STABILITY ANALYSIS OF EXPLICIT ENTROPY VISCOSITY METHODS FOR NON-LINEAR SCALAR CONSERVATION EQUATIONS*

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Abstract. We establish the L^2-stability of an entropy viscosity technique applied to nonlinear scalar conservation equations. First- and second-order explicit time-stepping techniques using continuous finite elements in space are considered. The method is shown to be stable independently of the polynomial degree of the space approximation under the standard CFL condition.

Key words. nonlinear conservation equations, finite elements, entropy, viscous approximation, stability, time stepping, strong stability preserving time stepping, Runge-Kutta.

AMS subject classifications. 35F25, 65M12, 65N30, 65N22

1. Introduction.

Owing to a classical theorem by Godunov, it is now well understood that nonlinear approximation is required to approximate solutions of first-order hyperbolic equations with higher-order accuracy (i.e., larger than first-order). One can roughly distinguish two categories of nonlinear methods; the first one uses limiters and non-oscillatory reconstructions, see for example [13, 14, 12, 20] and the second one uses nonlinear viscosities [24, 22, 15, 18, 4]. (This categorization is fuzzy as observed in Remark 4.1 of [4].) The purpose of this paper is to analyze the stability properties of a method of the second category which we call entropy viscosity. This method has been introduced in [9, 11] and is based on a research program exposed in [8].

The entropy viscosity technique is a new class of high-order numerical methods for approximating conservation equations. This approach does not use any flux or slope limiters, applies to equations or systems supplemented with one or more entropy inequalities and is easy to implement on a large variety of meshes and polynomial approximations. The use of limiters and non-oscillatory reconstructions is avoided by adding a degenerate nonlinear dissipation to the numerical discretization of the equation or system at hand. The numerical viscosity is set to be proportional to the local size of an entropy production. Scalar conservation equations have many entropy pairs and most physical systems have at least one entropy function satisfying an auxiliary entropy inequality. The entropy satisfies a conservation equation in the regions where the solution is smooth and satisfies an inequality in shocks; this inequality then becomes a selection principle for the physically relevant solution. The amount of violation of the entropy equation is called entropy production. By making the numerical diffusion proportional to the entropy production, the numerical dissipation becomes large in the regions of shock and small in the regions where the solution remains smooth.

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The method has been implemented with Fourier approximation in [9], with spectral finite elements in [10], with continuous finite elements in [11], with discontinuous finite elements in [26] and various entropy functionals. The method seems to perform well on various benchmarks for a large class of approximation techniques but no theoretical result has yet been produced so far to justify the performance of the method. The present paper is our very first attempt in this direction.

The convergence analysis of nonlinear schemes for conservation equations is complicated even for the one-dimensional linear transport equation. For instance, it was only recently that the convergence rate of the second-order Nessyahu-Tadmor scheme [20] was shown to be better than that of a first-order monotone scheme for the linear transport equation in one space dimension [21]. In the present paper we restrict ourselves to the L^2-stability of the entropy viscosity method applied to scalar nonlinear conservation equations with various explicit time-stepping techniques using continuous finite elements in space of any degree.

The paper is organized as follows. The problem and the discrete setting at hand are described in §2. The stability of the first-order forward Euler method using a formally second-order viscosity based on the quadratic entropy $E(u) = \frac{1}{2}u^2$ is investigated in §3. Two second-order Runge-Kutta (RK2) time stepping techniques are analyzed in §4 and §5. In §4 we focus on the Heun method which is an example of a strong-stability preserving scheme (SSP). Stability is obtained upon adding an entropy viscosity at each step of this two-step method. The viscosity used in the first step depends on the solution from the previous time interval. We prove L^2-stability using the linear entropy $E(u) = u$, i.e., the entropy equation is the residual of the conservation equation. In §5 we analyze the mid-point scheme using again the linear entropy $E(u) = u$ to construct the viscosity. The particularity of this two-step method is that the entropy viscosity is built on the fly; i.e., it is added only at the second step and uses the solution from the first step. This feature could be useful when adaptive refinement is performed. Concluding remarks and numerical illustrations are reported in §6. The three key results from this paper are Theorem 3.1, Theorem 4.1 and Theorem 5.1.

2. Preliminaries. We describe in this section the functional setting used in this paper and we establish preliminary results.

2.1. The scalar conservation equation. Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be an open connected domain with Lipschitz boundary. The outward unit normal of Ω is denoted by $\mathbf{n}$. We consider the scalar-valued conservation equation

$$\partial_t u + \nabla{\cdot}\mathbf{f}(u) = 0, \qquad u(x,0) = u_0(x), \quad (x,t) \in \Omega{\times}\mathbb{R}_+, \tag{2.1}$$

where $\mathbf{f} \in \mathcal{C}^1(\mathbb{R};\mathbb{R}^d)$. The uniform Lipschitz condition on the flux might seem to be restrictive. For instance, to be useful this condition requires uniform a-priori bounds on the discrete solution when $f(v) = \frac{1}{2}v^2$. However, since the solution u of (2.1) satisfies such uniforms bound, say

$$m := \operatorname*{ess\,inf}_{y\in\Omega} u_0(y) \leq u(x,t) \leq \operatorname*{ess\,sup}_{y\in\Omega} u_0(y) =: M, \qquad \forall (x,t) \in \Omega{\times}(0,T), \tag{2.2}$$

a standard way to by-pass the uniformly Lipschitz condition at the discrete level consists of replacing $\mathbf{f}$ by $\widetilde{\mathbf{f}}$ so that $\widetilde{\mathbf{f}}(v) = \mathbf{f}(v)$ for all $v \in [m,M]$ and $\widetilde{\mathbf{f}}'(v) = \mathbf{f}'(m)$ when $v \in (-\infty, m]$ and $\widetilde{\mathbf{f}}'(v) = \mathbf{f}'(M)$ when $v \in [M,\infty)$.

To avoid boundary condition issues that can be very difficult to handle, we assume that there exists some time $T > 0$ so that

$$\int_\Omega u(\mathbf{x},t)\nabla\cdot\mathbf{f}(u(\mathbf{x},t))\,\mathrm{d}\mathbf{x} \geq 0, \qquad \forall t \in [0,T). \tag{2.3}$$

Note that provided $\mathbf{f} \in \mathcal{C}^1(\mathbb{R};\mathbb{R}^d)$, (2.3) is just the requirement that $\int_\Omega \nabla\cdot\mathbf{G}(u)\,\mathrm{d}\mathbf{x} \geq 0$ where $\mathbf{G}(u) := \int_0^u v\mathbf{f}'(v)\,\mathrm{d}v$ is the entropy flux associated with the entropy $E(u) = \frac{1}{2}u^2$ (see below). This condition holds with $T = +\infty$ if the boundary conditions are periodic. It also holds if the initial data is compactly supported, and in this case T is the time at which the domain of dependence of u_0 reaches the boundary of Ω. Dealing with the general case can be done by enforcing entropy compatible boundary conditions à la Bardos, Leroux, Nédélec [1], instead of Condition (2.3). We choose not to take this path to avoid additional technicalities.

It is known that the scalar-valued Cauchy problem (2.1) may have infinitely many weak solutions, but only one of them is physical and satisfies the additional inequalities

$$\partial_t E(u) + \nabla\cdot\mathbf{F}(u) \leq 0, \tag{2.4}$$

for all strictly convex functions $E \in \mathcal{C}^1(\mathbb{R};\mathbb{R})$, where $\mathbf{F}(u) := \int E'(v)\mathbf{f}'(v)\,\mathrm{d}v$, see [19]. This physical solution is henceforth called the entropy solution. The function $E(u)$ is called entropy and $\mathbf{F}(u)$ is the associated entropy flux. The most well known entropy pairs are the Kružkov pairs generated by $\{E(u) = |u-c|, c \in \mathbb{R}\}$. It is also known for strictly convex fluxes in one space dimension that if the entropy inequality (2.4) holds for one entropy pair and one weak solution u (provided the entropy E is strictly convex), then it also holds for all possible pairs and u is the unique entropy solution.

The objective of this paper is to perform the L^2-stability analysis of the entropy viscosity method applied to the nonlinear conservation equation (2.1) with forward Euler time stepping and with RK2 time stepping using continuous finite elements in space.

2.2. Functional Spaces. We call a mesh $\mathcal{T}$ a subdivision of Ω into disjoint and closed elements K such that $\overline{\Omega} = \cup_{K\in\mathcal{T}} K$; $\overline{\Omega}$ is the closure of Ω. The mesh is assumed to be affine to avoid unnecessary technicalities, i.e., Ω is assumed to be a polygon in two space dimensions or a polyhedron in three space dimensions. For any $K \in \mathcal{T}$, we denote by $h_K = \mathrm{diam}(K)$ the diameter of K and by ρ_K the diameter of the largest ball inscribed in K. Also, we denote $h_\mathcal{T} : \Omega \to \mathbb{R}$ the meshsize function defined by

$$h_\mathcal{T}|_K := h_K, \qquad K \in \mathcal{T}.$$

The subscript $\mathcal{T}$ is omitted when no confusion is possible. We suppose that we have at hand a family of meshes $\{\mathcal{T}_i\}_{i=1}^\infty$ and that this family is shape-regular, meaning that the quantity

$$c_s := \sup_{i\geq 1} \max_{K\in\mathcal{T}_i} h_K/\rho_K \tag{2.5}$$

is finite, i.e., the elements are not too flat. For all $K \in \mathcal{T}_i$, the collection of elements in $\mathcal{T}_i$ that touch K is denoted Δ_K. We assume also that the mesh family is locally quasi-uniform in the sense that the quantity

$$c_u := \sup_{i\geq 1} \max_{K\in\mathcal{T}_i} \left(h_K/(\min_{K'\in\Delta_K} h_{K'}) \right) \tag{2.6}$$

is finite, i.e. all the elements that touch K have diameters of order h_K.

Given a mesh $\mathcal{T}$, we define $\mathbb{V}(\mathcal{T})$ the space of piecewise polynomials by

$$\mathbb{V}(\mathcal{T}) := \left\{ V \in \mathcal{C}^0(\overline{\Omega}) : \ V|_K \in \mathcal{P}(K), \ \forall K \in \mathcal{T} \right\}, \tag{2.7}$$

where the local finite-dimensional space $\mathcal{P}(K)$ is assumed to contain the multivariate polynomials of total degree at most $k \geq 1$ over K, where k is a fixed integer. As a general rule, we will use capital letters to denote discrete functions. Finally, the L^2-scalar product over a domain $S \subset \mathcal{T}$ is denoted by $(\cdot,\cdot)_S$, and we abuse the notation by using $(\cdot,\cdot)_\Omega$ instead of $(\cdot,\cdot)_\mathcal{T}$. We often use the shorter notation $\|\cdot\|_{L^2}$ for $\|\cdot\|_{L^2(\Omega)}$ whenever it is unambiguous to do so.

We denote $\Pi^0_\mathcal{T}$ the L^2-projection onto constants, i.e., $\Pi^0_\mathcal{T}\varphi|_K := \frac{1}{|K|}\int_K \varphi$ for $K \in \mathcal{T}$, and $\Pi_\mathcal{T}$ the L^2-projection onto $\mathbb{V}(\mathcal{T})$. We will frequently use the following inverse inequality

$$h_K^{-\frac{1}{2}}\|V\|_{L^2(\partial K)} + \|\nabla V\|_{L^2(K)} \leq c_i h_K^{-1}\|V\|_{L^2(K)}, \qquad \forall V \in \mathbb{V}(\mathcal{T}), \ \forall K \in \mathcal{T}, \tag{2.8}$$

$$\|V\|_{L^\infty(K)} \leq c_\infty |K|^{-\frac{1}{2}}\|V\|_{L^2(K)}, \quad \forall V \in \mathbb{V}(\mathcal{T}), \ \forall K \in \mathcal{T}, \tag{2.9}$$

and approximation estimate

$$\|v - \Pi^0_\mathcal{T} v\|_{L^2(\Omega)} \leq c_0 \|h_\mathcal{T}\nabla v\|_{L^2(\Omega)}, \qquad \forall v \in H^1(\Omega). \tag{2.10}$$

The above constants c_i, c_∞, c_0 solely depend on the polynomial degree k, the domain Ω and the mesh shape regularity constant c_u and c_s defined in (2.5)-(2.6). In the rest of this manuscript, c, c', c'' denote generic constants that may depend solely on the above constants if not stated otherwise. In order to simplify the presentation, we shall explicitly mention the specific constants only after the step invoking the corresponding estimate. When confusion is not possible, we omit the dependency in $\mathcal{T}$ using the abbreviation $\Pi := \Pi_\mathcal{T}$, $\Pi^0 := \Pi^0_\mathcal{T}$ and $h := h_\mathcal{T}$.

