Exercise 1  30% (BY HAND)

Let \( f(x) = x \ln |x| \) and \( x_0 = 7.4, x_1 = 7.6, x_2 = 7.8, x_3 = 8.0 \). Determine the most accurate three point formula approximation of \( f'(7.8) \) and use the error bound formula to determine the error.

Exercise 2  30% (BY HAND)

Consider the second difference approximation
\[
\frac{f(x + h) - 2f(x) + f(x - h)}{h^2}
\]
of \( f''(x) \). Recall that there exists \( \xi \in [x - h, x + h] \) such that
\[
f''(x) - \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} = -f^{(4)}(\xi) \frac{h^2}{12}
\]

- Assume that every evaluation \( f(y) \) is perturbed by the roundoff error
  \[
  \bar{f}(y) = f(y) + e(y)
  \]
  where \( |e(y)| \leq \epsilon \). Determine an error bound for
  \[
  \left| f''(x) - \frac{\bar{f}(x + h) - 2\bar{f}(x) + \bar{f}(x - h)}{h^2} \right|
  \]
  for any \( x \in [a + h, b - h] \) provided \( \max_{s \in [a,b]} |f^{(4)}(s)| \leq M \).

- Determine the optimal value of \( h \) (as a function of \( \epsilon \)) which minimizes the error and deduce the smallest error achievable.

Exercise 3  40% (MATLAB)

(MATLAB)

Let \( f(x) = e^x \) and consider the following two approximations of \( f'(0) \):

- \( f'(0) \approx r_1(h) = \frac{f(h) - f(0)}{h} \);
- \( f'(0) \approx r_2(h) = \frac{1}{h} \left( -\frac{3}{2} f(0) + 2 f(h) - \frac{1}{2} f(2h) \right) \).

1. Compute the above two difference approximations and report the results and errors \( e_i(h) = |r_i(h) - 1| \) for \( i = 1, 2 \) with \( h = 0.1, 0.01, 0.001, \ldots, 10^{-16} \). Make sure that you print your results in scientific notation. You will have a table with 16 lines, each containing

\( h, r_1(h), e_1(h), r_2(h), e_2(h) \).

Also plot in a log-log scale \( e_1(h) \) and \( e_2(h) \) vs \( h \). What do you deduce from this plot? The log-log plot can be easily obtained in matlab using the command
\texttt{loglog(H,E1,H,E2)}

where $H$, $E1$ and $E2$ are the arrays (of dimension 16) containing the values of $h$, $e1$ and $e2$. 