad \& \quad dp = \frac{1}{a^n} \varepsilon^{\frac{1}{n}-1} d\varepsilon$$

so $\boxed{C_V =} A \frac{1}{a^{\frac{D-1}{n}}} \frac{1}{a^n} \int_0^{\varepsilon_{max}} \underbrace{\varepsilon^{\frac{D-1}{n}} \varepsilon^{\frac{1}{n}-1}}_{= \varepsilon^{\frac{D}{n}-1}} d\varepsilon \cdot k \frac{x^2 e^x}{(e^x - 1)^2}$

$$= A \frac{1}{a^{\frac{D-1}{n}+n}} (kT)^{D/n} \int_0^{\varepsilon_{max}/kT} x^{\frac{D}{n}-1} dx \cdot k \frac{x^2 e^x}{(e^x - 1)^2}$$

$$\approx \boxed{\text{constant} \times T^{D/n}} \quad \text{for } \frac{\varepsilon_{max}}{kT} \gg 1$$

where

$$\boxed{\text{constant} = k \frac{A}{a^{\frac{D-1}{n}+n}} k^{D/n} \int_0^\infty dx \frac{x^{\frac{D}{n}+1} e^x}{(e^x - 1)^2}}$$

check: for Debye model with $D = 3$ & $n = 1$, $a = \bar{v}$, this gives

$$C_V = k \frac{A}{\bar{v}^3} k^3 \int_0^\infty dx \frac{x^4 e^x}{(e^x - 1)^2} \times T^3 \quad ✓$$

4. (a) $\boxed{E} = pc\left(1+\frac{m^2c^4}{p^2c^2}\right)^{1/2} \boxed{\approx} pc\left(1+\frac{1}{2}\frac{m^2c^4}{p^2c^2}\right) = \boxed{pc + \frac{ac}{p}}$, $\boxed{a \equiv \frac{1}{2} m^2 c^2}$

(b) $\frac{\partial E}{\partial p} = c - \frac{ac}{p^2} \Rightarrow \boxed{P = \frac{1}{3} n c \left\langle p - \frac{a}{p} \right\rangle}$

(c) $\underset{\text{spins}}{2}\cdot\frac{4\pi p^2 dp}{h^3/V} = \frac{2V}{h^3}\cdot 4\pi p^2 dp$ or $\boxed{\rho(p) = \frac{2V}{h^3}\cdot 4\pi p^2}$ from usual argument

(d) $N = \frac{2V}{h^3}\cdot 4\pi \int_0^{p_F} p^2 dp = \frac{2V}{h^3}\cdot\frac{4\pi}{3}p_F^3 \Rightarrow \boxed{p_F = h\left(\frac{3}{8\pi}n\right)^{1/3}}$, $n \equiv \frac{N}{V}$

(e) $\boxed{P =} \frac{V}{N}\cdot\frac{2\cancel{V}}{h^3}\cdot 4\pi \int_0^{p_F} dp\, p^2 \cdot \frac{1}{3}\cancel{n} c\left(p - \frac{a}{p}\right)$

$= \frac{8\pi}{3}\frac{c}{h^3}\left(\frac{p_F^4}{4} - a\frac{p_F^2}{2}\right) = \frac{2\pi}{3}\frac{c}{h^3}p_F^2\left(p_F^2 - m^2c^2\right)$

$\boxed{= \frac{2\pi}{3}\frac{c}{h^3}h^2\left(\frac{3}{8\pi}n\right)^{2/3}\left[h^2\left(\frac{3}{8\pi}n\right)^{2/3} - (mc)^2\right]}$

$\lambda \equiv \frac{h}{mc}$

(f) Note that $\frac{h^2}{(mc)^2}\left(\frac{3}{8\pi}n\right)^{2/3} = \lambda^2\left(\frac{3}{8\pi}\frac{M/m'}{\frac{4}{3}\pi R^3}\right)^{2/3} = \underbrace{\lambda^2\left(\left(\frac{3}{8\pi}\right)^2\frac{2}{m'}\frac{M}{R^3}\right)^{2/3}}_{\equiv \lambda^2 B \frac{M^{2/3}}{R^2}}$

and that $\frac{2\pi}{3}\frac{hc}{h^2}\left(\frac{3}{8\pi}n\right)^{2/3}(mc)^2 = \underbrace{\frac{2\pi}{3}hc\frac{1}{\lambda^2}B\frac{M^{2/3}}{R^2}}_{\equiv \frac{DB}{\lambda^2}\frac{M^{2/3}}{R^2}}$

Then

$$\underbrace{\frac{\alpha G}{4\pi}}_{\equiv \bar{a}}\frac{M^2}{R^4} = \underbrace{\frac{DB}{\lambda^2}}_{\equiv \bar{b}}\frac{M^{2/3}}{R^2}\left(\underbrace{\lambda^2 B}_{\equiv \bar{c}}\frac{M^{2/3}}{R^2} - 1\right)$$