For any subset $S \subset \mathcal{T}$ we define the two sets $\overline{S}$ and $\dot{S}$ as

$$\overline{S} := \cup_{K \in S}\Delta_K = \{K' \in \mathcal{T} : \exists K \in S, \ K' \cap K \neq \emptyset\}, \tag{2.11}$$

$$\dot{S} := \mathcal{T}\backslash(\overline{\mathcal{T}\backslash S}). \tag{2.12}$$

The set $\overline{S}$ is composed of S plus the layer of elements surrounding S (not to be confused with the closure of S). The set $\dot{S}$ is the complement in $\mathcal{T}$ of $\overline{S^c}$, where $S^c := \mathcal{T}\backslash S$ (not to be confused with the interior of S).

For all subset $S \subset \mathcal{T}$, we define the restriction operator $\mathcal{R}_S : \mathbb{V}(\mathcal{T}) \longrightarrow \mathbb{V}(\mathcal{T})$ as follows. Let $\{\psi_1, \ldots, \psi_M\}$ be the global shape functions spanning $\mathbb{V}(\mathcal{T})$. Let I be the set of indices, i, so that the support of ψ_i has a non empty intersection with $\dot{S}$ for all $i \in I$. Then for all $V := \sum_{i=1}^M V_i\psi_i \in \mathbb{V}(\mathcal{T})$, we set $\mathcal{R}_S V = \sum_{i \in I} V_i\psi_i$. This definition implies that

$$\mathcal{R}_S V \in \mathbb{V}(\mathcal{T}), \qquad \text{and} \qquad \mathcal{R}_S V(\mathbf{x}) := \begin{cases} 0 & \text{if } \mathbf{x} \in S^c := \mathcal{T}\backslash S, \\ V(\mathbf{x}) & \text{if } \mathbf{x} \in \dot{S}. \end{cases} \tag{2.13}$$

Lemma 2.1. *There is a uniform constant $c_\mathcal{R}$ depending on c_∞ and the polynomial degree k so that the following holds*

$$\|\mathcal{R}_S V\|_{L^2(S\backslash\dot{S})} \leq c_\mathcal{R}\|V\|_{L^2(S\backslash\dot{S})}, \qquad \forall V \in \mathbb{V}(\mathcal{T}), \quad \forall S \subset \mathcal{T}. \tag{2.14}$$

Proof. Let $K \in S\backslash\dot{S}$. Using (2.9) and the definition of $\mathcal{R}_{\mathcal{T}}V$ we infer that

$$\|\mathcal{R}_S V\|_{L^2(K)} \leq |K|^{1/2}\|\mathcal{R}_S V\|_{L^\infty(K)} \leq c\,|K|^{1/2}\|V\|_{L^\infty(K)} \leq c\,c_\infty\|V\|_{L^2(K)}.$$

The desired result follows readily. □

3. Forward Euler stability. We approximate in time the nonlinear conservation equation (2.1) using the first-order forward Euler method and we establish the L^2-stability of the method.

3.1. The algorithm. Let $\mathcal{T}$ be a mesh and let $U^0 \in \mathbb{V}(\mathcal{T})$ be an approximation of u_0. Let us set $\delta t_{-1} = +\infty$ and $t_0 = 0$. The forward Euler discretization of the equation (2.1) is constructed as follows. Let $U^n \in \mathbb{V}(\mathcal{T})$ be the approximation of u at time t_n, $n \geq 0$. To avoid boundary condition issues we assume that the following conservation property holds

$$(\nabla\cdot\mathbf{f}(U^n), U^n)_\Omega \geq 0. \tag{3.1}$$

As mentioned in §2.1, this property is known to hold if u_0 is compactly supported and t_n is small; it also holds if the boundary conditions are periodic.

Let $c_\tau \geq 1$, be a number and let $\lambda > 0$ be another positive number that we henceforth call CFL number; we select the time step δt_n so that

$$\delta t_n \leq \min(\lambda \min_{K\in\mathcal{T}} \frac{h_K}{\|\mathbf{f}'(U^n)\|_{L^\infty(K)}}, c_\tau \delta t_{n-1}). \tag{3.2}$$

Note that the quantity $\min_{K\in\mathcal{T}} \frac{h_K}{\|\mathbf{f}'(U^n)\|_{L^\infty(K)}} \geq \frac{1}{\|\mathbf{f}'\|_{L^\infty(\mathbb{R};\mathbb{R}^d)}} \min_{K\in\mathcal{T}} h_K$ is bounded away from zero since $\mathbf{f}$ is assumed to be uniformly Lipschitz; as a result, it is always possible to select $\delta t_n > 0$ satisfying (3.2) and to advance in time. The condition $\delta t_n \leq c_\tau \delta t_{n-1}$ ensures the time stepping is quasi-uniform. Let $t_{n+1} = t_n + \delta t_n$ and let $U^{n+1} \in \mathbb{V}(\mathcal{T})$ be such that

$$\left(U^{n+1} - U^n + \delta t_n \nabla\cdot\mathbf{f}(U^n), V\right)_\Omega + \delta t_n \left(\nu^n \nabla U^n, \nabla V\right)_\Omega = 0, \qquad \forall V \in \mathbb{V}(\mathcal{T}), \tag{3.3}$$

where ν^n is the entropy viscosity that we now define. Three different residuals are used to construct the entropy viscosity ν^n. We define the residual of the equation R^n,

$$R^n := \frac{U^n - U^{n-1}}{\delta t_{n-1}} + \nabla\cdot\mathbf{f}(U^n), \tag{3.4}$$

and we define two entropy residuals R^n_{E1}, R^n_{E2},

$$R^n_{E1} := R^n\, U^n, \quad \text{and} \quad R^n_{E2} := \frac{E(U^n) - E(U^{n-1})}{\delta t_{n-1}} + \mathbf{f}'(U^n)\cdot\nabla E(U^n). \tag{3.5}$$

where $E(v) = \frac{1}{2}v^2$ is the quadratic entropy. Let R^n_E be the total entropy residual defined as follows:

$$R^n_E|_K := \|R^n_{E1}\|_{L^\infty(K)} + \|R^n_{E2}\|_{L^\infty(K)} + \delta t_n \|R^n\|^2_{L^\infty(K)}. \tag{3.6}$$

We then define the entropy viscosity over each cell K as follows:

$$\nu^n|_K := h_K \min(c_M \|\mathbf{f}'(U^n)\|_{L^\infty(K)}, c_E R^n_E|_K), \tag{3.7}$$

where $c_M > 0$ and $c_E > 0$ are user-defined constants.

Remark 3.1. *(Choice of Parameters)* Usually we take $c_M = \frac{1}{2k}$ in one space dimension and $c_M = \frac{1}{4k}$ in two space dimensions (recall that k is the polynomial degree used in the local approximation space $\mathcal{P}$). The constant c_E is dimensional and is also user-defined; for instance it can be defined as follows:

$$c_E := \mathfrak{c}_E \frac{D}{\frac{1}{|\Omega|}\int_\Omega |E(U^0)|}, \tag{3.8}$$

or equivalently $c_E := \mathfrak{c}_E |\Omega| \|\nabla E(U^0)\|^{-1}_{L^1(\Omega)}$, or $c_E := \mathfrak{c}_E D \|E(U^0)\|^{-1}_{L^\infty(\Omega)}$, where $|\Omega| :=$ meas(Ω), $D := \text{diam}(\Omega)$ and $\mathfrak{c}_E$ is a non-dimensional constant of order one.

Remark 3.2. *(Consistency of the Entropy Residual)* Note that R_E is formally first-order, $\mathcal{O}(\delta t_n + h_K^k)$, in the region where u is smooth. That is, the entropic viscosity is formally second-order, i.e., $\mathcal{O}(h_K(\delta t_n + h_K^k))$, which is greater than the overall consistency order of the first-order Euler method. As a result, we expect the method to be as accurate as the first-order Euler method for smooth solutions, i.e., the error should be formally $\mathcal{O}(\delta t + h^k)$ in L^p-norms, $1 \le p < \infty$, provided some stability is established.

The entropic viscosity naturally splits the mesh $\mathcal{T}$ into a viscous and a smooth set as follows:

$$\mathcal{T} = \mathcal{T}_V^n \cup \mathcal{T}_S^n, \qquad \begin{cases} \mathcal{T}_V^n := \left\{K \in \mathcal{T} : \nu^n|_K = c_M h_K \|\mathbf{f}'(U^n)\|_{L^\infty(K)}\right\}, \\ \mathcal{T}_S^n := \mathcal{T}\backslash\mathcal{T}_V^n := \left\{K \in \mathcal{T} : \nu^n|_K = c_E h_K R_E|_K\right\}. \end{cases} \tag{3.9}$$

This decomposition will arise in the stability analysis below. For the moment, note that no stability issue should arise on $\mathcal{T}_V^n$ due to the presence of the first-order viscosity $\nu^n|_K = c_M h_K \|\mathbf{f}'(U)\|_{L^\infty(K)}$, $\forall K \in \mathcal{T}_V^n$. Establishing stability on $\mathcal{T}_S^n$ will turn out to be the more technical part of the proof; it will be essential to observe that the discrete time derivative satisfies

$$\left(\frac{U^n - U^{n-1}}{\delta t_n}\right)^2 = 2\frac{R_{E1}^n - R_{E2}^n}{\delta t_n} \le 2\frac{|R_{E1}^n| + |R_{E2}^n|}{\delta t_n}, \tag{3.10}$$

which justifies the introduction on the two entropy residuals R_{E1}^n and R_{E2}^n.

3.2. Stability Analysis of Forward Euler. We are now in position to prove the stability estimate for the forward Euler scheme (3.3).

Theorem 3.1 (Stability of the Forward-Euler Scheme). *Assume that the conditions* (3.1)-(3.2) *are satisfied. There is $\Lambda_0 > 0$ that depends only on the user-defined parameters c_M, $\mathfrak{c}_E$, the Lipschitz constant of the flux, and on the mesh family constants c_0, c_i, and there is a constant c that additionally depends linearly on the final time T so that the solution to* (3.3) *satisfies the following L^2-stability estimate for all $\lambda \le \Lambda_0$*

$$\|U^n\|^2_{L^2(\Omega)} + \sum_{i=0}^n \|\sqrt{\nu^i}\nabla U^i\|^2_{L^2(\Omega)} \le \|U^0\|^2_{L^2(\Omega)}(1 + c\,\lambda), \qquad \forall t_n \le T. \tag{3.11}$$

Proof. Step 1. Using $V = 2U^n$ in (3.3) together with the conservation property (3.1), we obtain

$$\|U^{n+1}\|^2_{L^2(\Omega)} - \|U^n\|^2_{L^2(\Omega)} + 2\delta t_n \|\sqrt{\nu^n}\nabla U^n\|^2_{L^2(\Omega)} \le \|U^{n+1} - U^n\|^2_{L^2(\Omega)}. \tag{3.12}$$

We now estimate the right-hand side of (3.12). Defining $B := \{V \in \mathbb{V}(\mathcal{T}) \mid \|V\|_{L^2(\Omega)} = 1\}$, and using $\nu^n|_K \le c_M \|\mathbf{f}'(U^n)\|_{L^\infty(K)} h_K$, (3.3) yields

$$\begin{aligned}\|U^{n+1} - U^n\|^2_{L^2(\Omega)} &= \sup_{V\in B} \left(U^{n+1} - U^n, V\right)^2_\Omega \le 2\delta t_n^2 \sup_{v\in B} \left((\nabla\cdot \mathbf{f}(U^n), V)^2_\Omega + (\nu^n \nabla U, \nabla V)^2_\Omega\right)\\ &\le 2\delta t_n^2 \|\nabla\cdot \mathbf{f}(U^n)\|^2_{L^2(\Omega)} + 2\delta t_n c_M c_i^2 \lambda \|\sqrt{\nu^n}\nabla U^n\|^2_{L^2(\Omega)}.\end{aligned}$$

Therefore we can re-write (3.12) as follows:

$$\|U^{n+1}\|^2_{L^2(\Omega)} - \|U^n\|^2_{L^2(\Omega)} + 2\delta t_n (1 - c_M c_i^2 \lambda) \|\sqrt{\nu^n}\nabla U^n\|^2_{L^2(\Omega)} \le 2\delta t_n^2 \|\nabla\cdot \mathbf{f}(U^n)\|^2_{L^2(\Omega)}. \tag{3.13}$$

The remainder of the proof consists of estimating a bound on $\|\nabla\cdot \mathbf{f}(U^n)\|^2_{L^2(\Omega)}$, and we are going to invoke the partition $\mathcal{T} = \mathcal{T}_V^n \cup \mathcal{T}_S^n$ for that purpose.

Step 2. (Control over $\mathcal{T}_V^n$) The viscosity is large enough to control $\delta t_n \|\nabla\cdot \mathbf{f}(U^n)\|_{L^2(\Omega)}$ on the viscous set $\mathcal{T}_V^n$, and we have:

$$\begin{aligned}\delta t_n^2 \int_{\mathcal{T}_V^n} |\nabla\cdot \mathbf{f}(U^n)|^2 &\le \|h^{-1}\mathbf{f}'(U^n)\|_{L^\infty(\Omega)} \delta t_n^2 c_M^{-1} \int_{\mathcal{T}_V^n} \nu^n |\nabla U^n|^2\\ &\le c_M^{-1}\, \delta t_n \lambda \|\sqrt{\nu^n}\nabla U^n\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.14}$$

Step 3. (Control over $\mathcal{T}_S^n$) Recalling the bound (3.10), we infer that

$$\begin{aligned}\delta t_{n-1} |\nabla\cdot \mathbf{f}(U^n)| &= |\delta t_{n-1} R^n - (U^n - U^{n-1})|\\ &\le \delta t_{n-1}|R^n| + \sqrt{2}\delta t_{n-1}^{\frac12} (|R^n_{E1}|^{\frac12} + |R^n_{E2}|^{\frac12}).\end{aligned}$$

With this estimate in hand we infer that the following estimate holds on the smooth set $\mathcal{T}_S^n$

$$\begin{aligned}\delta t_n^2 \int_{\mathcal{T}_S^n} |\nabla\cdot \mathbf{f}(U^n)|^2 &\le \delta t_n^{\frac32} \int_{\mathcal{T}_S^n} |\nabla U^n| |\mathbf{f}'(U^n)| \left(\delta t_n^{\frac12} |R^n| + \sqrt{2}\delta t_n^{\frac12} \delta t_{n-1}^{-\frac12} (|R^n_{E1}|^{\frac12} + |R^n_{E2}|^{\frac12})\right)\\ &\le c\delta t_n^{\frac32} \int_{\mathcal{T}_S^n} |\nabla U^n| |\mathbf{f}'(U^n)|\ (R^n_E)^{1/2},\end{aligned}$$

where we have used the quasi-uniformity assumption (3.2) of the time stepping. Hence, we obtain

$$\delta t_n^2 \|\nabla\cdot \mathbf{f}(U^n)\|^2_{L(\mathcal{T}_S^n)} \le c c_E^{-1} \lambda \delta t_n |\mathcal{T}_S^n| \|\mathbf{f}'(U^n)\|_{L^\infty(\Omega)} + \frac{1}{2} \delta t_n^2 \lambda^{-1} c_E \int_{\mathcal{T}_S^n} |\nabla U^n|^2 |\mathbf{f}'(U^n)| R^n_E,$$

which after using that $\mathbf{f}$ is uniformly Lipschitz together with the expression of the viscosity $\nu_K^n = c_E h_K R_E^n$ on $\mathcal{T}_S^n$, leads to

$$\delta t_n^2 \|\nabla\cdot \mathbf{f}(U^n)\|^2_{L^2(\mathcal{T}_S^n)} \le c\lambda g_E \delta t_n \|U^0\|^2_{L^2(\Omega)} + \frac{1}{2} \delta t_n \|\sqrt{\nu^n}\nabla U^n\|^2_{L^2(\Omega)}, \tag{3.15}$$

where we set $g_E := \|\mathbf{f}'\|_{L^\infty(\mathbb{R})} |\Omega| c_E^{-1} \|U^0\|^{-2}_{L^2(\Omega)}$.

Step 4. Setting $\Lambda_0 := \frac{c_M}{4(c_M^2 c_i^2 + 1)}$ and inserting (3.14) and (3.15) into (3.13), we finally obtain that the following holds for all $\lambda \le \Lambda_0$,

$$\|U^{n+1}\|^2_{L^2(\Omega)} - \|U^n\|^2_{L^2(\Omega)} + \delta t_n \|\sqrt{\nu^n}\nabla U^n\|^2_{L^2(\Omega)} \le c\, \lambda g_E \delta t_n,$$

which immediately leads to

$$\|U^n\|^2_{L^2(\Omega)} + \sum_{i=0}^{n} \|\sqrt{\nu^i}\nabla U^i\|^2_{L^2(\Omega)} \le \|U^0\|^2_{L^2(\Omega)}(1 + c\,\lambda g_E t_n), \qquad \forall n \in \mathbb{N}\ .$$

Observe that $\lambda g_E t_n$ is a dimensionless constant. This is completes the proof. □

4. Runge-Kutta 2 (Heun). We now turn our attention to the second-order RK2/Heun time discretization to approximate (2.1). This time stepping is known to be a Strong-Stability-Preserving method [7]. The viscosity considered in this section is mainly based on the the linear entropy $E(u)$, i.e., the residual of equation at the previous time step. We analyze another second-order method with the viscosity computed on the fly in §5. The present scheme and that in §5 do not require the quasi-uniformity assumption that had to be invoked for the forward Euler scheme, see (3.2).

4.1. The Algorithm. Let us set $t_0 = 0$ and let $U^0 \in \mathbb{V}(\mathcal{T})$ be an approximation of u_0. Let $\lambda > 0$ be a CFL number. Let $U^n \in \mathbb{V}(\mathcal{T})$ be the approximation of u at time t_n, $n \ge 0$. Let δt_n be a given time step possibly restricted later by the CFL number, see (4.4), and set $t_{n+1} = t_n + \delta t_n$. The fully discrete RK2/Heun algorithm that we consider is formulated as follows: find $W^n \in \mathbb{V}(\mathcal{T})$ and $U^{n+1} \in \mathbb{V}(\mathcal{T})$ satisfying

$$(W^n, V)_\Omega - (U^n, V)_\Omega + \delta t_n\,(\nabla{\cdot}\mathbf{f}(U^n), V)_\Omega + \delta t_n\,(\nu_1^n \nabla U^n, \nabla V)_\Omega = 0, \tag{4.1}$$

$$\big(U^{n+1} - \tfrac{1}{2}(W^n + U^n), V\big)_\Omega + \frac{\delta t_n}{2}\,(\nabla{\cdot}(\mathbf{f}(W^n), V)_\Omega + \frac{\delta t_n}{2}\,(\nu_2^n \nabla W^n, \nabla V)_\Omega = 0, \tag{4.2}$$

for all $V \in \mathbb{V}(\mathcal{T})$, where the viscosities ν_1^n, ν_2^n are defined below. To avoid issues induced by the boundary condition we assume that both U^n and W^n satisfy the following conservation properties

$$(\nabla{\cdot}\mathbf{f}(U^n), U^n)_\Omega \ge 0, \qquad (\nabla{\cdot}\mathbf{f}(W^n), W^n)_\Omega \ge 0. \tag{4.3}$$

We refer to §2.1 for a discussion on the validity of this assumption. We assume that δt_n satisfies the additional condition

$$\delta t_n \le \lambda \min_{K\in\mathcal{T}} \frac{h_K}{\max(\|\mathbf{f}'(U^n)\|_{L^\infty(K)}, \|\mathbf{f}'(W^n)\|_{L^\infty(K)})}. \tag{4.4}$$

If this condition is not satisfied at the end of the time step, the computation of W^n and U^{n+1} is redone with a smaller time step, say δt_n is divided by 1.5. Note that due to the uniform Lipschitz assumption on $\mathbf{f}$, picking δt_n smaller than $\frac{1}{\|\mathbf{f}'\|_{L^\infty(\mathbb{R})}} \min_{K\in\mathcal{T}} h_K$ always guarantees that (4.4) holds.

Let us now construct the viscosities ν_1^n, ν_2^n. Let $U^{-1} = U^0$, and consider the residual R^n, $n \ge 0$, defined by

$$R^n := \frac{U^n - U^{n-1}}{\delta t_{n-1}} + \nabla{\cdot}\mathbf{f}(U^n). \tag{4.5}$$

Let $c_M > 0$, $c_E > 0$, $\alpha \ge 0$ be three real numbers and let us consider the partition of $\mathcal{T}$ defined at time step t_n as follows:

$$\mathcal{T} = \mathcal{T}_V^n \cup \mathcal{T}_S^n, \qquad \begin{cases} \mathcal{T}_S^n := \left\{K \in \mathcal{T} : c_E h_K^\alpha \|R^n\|_{L^\infty(K)} \le c_M \|\mathbf{f}'(U^n)\|_{L^\infty(K)}\right\}, \\ \mathcal{T}_V^n := \mathcal{T}_h \backslash \mathcal{T}_S^n. \end{cases} \tag{4.6}$$

We now define the viscosities ν_1^n, ν_2^n, $n \geq 0$, to be piecewise constant functions on the mesh cells. For any $K \in \mathcal{T}$ we set $\nu_1^0|_K = c_M h_K \|\mathbf{f}'(U^0)\|_{L^\infty(K)}$ and for $n \geq 1$

$$\nu_1^n|_K := \begin{cases} c_M h_K \|\mathbf{f}'(U^n)\|_{L^\infty(K)} & \text{if } K \in \overline{\mathcal{T}_V^n}, \\ h_K \max\left(c_E h_K^\alpha \|R^n\|_{L^\infty(K)}, c_M \mathrm{osc}_K(\mathbf{f}, U^n)\right) & \text{if } K \in \dot{\mathcal{T}}_S^n := \mathcal{T} \backslash \overline{\mathcal{T}_V^n}, \end{cases} \tag{4.7}$$

where

$$\mathrm{osc}_K(\mathbf{f}, U^n) := \frac{\|\nabla\cdot\mathbf{f}(U^n) - \Pi^0(\nabla\cdot\mathbf{f}(U^n))\|^2_{L^\infty(K)}}{4\|\mathbf{f}'(U^n)\|_{L^\infty(K)}\|\nabla U^n\|^2_{L^\infty(K)}}. \tag{4.8}$$

Note that $\mathrm{osc}_K(\mathbf{f}, U^n) \leq \|\mathbf{f}'(U^n)\|_{L^\infty(K)}$. The second sub-step viscosity ν_2^n is defined as follows for all $n \geq 0$:

$$\nu_2^n|_K := c_M h_K \mathrm{nl}_K(\mathbf{f}, W^n, U^n), \tag{4.9}$$

$$\mathrm{nl}_K(\mathbf{f}, W^n, U^n) := \frac{1}{2} \frac{\|\mathbf{f}'(W^n) - \mathbf{f}'(U^n)\|^2_{L^\infty(K)}}{\||\mathbf{f}'(W^n)| + |\mathbf{f}'(U^n)|\|_{L^\infty(K)}}. \tag{4.10}$$

Several comments are in order regarding the definition of the viscosities.