$\Rightarrow \bar{a} M^{4/3} = \bar{b}\bar{c} M^{2/3} - \bar{b} R^2$

$\Rightarrow \boxed{R^2 =} \bar{c} M^{2/3}\left(1 - \frac{\bar{a}}{\bar{b}\bar{c}}M^{2/3}\right)$

$= \lambda^2\left(\left(\frac{3}{8\pi}\right)^2\frac{2}{m'}\right)^{2/3} M^{2/3}\left(1 - \frac{\alpha G/4\pi}{DB\cdot B}M^{2/3}\right)$

$= \boxed{\left(\frac{h}{mc}\right)^2\left(\left(\frac{3}{8\pi}\right)^2\frac{2}{m'}\right)^{2/3} M^{2/3}\left(1 - \left(\frac{M}{M_0}\right)^{2/3}\right)}$

where $\boxed{M_0} \equiv \left(\frac{DB^2}{\alpha G/4\pi}\right)^{3/2} \boxed{= \left(\frac{2\pi}{3}hc\right)^{3/2}\left(\left(\frac{3}{8\pi}\right)^2\frac{2}{m'}\right)^2\left(\frac{4\pi}{\alpha G}\right)^{3/2}}$

(g) $\boxed{M_0} = \frac{3}{4}\left(\frac{\hbar c}{G}\right)^{3/2}\left(\frac{\pi}{2}\right)^{3/2}\frac{1}{m'^2}$

$\boxed{= \frac{3}{4}\left(\frac{\pi}{2}\right)^{3/2}\frac{m_{Planck}^3}{m_n'^2}} \boxed{= 3.49\times 10^{30}\ \text{kg}} \boxed{= 1.75\ M_\odot}$

(h) $R^2 = \lambda_n^2\left(\left(\frac{3}{8\pi}\right)^2\frac{2M}{m_n}\right)^{2/3}\left(1 - \left(\frac{1}{2}\right)^{2/3}\right)$; $M = \frac{1}{2}M_0$

$\Rightarrow \boxed{R =} \lambda_n\left(\left(\frac{3}{8\pi}\right)^2\frac{M_0}{m}\right)^{1/3}(0.6083)$

$= \boxed{2.5\ \text{km}}$

[true values are $\sim 2.5\ M_\odot$ and ~ 10 km. This treatment, with $\alpha = \frac{3}{2}$, gives $0.44\ M_\odot$ for white dwarf versus true Chandrasekhar limit of $1.44\ M_\odot$.]

5. (a) $\Delta(T_c) = 0 \Rightarrow 1 = N(0)V_0 \int_0^{\hbar\omega_D} d\varepsilon \, \dfrac{\tanh\left(\frac{\varepsilon}{2kT}\right)}{\varepsilon}$

$$= N(0)V_0 \int_0^{\hbar\omega_D/2kT_c} dx \, \frac{\tanh x}{x} \quad , \; x \equiv \frac{\varepsilon}{2kT_c}$$

$$\approx N(0)V_0 \ln\left(1.134 \frac{\hbar\omega_D}{kT_c}\right)$$

$$\Rightarrow e^{-\frac{1}{N(0)V_0}} = \frac{kT_c}{1.134\,\hbar\omega_D} \Rightarrow \boxed{kT_c = 1.134\,\hbar\omega_D\, e^{-\frac{1}{N(0)V_0}}}$$

(b) $T = 0 \Rightarrow 1 = N(0)V_0 \int_0^{\hbar\omega_D} d\varepsilon \, \dfrac{\tanh(\infty)}{(\varepsilon^2 + \Delta(0)^2)^{1/2}}$

$$= N(0)V_0 \int_0^{\hbar\omega_D/\Delta(0)} \quad , \; x \equiv \frac{\varepsilon}{\Delta(0)}$$

taken to be real & positive

$$= N(0)V_0 \sinh^{-1}\left(\frac{\hbar\omega_D}{\Delta(0)}\right)$$

$$\Rightarrow \frac{\hbar\omega_D}{\Delta(0)} = \sinh\left(\frac{1}{N(0)V_0}\right) \approx \frac{1}{2} e^{\frac{1}{N(0)V_0}}$$

$$\Rightarrow \boxed{\Delta(0) = 2\hbar\omega_D\, e^{-\frac{1}{N(0)V_0}}}$$

$$\boxed{\frac{2\Delta(0)}{kT_c}} = \frac{4\hbar\omega_D\, e^{-\frac{1}{N(0)V_0}}}{1.134\,\hbar\omega_D\, e^{-\frac{1}{N(0)V_0}}} \boxed{= 3.53}$$