Remark 4.1. *(Oscillation of $\nabla\cdot\mathbf{f}(U^n)$)* The oscillation of $\nabla\cdot\mathbf{f}(U^n)$, denoted $\mathrm{osc}_K(\mathbf{f}, U^n)$, and the nonlinear variation of $\mathbf{f}$, denoted $\mathrm{nl}_K(\mathbf{f}, W^n, U^n)$, are both zero for the linear transport equation, $\mathbf{f}(u) := \boldsymbol{\beta} u$, $\boldsymbol{\beta} \in \mathbb{R}^d$. The purpose of these two terms is to help control the nonlinearity of the flux. To the best of our knowledge, stability under the usual CFL condition of the Heun discretization of the linear transport equation with continuous finite elements is known so far only for the piecewise linear approximation [5]. This issue with the piecewise linear approximation does not seem to arise for higher-order time stepping [5, 25]. The oscillation term $\mathrm{osc}_K(\mathbf{f}, U^n)$ in the definition of ν_1^n seems to be necessary to handle finite elements of polynomial degrees larger than one.

Remark 4.2. *(Alternative Expression of ν_1^n)* The viscosity ν_1^n can be rewritten in the following alternative form

$$\nu_1^n|_K := h_K \min\left(c_M \|\mathbf{f}'(U^n)\|_{L^\infty(K)}, \max(c_E h_K^\alpha \|R^n\|_{L^\infty(K)}, c_M \mathrm{osc}_K(\mathbf{f}, U^n))\right),$$

for all $K \in \mathcal{T}_V^n \cup \dot{\mathcal{T}}_S^n$, $n \geq 1$, and $\nu_1^n|_K := c_M h_K \|\mathbf{f}'(U^n)\|_{L^\infty(K)}$, for $K \in \mathcal{L}^n$, where we have defined $\mathcal{L}^n := \mathcal{T}_S^n \setminus \dot{\mathcal{T}}_S^n$. The viscosity saturates to first-order in the so-called viscous set $\mathcal{T}_V^n \cup \mathcal{L}^n$ and is small in the so-called smooth set $\dot{\mathcal{T}}_S^n$, see Figure 4.1.

Remark 4.3. *(Consistency of Viscosities)* Note that the terms $c_M h_K \mathrm{osc}_K(\mathbf{f}, U^n)$ and $c_M h_K |R^n|$ are formally $\mathcal{O}(h_K^3)$ and $\mathcal{O}(h_K^{1+\alpha}(\delta t_n + h_K^k))$, respectively. This mean that the viscosity $\nu_1|_K$ is $\mathcal{O}(h_K^{2+\alpha})$ under the CFL condition. The viscosity $\nu_2|_K = c_M h_K \mathrm{nl}_K(\mathbf{f}, W^n, U^n)$ is formally $\mathcal{O}(\delta t_n^2 h_K)$, i.e., it is third-order in the smooth region $\dot{\mathcal{T}}_S^n$. Overall the consistency order of the artificial viscosities is higher than the overall $\mathcal{O}(\delta t_n^2)$ consistency of the Heun method under the CFL condition. The accuracy order of the method is expected to be at least $\mathcal{O}(\delta t^2 + h^{\min(2+\alpha,k)})$.

Remark 4.4. *(Constants c_M and c_E)* The constant c_M is user-defined, non-dimensional and of order one. The constant c_E is also user-defined but dimensional; for instance it can be defined as follows:

$$c_E := \mathfrak{c}_E \frac{D^{1-\alpha}}{|\Omega|^{-1/2}\|U^0\|_{L^2(\Omega)}}, \tag{4.11}$$

or $c_E := \mathfrak{c}_E D^{1-\alpha}\|U^0\|^{-1}_{L^\infty}$, where $D := \mathrm{diam}(\Omega)$ and $\mathfrak{c}_E$ is a user-defined non-dimensional constant of order one; see also Remark 3.1.

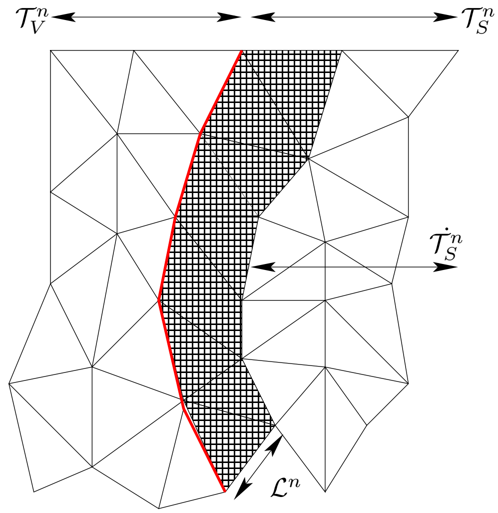


FIG. 4.1. *Schematic representation of the partition* $\mathcal{T} = \dot{\mathcal{T}}_S^n \cup \mathcal{L}^n \cup \mathcal{T}_V^n$.

4.2. Stability Analysis of RK2/Heun. We establish in this section the L^2-stability of the RK2/Heun time discretization of (2.1).

THEOREM 4.1 (Stability of the RK2/Heun). *There is* $\Lambda_0 > 0$ *that depends only on the user-defined parameters* c_M, $\mathfrak{c}_E$, *the Lipschitz constant of the flux, and on the mesh family constants* c_0, c_i, *and there is a constant* c *that additionally depends linearly on* $T^{2(1-\alpha)}$ *so that the solution to* (4.1)-(4.2) *satisfies the following* L^2*-stability estimate for all* $\lambda \le \Lambda_0$*:*

$$\|U^{n+1}\|^2_{L^2(\Omega)} + \sum_{i=0}^{n} \delta t_n \left(\|\sqrt{\nu_1^i}\nabla U^i\|^2_{L^2(\Omega)} + \|\sqrt{\nu_2^i}\nabla W^i\|^2_{L^2(\Omega)} \right) \le \|U_0\|^2_{L^2(\Omega)} \left(1 + c\lambda^{2(1+\alpha)}(\delta t/T)^{1-2\alpha}\right), \qquad \forall t_n \le T, \tag{4.12}$$

where $\delta t := \max_{i=0,\dots,n} \delta t_n$. *In particular,* (4.1)-(4.2) *is stable provided* $\alpha \le \frac{1}{2}$.

Proof. Step 1. Choosing $V = U^n$ in (4.1), $V = 2W^n$ in (4.2), using the conservation property (4.3), and adding the two results we obtain that

$$\|U^{n+1}\|^2_{L^2(\Omega)} - \|U^n\|^2_{L^2(\Omega)} + \delta t_n \left(\|\sqrt{\nu_1^n}\nabla U^n\|^2_{L^2(\Omega)} + \|\sqrt{\nu_2^n}\nabla W^n\|^2_{L^2(\Omega)} \right) \le \|U^{n+1} - W^n\|^2_{L^2(\Omega)}. \tag{4.13}$$

The rest of the proof consists of deriving a bound on the time increment $\|U^{n+1} - W^n\|^2_{L^2(\Omega)}$. Note that this time increment is formally second-order as can be observed by constructing (4.2) $- \frac{1}{2}$(4.1):

$$\left(U^{n+1} - W^n, V\right)_\Omega = -\frac{\delta t_n}{2} \left(\nabla\cdot(\mathbf{f}(W^n) - \mathbf{f}(U^n)), V\right)_\Omega - \frac{\delta t_n}{2} \left(\nu_2^n \nabla W^n - \nu_1^n \nabla U^n, \nabla V\right)_\Omega . \tag{4.14}$$

Step 2. We set $Z^n := W^n - U^n$ and test (4.14) with $V = U^{n+1} - W^n$. The first term in the right hand side is handled as follows:

$$
\begin{aligned}
-\frac{\delta t_n}{2}(\nabla{\cdot}(\mathbf{f}(W^n) - \mathbf{f}(U^n)), V)_\Omega &= -\frac{\delta t_n}{2}\left((\mathbf{f}'(W^n) - \mathbf{f}'(U^n)){\cdot}\nabla W^n, V\right)_\Omega \\
&\quad - \frac{\delta t_n}{2}\left(\mathbf{f}'(U^n){\cdot}\nabla(W^n - U^n), V\right)_\Omega \\
&\le c_M^{-\frac12}\delta t_n^{\frac12}\lambda^{\frac12}\|\sqrt{\nu_2^n}\nabla W^n\|_{L^2(\Omega)}\|V\|_{L^2(\Omega)} + \frac12\lambda\|h\nabla Z^n\|_{L^2(\Omega)}\|V\|_{L^2(\Omega)}
\end{aligned}
$$

where we used the definition of ν_2^n to deduce that

$$
\|\mathbf{f}'(W^n) - \mathbf{f}'(U^n)\|_{L^\infty(K)}^2 \le 4\nu_2^n|_K h_K^{-1} c_M^{-1} \max(\|\mathbf{f}'(U^n)\|_{L^\infty(K)}, \|\mathbf{f}'(W^n)\|_{L^\infty(K)}).
$$

The second term in (4.14) is estimated as follows:

$$
-\frac{\delta t_n}{2}\left(\nu_2^n\nabla W^n - \nu_1^n\nabla U^n, \nabla V\right)_\Omega \le \frac{c_i}{2}\delta t_n^{\frac12}\lambda^{\frac12}c_M^{\frac12}\left(\|\sqrt{\nu_2^n}\nabla W^n\|_{L^2} + \|\sqrt{\nu_1^n}\nabla U^n\|_{L^2}\right)\|V\|_{L^2}.
$$

Combining the above estimates gives

$$
\begin{aligned}
\|U^{n+1} - W^n\|_{L^2}^2 &\le \lambda^2\|h\nabla Z^n\|_{L^2}^2 \\
&\quad + (c_i^2 c_M + 4c_M^{-1})\lambda\delta t_n\left(\|\sqrt{\nu_2^n}\nabla W^n\|_{L^2}^2 + \|\sqrt{\nu_1^n}\nabla U^n\|_{L^2}^2\right).
\end{aligned}
\tag{4.15}
$$

The two viscous terms in the right-hand side can be absorbed in the left-hand side of (4.13) provided Λ_0 is small enough. The remaining term $\|h\nabla Z^n\|_{L^2}$ is critical. To control this term we borrow an argument from [5] and adapt it to make it work for any polynomial degree (see Remark 4.5). The argument is based on the following two properties:

$$
\|h\nabla Z^n\|_{L^2(K)} \le c_i\|Z^n - \Pi^0 Z^n\|_{L^2(K)}, \quad \forall K \in \mathcal{T}, \tag{4.16}
$$

$$
\int_K \Pi^0(\nabla{\cdot}\mathbf{f}(U^n))(Z^n - \Pi^0 Z^n)\,\mathrm{d}\mathbf{x} = 0, \quad \forall K \in \mathcal{T}, \tag{4.17}
$$

where Π^0 is the L^2-projection onto piecewise constants, i.e., $\Pi^0 v$ is defined on each mesh cell by $\Pi^0 v|_K = |K|^{-1}\int_K v\,\mathrm{d}\mathbf{x}$, for all $v \in L^2(\Omega)$. Using inequality (4.16) in (4.15) implies that

$$
\begin{aligned}
\|U^{n+1} - W^n\|_{L^2}^2 &\le c_i^2\lambda^2\|Z^n - \Pi^0 Z^n\|_{L^2}^2 \\
&\quad + (c_i^2 c_M + 4c_M^{-1})\lambda\delta t_n\left(\|\sqrt{\nu_2^n}\nabla W^n\|_{L^2(\Omega)}^2 + \|\sqrt{\nu_1^n}\nabla U^n\|_{L^2(\Omega)}^2\right).
\end{aligned}
\tag{4.18}
$$

Step 3. We now focus our attention on the first term in the right-hand side of (4.18) and we denote $X^n := Z^n - \Pi^0 Z^n$. The defining properties of Π^0 and Π imply

$$
\lambda^2\|X^n\|_{L^2}^2 = \lambda^2\left(X^n, Z^n\right)_\Omega = \lambda^2\left(\Pi X^n, Z^n\right)_\Omega. \tag{4.19}
$$

Note that from (4.1) we have

$$
(Z^n, V)_\Omega = -\delta t_n\left(\nabla{\cdot}\mathbf{f}(U^n), V\right)_\Omega - \delta t_n\left(\nu_1^n\nabla U^n, \nabla V\right)_\Omega, \qquad \forall V \in \mathbb{V}(\mathcal{T}).
$$

Hence, by choosing the test function $V = \Pi X^n$ in this equation, we obtain

$$
\begin{aligned}
\lambda^2\|X^n\|_{L^2}^2 &= \lambda^2\left(\Pi X^n, Z^n\right)_\Omega \\
&= -\lambda^2\delta t_n\left(\nabla{\cdot}\mathbf{f}(U^n), \Pi X^n\right)_\Omega - \delta t_n\lambda^2\left(\nu_1^n\nabla U^n, \nabla\Pi X^n\right)_\Omega.
\end{aligned}
$$

The L^2-stability of Π and the boundedness of ν_1^n imply that the last term above can be bounded as follows:

$$\begin{aligned}-\delta t_n\lambda^2\left(\nu_1^n\nabla U^n,\nabla\Pi X^n\right)_\Omega &\le \delta t_n\lambda^2\|\sqrt{\nu_1^n}\nabla U^n\|_{L^2(\Omega)}\|\sqrt{\nu_1^n}\nabla\Pi X^n\|_{L^2}\\ &\le c_i c_M^{\frac12}\delta t_n^{\frac12}\lambda^{\frac52}\|\sqrt{\nu_1^n}\nabla U^n\|_{L^2(\Omega)}\|\Pi X^n\|_{L^2}\\ &\le c_i c_M^{\frac12}\delta t_n^{\frac12}\lambda^{\frac52}\|\sqrt{\nu_1^n}\nabla U^n\|_{L^2(\Omega)}\|X^n\|_{L^2}.\end{aligned}$$

Gathering the above estimates, we can recast (4.19) into

$$\lambda^2(1-\frac{c_i^2c_M}{2}\lambda^2)\|X^n\|_{L^2}^2 \le \frac12\delta t_n\lambda\|\sqrt{\nu_1^n}\nabla U^n\|^2_{L^2(\Omega)} - \lambda^2\delta t_n\left(\nabla\cdot\mathbf{f}(U^n),\Pi X^n\right)_\Omega.$$

If Λ_0 is chosen so that $\Lambda_0\le c_i^{-1}c_M^{-\frac12}$, then for all $\lambda\le\Lambda_0$

$$\lambda^2\|X^n\|^2_{L^2}\le\delta t_n\lambda\|\sqrt{\nu_1^n}\nabla U^n\|^2_{L^2(\Omega)}-2\lambda^2\delta t_n\left(\nabla\cdot\mathbf{f}(U^n),\Pi X^n\right)_\Omega. \tag{4.20}$$

The last term in the right-hand side of the above expression is the most complicated to estimate, and this is done by invoking the decomposition $\mathcal{T}=\mathcal{T}_V^n\cup\mathcal{T}_S^n$.

Step 4. (Control over $\mathcal{T}_V^n$). We use the fact that $\nu_1^n|_K = c_M h_K\|\mathbf{f}'(U^n)\|_{L^\infty(K)}$ over $\mathcal{T}_V^n$ and the L^2-stability of Π to obtain

$$-2\lambda^2\delta t_n\left(\mathbf{f}'(U^n)\cdot\nabla U^n,\Pi X^n\right)_{\mathcal{T}_V^n}\le\lambda\delta t_n\|\sqrt{\nu_1^n}\nabla U^n\|^2_{L^2(\Omega)}+c_M^{-1}\lambda^4\|X^n\|^2_{L^2(\Omega)}. \tag{4.21}$$

Step 5. (Control over $\mathcal{T}_S^n$). We handle the term $I:=-2\lambda^2\delta t_n\left(\nabla\cdot\mathbf{f}(U^n),\Pi X^n\right)_{\mathcal{T}_S^n}$ as follows:

$$\tfrac12 I = -\lambda^2\delta t_n\left(\Pi^0(\nabla\cdot\mathbf{f}(U^n)),\Pi X^n\right)_{\mathcal{T}_S^n}-\lambda^2\delta t_n\left(\nabla\cdot\mathbf{f}(U^n)-\Pi^0(\nabla\cdot\mathbf{f}(U^n)),\Pi X^n\right)_{\mathcal{T}_S^n}.$$

We now need to control $-\lambda^2\delta t_n\left(\Pi^0\nabla\cdot\mathbf{f}(U^n),\Pi X^n\right)_{\mathcal{T}_S^n}$; the key to the whole proof is here. Let us first recall that $X^n:=Z^n-\Pi^0Z^n$ and (4.17) holds since $\Pi^0\nabla\cdot\mathbf{f}(U^n)$ is piecewise constant; this property in turn implies that

$$-\lambda^2\delta t_n\left(\Pi^0\nabla\cdot\mathbf{f}(U^n),\Pi X^n\right)_{\mathcal{T}_S^n}=-\lambda^2\delta t_n\left(\Pi^0\nabla\cdot\mathbf{f}(U^n),\Pi X^n-X^n\right)_{\mathcal{T}_S^n}.$$

It is at this point that we use the fact that we are testing with $\Pi X^n - X^n$. In particular we are going to use the following key property

$$(\mathcal{R}_{\mathcal{T}_S^n}(U^n-U^{n-1}),\Pi X^n-X^n)_{\mathcal{T}_S^n}=0,$$

where the restriction operator $\mathcal{R}_{\mathcal{T}_S^n}$ is defined in (2.13). The above orthogonality property allows us to construct a residual $R^n:=\delta t_{n-1}^{-1}(U^n-U^{n-1})+\nabla\cdot\mathbf{f}(U^n)$ so that

$$\begin{aligned}\tfrac12 I &= -\lambda^2\delta t_n\left(\Pi^0(\nabla\cdot\mathbf{f}(U^n))+\delta t_{n-1}^{-1}\mathcal{R}_{\mathcal{T}_S^n}(U^n-U^{n-1}),\Pi X^n-X^n\right)_{\mathcal{T}_S^n}\\ &\quad-\lambda^2\delta t_n\left((\nabla\cdot\mathbf{f}(U^n)-\Pi^0(\nabla\cdot\mathbf{f}(U^n)),\Pi X^n\right)_{\mathcal{T}_S^n}\\ &= -\lambda^2\delta t_n\left(R^n,\Pi X^n-X^n\right)_{\dot{\mathcal{T}}_S^n}-\lambda^2\delta t_n\left((\Pi^0(\nabla\cdot\mathbf{f}(U^n))-\nabla\cdot\mathbf{f}(U^n),\Pi X^n-X^n\right)_{\dot{\mathcal{T}}_S^n}\\ &\quad-\lambda^2\delta t_n\left(\Pi^0(\nabla\cdot\mathbf{f}(U^n))+\delta t_{n-1}^{-1}\mathcal{R}_{\mathcal{T}_S^n}(U^n-U^{n-1}),\Pi X^n-X^n\right)_{\mathcal{L}^n}\\ &\quad-\lambda^2\delta t_n\left((\nabla\cdot\mathbf{f}(U^n)-\Pi^0(\nabla\cdot\mathbf{f}(U^n)),\Pi X^n\right)_{\mathcal{T}_S^n},\end{aligned}$$

where $\mathcal{L}^n$ is the layer of elements in $\mathcal{T}_S^n$ that is between $\dot{\mathcal{T}}_S^n$ and $\mathcal{T}_V^n$, i.e., $\dot{\mathcal{T}}_S^n \cup \mathcal{L}^n = \mathcal{T}_S^n$. We reorganize the above identity as follows:

$$\begin{aligned}\tfrac{1}{2} I = &-\lambda^2 \delta t_n \left(R^n, \Pi X^n - X^n\right)_{\dot{\mathcal{T}}_S^n} + \lambda^2 \delta t_n \left((\Pi^0(\nabla{\cdot}\mathbf{f}(U^n)) - \nabla{\cdot}\mathbf{f}(U^n), X^n\right)_{\mathcal{T}_S^n} \\ &- \lambda^2 \delta t_n \left(\nabla{\cdot}\mathbf{f}(U^n) + \delta t_{n-1}^{-1} \mathcal{R}_{\mathcal{T}_S^n}(U^n - U^{n-1}), \Pi X^n - X^n\right)_{\mathcal{L}^n}.\end{aligned}$$

Let us denote I_1, I_2 and I_3 the three terms in the right-hand side. We know that $c_E h_K^\alpha \|R^n\|_{L^\infty(K)} \le c_M \|\mathbf{f}'(U^n)\|_{L^\infty(K)}$, for all $K \in \mathcal{T}_S$; this implies that

$$\begin{aligned}I_1 := -\lambda^2 \delta t_n \left(R^n, \Pi X^n - X^n\right)_{\dot{\mathcal{T}}_S^n} &\le 2\lambda^2 \delta t_n \|R^n\|_{L^2(\dot{\mathcal{T}}_S^n)} \|X^n\|_{L^2(\Omega)} \\ &\le \epsilon \lambda^2 \|X^n\|_{L^2(\Omega)}^2 + \frac{c_M^2}{c_E^2 \epsilon} \lambda^{2(1+\alpha)} \delta t_n^{2(1-\alpha)} \|\mathbf{f}'\|_{L^\infty(\Omega)}^{2(1-\alpha)} |\Omega| \\ &\le \epsilon \lambda^2 \|X^n\|_{L^2(\Omega)}^2 + \frac{c_M^2 g_E}{\epsilon} \lambda^{2(1+\alpha)} \delta t_n^{2(1-\alpha)} \|U^0\|_{L^2(\Omega)}^2,\end{aligned}$$

where we set

$$g_E := \|\mathbf{f}'\|_{L^\infty(\mathbb{R})}^{2(1-\alpha)} |\Omega| c_E^{-2} \|U^0\|_{L^2(\Omega)}^{-2}, \tag{4.22}$$

and $\epsilon > 0$ is a constant yet to be chosen. To control I_2, we first observe that if $\nabla U^n|_K = 0$ or $\mathbf{f}'(U^n)|_K = 0$ then $\delta t_n \|\nabla{\cdot}\mathbf{f}(U^n) - \Pi^0(\nabla{\cdot}\mathbf{f}(U^n))\|_{L^\infty(K)}^2 = 0$, otherwise,

$$\begin{aligned}\delta t_n \|\nabla{\cdot}\mathbf{f}(U^n) - \Pi^0(\nabla{\cdot}\mathbf{f}(U^n))\|_{L^\infty(K)}^2 &\le \lambda h_K 4 \mathrm{osc}_K(\mathbf{f}, U^n) \|\nabla U^n\|_{L^\infty(K)}^2 \\ &\le 4 \frac{\lambda}{c_M} \nu_1^n|_K \|\nabla U^n\|_{L^\infty(K)}^2.\end{aligned}$$

Since the mesh is affine (2.9) also holds for ∇U^n, i.e., $\|\nabla U^n\|_{L^\infty(K)}^2 \le c_\infty^2 |K|^{-1} \|\nabla U^n\|_{L^2(K)}^2$. Upon using this inequality and the L^2-stability of Π we infer that

$$\begin{aligned}I_2 &:= -\lambda^2 \delta t_n \left(\nabla{\cdot}\mathbf{f}(U^n) - \Pi^0(\nabla{\cdot}\mathbf{f}(U^n)), \Pi X^n - X^n\right)_{\mathcal{T}_S^n} \\ &\le 4 c_\infty c_M^{-\frac{1}{2}} \lambda^{\frac{5}{2}} \delta t_n^{\frac{1}{2}} \|\sqrt{\nu_1^n} \nabla U^n\|_{L^2(\Omega)} \|X^n\|_{L^2(\Omega)} \le \epsilon \lambda^2 \|X^n\|_{L^2(\Omega)}^2 + \frac{4 c_\infty^2}{\epsilon c_M} \lambda^3 \delta t_n \|\sqrt{\nu_1^n} \nabla U^n\|_{L^2(\Omega)}^2.\end{aligned}$$

We proceed as follows to control I_3:

$$\begin{aligned}I_3 &:= -\lambda^2 \delta t_n \left(\nabla{\cdot}\mathbf{f}(U^n) + \delta t_{n-1}^{-1} \mathcal{R}_{\mathcal{T}_S^n}(U^n - U^{n-1}), \Pi X^n - X^n\right)_{\mathcal{L}^n} \\ &\le \lambda^2 \delta t_n \left(\|\nabla{\cdot}\mathbf{f}(U^n)\|_{L^2(\mathcal{L}^n)} + c_{\mathcal{R}} \|\delta t_{n-1}^{-1}(U^n - U^{n-1})\|_{L^2(\mathcal{L}^n)}\right) \|\Pi X^n - X^n\|_{L^2(\mathcal{L}^n)} \\ &\le \lambda^2 \delta t_n \left(\|\nabla{\cdot}\mathbf{f}(U^n)\|_{L^2(\mathcal{L}^n)} + c_{\mathcal{R}} \|\nabla{\cdot}\mathbf{f}(U^n)\|_{L^2(\mathcal{L}^n)} + c_{\mathcal{R}} \|R^n\|_{L^2(\mathcal{L}^n)}\right) \|\Pi X^n - X^n\|_{L^2(\Omega)} \\ &\le 2(1 + c_{\mathcal{R}}) c_M^{-\frac{1}{2}} \lambda^{\frac{5}{2}} \delta t_n^{\frac{1}{2}} \|\sqrt{\nu_1^n} \nabla U^n\|_{L^2(\Omega)} \|X^n\|_{L^2(\Omega)} + 2 c_{\mathcal{R}} \lambda^2 \delta t_n \|R^n\|_{L^2(\mathcal{L}^n)} \|X^n\|_{L^2(\Omega)},\end{aligned}$$

where we used that $\nu_1^n|_K = c_M h_k \|\mathbf{f}'(U^n)\|_{L^\infty(K)}$ for all $K \in \mathcal{L}^n$ together with the L^2-stability of Π^0, Π and $\mathcal{R}_{\mathcal{T}_S^n}$ (Lemma 2.1). Using again that $c_E h_K^\alpha \|R^n\|_{L^\infty(K)} \le c_M \|\mathbf{f}'(U^n)\|_{L^\infty(K)}$ for all $K \in \mathcal{L}^n \subset \mathcal{T}_S^n$ we infer that

$$I_3 \le \epsilon \lambda^2 \|X^n\|_{L^2} + \frac{c_{\mathcal{R}}^2 c_M^2 g_E}{\epsilon} \lambda^{2(1+\alpha)} \delta t_n^{2(1-\alpha)} \|U^0\|_{L^2}^2 + \frac{c'}{\epsilon c_M} \lambda^3 \delta t_n \|\sqrt{\nu_1^n} \nabla U^n\|_{L^2}^2,$$

where g_E is given by (4.22). Gathering the estimates on I_1, I_2, and I_3 we finally deduce the following estimate

$$
\begin{aligned}
(4.23)\qquad -2\lambda^2\delta t_n(\mathbf{f}'(U^n)\cdot\nabla U^n, \Pi X^n)_\Omega \le 6\epsilon\lambda^2\|X^n\|_{L^2} \\
+ \frac{c\, c_M^2 g_E}{\epsilon}\lambda^{2(1+\alpha)}\delta t_n^{2(1-\alpha)}\|U^0\|^2_{L^2(\Omega)} + \frac{c'}{\epsilon c_M}\lambda^3\delta t_n\|\sqrt{\nu_1^n}\nabla U^n\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Step 6. (Conclusion) Combining (4.21) and (4.23), and setting $\epsilon = \frac{1}{12}$ we can finally rewrite (4.20) as follows:

$$
\lambda^2\|X^n\|_{L^2(\Omega)} \le c\, c_M^2 g_E\lambda^{2(1+\alpha)}\delta t_n^{2(1-\alpha)}\|U^0\|^2_{L^2(\Omega)} + c'\lambda(1+\lambda^2 c_M^{-1})\delta t_n\|\sqrt{\nu_1^n}\nabla U^n\|^2_{L^2(\Omega)}.
$$

We now combine the above estimate with (4.18) to obtain

$$
\begin{aligned}
\|U^{n+1} - W^n\|^2_{L^2} &\le c\, c_M^2 g_E\lambda^{2(1+\alpha)}\delta t_n^{2(1-\alpha)}\|U^0\|^2_{L^2} \\
&\quad + c'(1 + c_M c_i^2 + (4+\lambda^2)c_M^{-1})\lambda\delta t_n(\|\sqrt{\nu_1^n}\nabla U^n\|^2_{L^2} + \|\sqrt{\nu_2^n}\nabla W^n\|^2_{L^2}).
\end{aligned}
$$

Provided Λ_0 is chosen so that $c'\Lambda_0(1 + c_M c_i^2 + (4+\Lambda_0^2)c_M^{-1}) \le \frac{1}{2}$, the above bound together with (4.13) implies that the following energy estimate holds for $\lambda \le \Lambda_0$,

$$
\begin{aligned}
\|U^{n+1}\|^2_{L^2} - \|U^n\|^2_{L^2} + \delta t_n\left(\|\sqrt{\nu_1^n}\nabla U^n\|^2_{L^2} + \|\sqrt{\nu_2^n}\nabla W^n\|^2_{L^2}\right) \\
\le c\, c_M^2 g_E\lambda^{2(1+\alpha)}\delta t_n^{2(1-\alpha)}\|U^0\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Summing this inequality from $n = 0$ to N gives

$$
\begin{aligned}
\|U^{n+1}\|^2_{L^2} + \sum_{i=0}^{n}\delta t_n\left(\|\sqrt{\nu_1^i}\nabla U^i\|^2_{L^2} + \|\sqrt{\nu_2^i}\nabla W^i\|^2_{L^2}\right) \\
\le \|U_0\|^2_{L^2}\left(1 + c\, c_M^2 g_E T^{2(1-\alpha)}\lambda^{2(1+\alpha)}(\delta t/T)^{1-2\alpha}\right)
\end{aligned}
$$

which is the desired estimates. Note that $g_E T^{2(1-\alpha)}$ is a dimensionless constant. ∎

Remark 4.5. *(No Restriction on the Polynomial Order)* We emphasize that one of the key step in the above stability proof is the control of the term $\|h\nabla Z^n\|_{L^2(K)}$. The two key arguments consist of the following: (i) subtracting the projection onto constants of Z^n, i.e., $\|h\nabla(Z^n - \Pi^0 Z^n)\|_{L^2(K)}$, so as to be able to use the inverse estimate (4.16); (ii) forming a residual relying on the orthogonality property (4.17). This argument is borrowed from [5], where it was restricted to piecewise linear finite elements. We have extended it to any polynomial degree $k \ge 1$ by taking advantage of the non-linear viscosity ν_1^n which satisfies

$$
c_M h_K \mathrm{osc}_K(\mathbf{f}, U^n) \le \nu_1^n|_K, \qquad \forall K \in \dot{\mathcal{T}}_S^n.
$$

Remark 4.6. *(Restriction on α)* The restriction $\alpha < \frac{1}{2}$ for stability in Theorem 4.1 seems to be purely technical. Thorough numerical experiments have shown that the method is stable and convergent with $\alpha = 1$. We then conjecture that Theorem 4.1 should hold in the range $\alpha \in [0, 1]$.

5. Midpoint RK2. The algorithm presented in §4.1 relies on a viscosity that is built from the previous time step (see (4.5)-(4.7)). This may seem a little odd since we are solving a Cauchy problem. We propose in this section an alternative technique that consists of constructing the viscosity on the fly. The method is implemented with the midpoint RK2 technique.

5.1. The Algorithm. Let $t_0 = 0$ and let $U^0 \in \mathbb{V}(\mathcal{T})$ be an approximation of u_0. Let $\lambda > 0$ be a CFL number. Let $U^n \in \mathbb{V}(\mathcal{T})$ be the approximation of u at time t_n, $n \geq 0$. Let δt_n be a given time step possibly restricted later by the CFL number, see (5.3), and set $t_{n+1} = t_n + \delta t_n$. The midpoint RK2 algorithm is formulated as follows: Seek $W^n \in \mathbb{V}(\mathcal{T})$ and $U^{n+1} \in \mathbb{V}(\mathcal{T})$ satisfying

$$(W^n, V)_\Omega - (U^n, V)_\Omega + \frac{\delta t_n}{2}(\nabla\cdot\mathbf{f}(U^n), V)_\Omega = 0, \tag{5.1}$$

$$(U^{n+1}, V)_\Omega - (U^n, V)_\Omega + \delta t_n(\nabla\cdot\mathbf{f}(W^n), V)_\Omega + \delta t_n(\nu^n \nabla W^n, \nabla V)_\Omega = 0, \tag{5.2}$$

for all $V \in \mathbb{V}(\mathcal{T})$, where the viscosity ν^n is defined below. We assume that the time step satisfies the condition

$$\delta t_n \leq \lambda \min_{K\in\mathcal{T}} \frac{h_K}{\max(\|\mathbf{f}'(U^n)\|_{L^\infty(K)}, \|\mathbf{f}'(W^n)\|_{L^\infty(K)})}. \tag{5.3}$$

Note that the above condition can only be verified a-posteriori. If the condition (5.3) is not satisfied, the computation of W^n and U^{n+1} is redone with a smaller time step, say δt_n is divided by 1.5. This procedure always terminates due to the uniform Lipschitz assumption on the flux $\mathbf{f}$; i.e., picking δt_n smaller than $\frac{1}{\|\mathbf{f}'\|_{L^\infty(\mathbb{R})}} \min_{K\in\mathcal{T}} h_K$ always guarantees that (5.3) holds. To avoid issues induced by the boundary condition we assume that W^n satisfy the following conservation properties

$$(\nabla\cdot\mathbf{f}(W^n), W^n)_\Omega \geq 0. \tag{5.4}$$

We refer to §2.1 for a discussion on the validity of this assumption.

Let $c_M > 0$, $c_E > 0$ and $\alpha \geq 0$ be three real numbers, and let introduce the following time-dependent partition of $\mathcal{T} = \mathcal{T}_V^n \cup \mathcal{T}_S^n$, $\mathcal{L}^n := \mathcal{T}_S^n \backslash \dot{\mathcal{T}}_S^n$,

$$\mathcal{T}_S^n := \{K \in \mathcal{T} : c_E \|h^\alpha R^n\|_{L^\infty(K)} \leq c_M \|\mathbf{f}'(U^n)\|_{L^\infty(K)}\}, \qquad \mathcal{T}_V^n := \mathcal{T} \backslash \mathcal{T}_S^n, \tag{5.5}$$

where the residual R^n is defined by

$$R^n := 2\frac{W^n - U^n}{\delta t_n} + \nabla\cdot\mathbf{f}(W^n). \tag{5.6}$$

We define the viscosity $\nu^n : \Omega \to \mathbb{R}$ at time t_n, $n \geq 1$, as follows:

$$\nu^n|_K = \begin{cases} \min(\nu_M^n|_K, \nu_1^n|_K), & \text{if } K \in \mathcal{T}_V^n \cup \dot{\mathcal{T}}_S^n, \\ \nu_M^n|_K & \text{if } K \in \mathcal{L}^n, \end{cases} \tag{5.7}$$

where

$$\nu_M^n|_K := c_M h_K \max(\|\mathbf{f}'(U^n)\|_{L^\infty(K)}, \|\mathbf{f}'(W^n)\|_{L^\infty(K)}), \tag{5.8}$$

$$\nu_1^n|_K := h_K \max(c_E \|h^\alpha R^n\|_{L^\infty(K)}, c_M \mathrm{osc}_K(\mathbf{f}, W^n), c_M \mathrm{nl}_K(\mathbf{f}, W^n, U^n)). \tag{5.9}$$

The oscillation $\mathrm{osc}_K(\mathbf{f}, W^n)$ is defined in (4.8) and the nonlinear variation $\mathrm{nl}_K(\mathbf{f}, W^n, U^n)$ is defined in (4.10).

Remark 5.1. *(Consistency of Viscosities)* The set $\mathcal{T}_V^n \cup \mathcal{L}^n$ is composed of the elements where the viscosity saturates to first-order, $\nu^n = c_M \max(\|h\, \mathbf{f}'(U^n)\|_{L^\infty(K)}, \|h\, \mathbf{f}'(W^n)\|_{L^\infty(K)})$, and $\mathcal{T}_S^n$ is composed of the elements where the viscosity is formally higher-order, $\nu^n \approx c_E \|h^{1+\alpha} R^n\|_{L^\infty(K)}$. Note that $\|h^{1+\alpha} R^n\|_{L^\infty(K)}$ is formally of order $\mathcal{O}(h_K^{1+\alpha} \delta t_n)$ whereas $h_K \mathrm{osc}_K(\mathbf{f}, W^n)$ and $h_K \mathrm{nl}_K(\mathbf{f}, W^n, U^n)$ are of order $\mathcal{O}(h_K^3)$ and $\mathcal{O}(h_K \delta t_n^2)$, respectively. We refer to Remarks 4.1-4.3 for discussions on the viscosities. Note again that the consistency error induced by the entropy viscosity is of higher order than that of the second-order RK2 method.

Remark 5.2. *(Definition of c_M and c_E)* The constants c_M and c_E are user-defined; c_M is non-dimensional and of order one, whereas c_E is dimensional. For instance just like in Remark 4.4 one can set

$$c_E := \mathfrak{c}_E \frac{D^{1-\alpha}}{|\Omega|^{-1/2}\|U^0\|_{L^2(\Omega)}}, \tag{5.10}$$

or $c_E := \mathfrak{c}_E D^{1-\alpha}\|u_0\|^{-1}_{L^\infty(\Omega)}$, where $D := \mathrm{diam}(\Omega)$ and $\mathfrak{c}_E$ is a user-defined non-dimensional constant of order one; see also Remark 3.1.

We mention two useful bounds that we will use repeatedly. On one hand

$$\delta t_n \|\mathbf{f}'(V)\cdot\nabla\varphi\|^2_{L^2(\tau)} \le c_M^{-1}\lambda\|\sqrt{\nu^n}\ \nabla\varphi\|^2_{L^2(\tau)}, \tag{5.11}$$

holds for $V = U^n$ or $V = W^n$ and for any subset $\tau \subset \mathcal{T}_V^n \cup \mathcal{L}^n$ and any $\varphi \in H^1(\tau)$. On the other hand,

$$\nu^n|_K \le c_M \max(\|h\ \mathbf{f}'(U^n)\|_{L^\infty(K)}, \|h\ \mathbf{f}'(W^n)\|_{L^\infty(K)}), \qquad \forall K \in \mathcal{T}, \tag{5.12}$$

$$\max(c_M h_K \mathrm{osc}_K(\mathbf{f}, W^n), c_M h_K \mathrm{nl}_K(\mathbf{f}, W^n, U^n)) \le \nu^n|_K, \qquad \forall K \in \mathcal{T}. \tag{5.13}$$

5.2. Stability Analysis of RK2/Midpoint.

We now analyze the L^2-stability of the Midpoint time discretization of (2.1).

Theorem 5.1 (Stability of RK2/Mid Point). *Let $(U^i)_{i=0}^{n+1}$, $(W^i)_{i=0}^{n}$ be the sequences produced by the algorithm* (5.1)–(5.2)–(5.7). *There is $\Lambda_0 > 0$ that depends only on the user-defined parameters c_M, $\mathfrak{c}_E$, the Lipschitz constant of the flux, and on the mesh family constants c_0, c_i, and there is a constant c that additionally depends linearly on $T^{2(1-\alpha)}$ so that the following L^2-stability estimate holds for all $t_n \le T$ and all $\lambda \in (0, \Lambda_0]$:*

$$\|U^{n+1}\|^2_{L^2(\Omega)} + \sum_{i=0}^{n} \delta t_i \|\sqrt{\nu^i}\nabla W^n\|^2_{L^2(\Omega)} \le \|U^0\|^2_{L^2(\Omega)}\left(1 + c\lambda^{2(1+\alpha)}(\delta t/T)^{1-2\alpha}\right),$$

where $\delta t := \max_{i=0,\dots,n}\delta t_i$. Moreover, the algorithm is L^2-stable if $\alpha \le \frac{1}{2}$.

Proof. The proof is similar to that provided in §4 for the Heun method and we only outline the main steps.

Step 1. Testing (5.2) with W^n gives

$$\|U^{n+1}\|^2_{L^2(\Omega)} - \|U^n\|^2_{L^2(\Omega)} + 2\delta t_n\|\sqrt{\nu^n}\nabla W^n\|^2_{L^2(\Omega)} \le (Y^n, U^{n+1} - U^n)_\Omega, \tag{5.14}$$

where we used the conservation property (5.4) and the notation

$$Y^n = U^{n+1} + U^n - 2W^n.$$

In view of (5.14), we need to establish a bound on $\|Y^n\|_{L^2(\Omega)}$. The linear combination (5.2)$-2\times$(5.1) gives us a way to control Y,

$$\begin{aligned}(Y^n, V)_\Omega = &-\delta t_n((\mathbf{f}'(W^n) - \mathbf{f}'(U^n))\cdot\nabla W^n, V)_\Omega \\ &- \delta t_n(\mathbf{f}'(U^n)\nabla(W^n - U^n), V)_\Omega - \delta t_n(\nu^n\nabla W^n, \nabla V)_\Omega, \quad \forall V \in \mathbb{V}(\mathcal{T})\end{aligned} \tag{5.15}$$

Owing to the definition of the viscosity ν^n (see (4.10) and (5.13)), we have

$$\delta t_n\|\mathbf{f}'(W^n) - \mathbf{f}'(U^n)\|^2_{L^\infty(K)} \le 4\nu^n|_K\lambda c_M^{-1},$$

which in turn gives

$$(5.16)\quad -\delta t_n((\mathbf{f}'(W^n)-\mathbf{f}'(U^n))\cdot\nabla W^n, V)_\Omega \le 2c_M^{-\frac{1}{2}}\lambda^{\frac{1}{2}}\delta t_n^{\frac{1}{2}}\|\sqrt{\nu^n}\nabla W^n\|_{L^2(\Omega)}\|V\|_{L^2(\Omega)}.$$

The second term in the right hand side of (5.15) is handled as follows:

$$-\delta t_n(\mathbf{f}'(U^n)\nabla(W^n-U^n), V)_\Omega \le \lambda\|h\nabla(W^n-U^n)\|_{L^2(\Omega)}\|V\|_{L^2(\Omega)}.$$

For the third term in the right hand side of (5.15), we use the bound (5.12) on $\nu^n|_K$ and an inverse estimate

$$-\delta t_n(\nu^n\nabla W^n, \nabla V)_\Omega \le c_i c_M^{\frac{1}{2}}\lambda^{\frac{1}{2}}\delta t_n^{\frac{1}{2}}\|\sqrt{\nu^n}\nabla W^n\|_{L^2(\Omega)}\|V\|_{L^2(\Omega)}.$$

Gathering the above three estimates we arrive at

$$(5.17)\quad \|Y^n\|_{L^2(\Omega)}^2 \le \lambda^2\|h\ \nabla(W^n-U^n)\|_{L^2(\Omega)}^2 + c\,(c_i^2c_M+c_M^{-1})\lambda\delta t_n\|\sqrt{\nu^n}\nabla W^n\|_{L^2(\Omega)}^2.$$

Then upon introducing the notation $Z^n := W^n - U^n$, we now realize that we must find a bound on $\lambda\|h\ \nabla Z^n\|_{L^2(\Omega)}$.

Step 2. Recalling that that Π^0 is the L^2-projection over constants and that Π is the L^2-projection onto $\mathbb{V}(\mathcal{T})$, we set $X^n := Z^n - \Pi^0 Z^n$ and we observe that

$$(5.18)\quad c_0^{-2}\|X^n\|_{L^2}^2 \le \|h\nabla Z^n\|_{L^2}^2 = \sum_{K\in\mathcal{T}}\|h\nabla X^n\|_{L^2(K)}^2 \le c_i^2\|X^n\|_{L^2}^2 = c_i^2(\Pi X^n, Z^n)_\Omega.$$

Then, (5.1) together with (5.18) and the stability of the L^2-projection yields

$$\begin{aligned}
\lambda^2\|h\nabla Z^n\|_{L^2(\Omega)}^2 &\le -\frac{1}{2}c_i^2\delta t_n\lambda^2(\Pi X^n, \nabla\cdot\mathbf{f}(U^n))_\Omega\\
&\le -\frac{1}{2}c_i^2\delta t_n\lambda^2(\Pi X^n, \mathbf{f}'(U^n)\cdot\nabla W^n)_\Omega + \frac{1}{2}c_i^2\delta t_n\lambda^2(\Pi X^n, \mathbf{f}'(U^n)\cdot\nabla Z^n)_\Omega\\
&\le -\frac{1}{2}c_i^2\delta t_n\lambda^2(\Pi X^n, \mathbf{f}'(U^n)\cdot\nabla W^n)_\Omega + \frac{1}{2}c_i^2c_0\lambda^3\|h\nabla Z^n\|_{L^2(\Omega)}^2.
\end{aligned}$$

Restricting $\Lambda_0 \le c_i^{-2}c_0^{-1}$, we deduce that

$$(5.19)\qquad \lambda^2\|h\nabla Z^n\|_{L^2(\Omega)}^2 \le -c_i^2\delta t_n\lambda^2(\Pi X^n, \mathbf{f}'(U^n)\cdot\nabla W^n)_\Omega.$$

We are now going to use different techniques to deduce a bound from above on the quantity $\delta t_n\lambda^2|(\Pi X^n, \mathbf{f}'(U^n)\cdot\nabla W^n)_\Omega|$ in the smooth and in the viscous regions.

Step 3. (Control over $\mathcal{T}_V$) Invoking (5.11) with $V = U^n$ and the stability of the L^2-projection, we write

$$\delta t_n\lambda^2|(\Pi X^n, \mathbf{f}'(U^n)\cdot\nabla W^n)_{\mathcal{T}_V^n}| \le c_M^{-\frac{1}{2}}\delta t_n^{\frac{1}{2}}\lambda^{\frac{5}{2}}\|X^n\|_{L^2(\Omega)}\|\sqrt{\nu^n}\ \nabla W^n\|_{L^2(\mathcal{T}_V^n)},$$

which, owing to (5.18), gives

$$(5.20)\qquad \begin{aligned}\delta t_n\lambda^2|(\Pi X^n, \mathbf{f}'(U^n)\cdot\nabla W^n)_{\mathcal{T}_V^n}| \le{}& \epsilon\lambda^2\|h\nabla Z^n\|_{L^2(\Omega)}^2\\ &+\frac{c_0^2c_M^{-1}}{4\epsilon}\lambda^3\delta t_n\|\sqrt{\nu^n}\nabla W^n\|_{L^2(\Omega)}^2,\end{aligned}$$

where ϵ is a constant yet to be chosen.

Step 4. (Control over $\mathcal{T}_S$) We now focus our attention on the smooth region, and we use the property that the residual (5.6) is small in the smooth region. We have

$$\delta t_n \lambda^2 (\Pi X^n, \mathbf{f}'(U^n)\cdot\nabla W^n)_{\mathcal{T}_S^n} = \delta t_n \lambda^2 (\Pi X^n, (\mathbf{f}'(U^n) - \mathbf{f}'(W^n))\cdot\nabla W^n)_{\mathcal{T}_S^n} + \delta t_n \lambda^2 (\Pi X^n, \nabla\cdot\mathbf{f}(W^n))_{\mathcal{T}_S^n}.$$

Note that the first term in the right hand side of the above expression is directly absorbed in the viscosity using (5.16). Indeed, the stability of Π and (5.18) imply that

$$\begin{aligned}\delta t_n \lambda^2 (\Pi X^n, (\mathbf{f}'(U^n) - \mathbf{f}'(W^n))\cdot\nabla W^n)_{\mathcal{T}_S^n} &\leq c_M^{-\frac{1}{2}} c_0 \lambda^{\frac{5}{2}} \delta t^{\frac{1}{2}} \|\sqrt{\nu^n}\nabla W^n\|_{L^2} \|h\nabla Z^n\|_{L^2} \\ &\leq \epsilon\lambda^2 \|h\ \nabla Z^n\|^2_{L^2(\Omega)} + \frac{c_0^2 c_M^{-1}}{4\epsilon}\lambda^3 \delta t_n \|\sqrt{\nu^n}\nabla W^n\|^2_{L^2(\Omega)},\end{aligned}$$

where $\epsilon > 0$ is yet to be chosen. The remaining term, $\delta t_n \lambda^2 (\Pi X^n, \nabla\cdot\mathbf{f}(W^n))_{\mathcal{T}_S^n}$, is the most critical one. We start by writing

$$\delta t_n \lambda^2 (\Pi X^n, \nabla\cdot\mathbf{f}(W^n))_{\mathcal{T}_S^n} = \delta t_n \lambda^2 (\Pi X^n, \Pi^0 \nabla\cdot\mathbf{f}(W^n))_{\mathcal{T}_S^n} + \delta t_n \lambda^2 (\Pi X^n, \nabla\cdot\mathbf{f}(W^n) - \Pi^0\nabla\cdot\mathbf{f}(W^n))_{\mathcal{T}_S^n}.$$

Taking advantage of the orthogonality of $X^n := Z^n - \Pi^0 Z^n$ with respect to piecewise constants and of the orthogonality of $X^n - \Pi X^n$ with respect to elements in $\mathbb{V}(\mathcal{T})$, we infer that

$$\begin{aligned}\delta t_n \lambda^2 (\Pi X^n, \nabla\cdot\mathbf{f}(W^n))_{\mathcal{T}_S^n} &= \delta t_n \lambda^2 (\Pi X^n - X^n, \Pi^0\nabla\cdot\mathbf{f}(W^n))_{\mathcal{T}_S^n} \\ &+ \delta t_n \lambda^2 (\Pi X^n, \nabla\cdot\mathbf{f}(W^n) - \Pi^0\nabla\cdot\mathbf{f}(W^n))_{\mathcal{T}_S^n}, \\ &= \delta t_n \lambda^2 (\Pi X^n - X^n, (\delta t_n)^{-1}\mathcal{R}_{\mathcal{T}_S^n}(W^n - U^n) + \nabla\cdot\mathbf{f}(W^n))_{\mathcal{T}_S^n} \\ &+ \delta t_n \lambda^2 (\Pi X^n - X^n, \Pi^0\nabla\cdot\mathbf{f}(W^n) - \nabla\cdot\mathbf{f}(W^n))_{\mathcal{T}_S^n} \\ &+ \delta t_n \lambda^2 (\Pi X^n, \nabla\cdot\mathbf{f}(W^n) - \Pi^0\nabla\cdot\mathbf{f}(W^n))_{\mathcal{T}_S^n},\end{aligned}$$

where $\mathcal{R}_{\mathcal{T}_S^n}$ is defined by (2.13) and is the identity operator over $\dot{\mathcal{T}}_S^n$. The direct decomposition of the domain partition into $\mathcal{T} = \dot{\mathcal{T}}_S^n \cup \mathcal{L}^n \cup \mathcal{T}_V^n$ (see Figure 4.1) yields

$$\begin{aligned}\delta t_n \lambda^2 (\Pi X^n, \nabla\cdot\mathbf{f}(W^n))_{\mathcal{T}_S^n} &= \delta t_n \lambda^2 (\Pi X^n - X^n, R^n)_{\dot{\mathcal{T}}_S^n} \\ &+ \delta t_n \lambda^2 (X^n, \nabla\cdot\mathbf{f}(W^n) - \Pi^0\nabla\cdot\mathbf{f}(W^n))_{\mathcal{T}_S^n} \\ &+ \delta t_n \lambda^2 (\Pi X^n - X^n, (\delta t_n)^{-1}\mathcal{R}_{\mathcal{T}_S^n}(W^n - U^n) + \nabla\cdot\mathbf{f}(W^n))_{\mathcal{L}^n} \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

Proceeding as in the proof of Theorem 4.1, we obtain the following bounds for each term:

$$\begin{aligned}I_1 &\leq \epsilon\lambda^2 \|X^n\|^2_{L^2(\Omega)} + \frac{c_M^2 g_E}{\epsilon}\lambda^{2(1+\alpha)}\delta t_n^{2(1-\alpha)} \|U^0\|^2_{L^2(\Omega)}, \\ I_2 &\leq \epsilon\lambda^2 \|X^n\|^2_{L^2(\Omega)} + \frac{4c_\infty^2}{\epsilon c_M}\lambda^3 \delta t_n \|\sqrt{\nu^n}\nabla W^n\|^2_{L^2(\Omega)}, \\ I_3 &\leq \epsilon\lambda^2 \|X^n\|_{L^2} + \frac{c c_M^2 g_E}{\epsilon}\lambda^{2(1+\alpha)}\delta t_n^{2(1-\alpha)} \|U^0\|^2_{L^2(\Omega)} + \frac{c'}{\epsilon c_M}\lambda^3 \delta t_n \|\sqrt{\nu^n}\nabla W^n\|^2_{L^2(\Omega)},\end{aligned}$$

where g_E is defined in (4.22). Gathering the above estimates and using (5.18) yields

$$\begin{aligned}(5.21)\quad \delta t_n \lambda^2 (\Pi X^n, \nabla{\cdot}\mathbf{f}(W^n))_{\mathcal{T}_S^n} &\le 3c_0^2\epsilon\lambda^2 \|h\ \nabla Z^n\|_{L^2} \\ &+ c\frac{c_M^2 g_E}{\epsilon}\lambda^{2(1+\alpha)}\delta t_n^{2(1-\alpha)}\|U^0\|^2_{L^2(\Omega)} + \frac{c'}{\epsilon c_M}\lambda^3\delta t_n\|\sqrt{\nu_2^n}\nabla U^n\|^2_{L^2(\Omega)}.\end{aligned}$$

Step 5. (Control of $\|h\nabla Z^n\|_{L^2(\Omega)}$ and $\|Y^n\|_{L^2(\Omega)}$) Combining (5.20) and (5.21), and setting $\epsilon = \frac{1}{8c_0^2}$, we can finally rewrite (5.19) as follows:

$$\begin{aligned}(5.22)\quad \lambda^2\|h\nabla Z^n\|^2_{L^2} &\le c\, c_M^2 g_E \lambda^{2(1+\alpha)}\delta t_n^{2(1-\alpha)}\|U^0\|^2_{L^2(\Omega)} \\ &+ c'(c_i^2 c_M + c_M^{-1})\lambda^3\delta t_n\|\sqrt{\nu^n}\nabla W^n\|^2_{L^2(\Omega)}.\end{aligned}$$

We now combine the above estimate with (5.17) to arrive at

$$\begin{aligned}(5.23)\quad \|Y^n\|^2_{L^2} &\le c\, c_M^2 g_E \lambda^{2(1+\alpha)}\delta t_n^{2(1-\alpha)}\|U^0\|^2_{L^2(\Omega)} \\ &+ c'\lambda(1+\lambda^2)(c_i^2 c_M + c_M^{-1})\delta t_n\|\sqrt{\nu^n}\nabla W^n\|^2_{L^2(\Omega)}.\end{aligned}$$

Step 6. We now conclude. Upon observing that

$$|(Y^n, U^{n+1}-U^n)_\Omega| = |\|Y^n\|^2_{L^2(\Omega)} + 2(Y, W^n - U^n)_\Omega| \le \|Y^n\|^2_{L^2(\Omega)} + \|Y^n\|_{L^2(\Omega)}\|Z^n\|_{L^2(\Omega)}.$$

and by using the estimates (5.22) and (5.23) in (5.14), we infer that

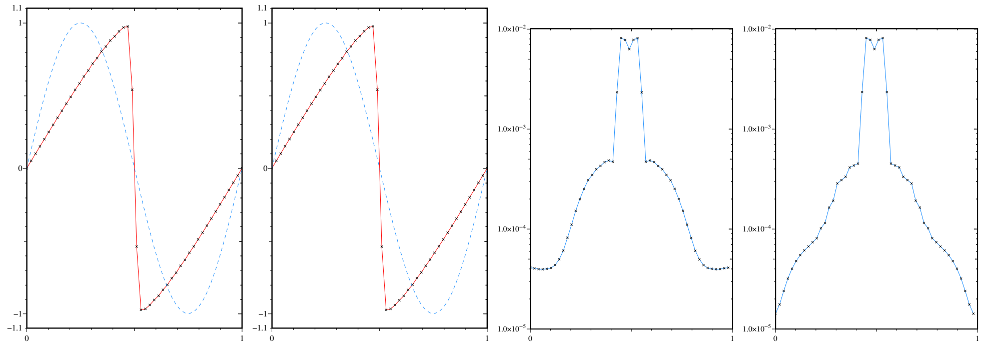
$$\begin{aligned}\|U^{n+1}\|^2_{L^2(\Omega)} - \|U^n\|^2_{L^2(\Omega)} + 2\delta t_n\|\sqrt{\nu^n}\nabla W^n\|^2_{L^2(\Omega)} &\le c\, c_M^2 g_E \lambda^{2(1+\alpha)}\delta t_n^{2(1-\alpha)}\|U^0\|^2_{L^2(\Omega)} \\ &+ c'\lambda(1+\lambda^2)(c_i^2 c_M + c_M^{-1})\delta t_n\|\sqrt{\nu^n}\nabla W^n\|^2_{L^2(\Omega)}.\end{aligned}$$

Upon further restricting Λ_0 so that $c'\Lambda_0(1+\Lambda_0^2)(c_i^2 c_M + c_M^{-1}) \le 1$, we conclude by using the usual telescoping argument. ∎

Remark 5.3. *(Restriction on α)* The stability restriction $\alpha < \frac{1}{2}$ in Theorem 5.1 seems to be technical. We conjecture again that Theorem 5.1 should hold in the range $\alpha \in [0, 1]$.

6. Discussion on entropies. The method discussed above bears some resemblance to the residual-based shock capturing techniques from [15, 23] when $E(u) = u$. The present method is however significantly different from that in [15, 23] in the sense that the viscosity is scaled differently, it is not allowed to exceed the first-order viscosity $c_M\|h\boldsymbol{\beta}\|_{L^\infty(K)}$, the time stepping is explicit, and our analysis does not require any sort of additional linear stabilization to work properly (Galerkin-Least-Squares, streamline diffusion [16], SUPG [3], Discontinuous Galerkin [17] or edge stabilization [6]). Our analysis is similar in spirit to that of [4], where convergence of a class of nonlinear viscosity methods for the one-dimensional Burgers equation is performed without using any type of linear stabilization. This idea was later applied to viscoelastic systems in [2]. However, our work differs from [4] in that the viscosity is built differently and the time is kept continuous in [4].

We illustrate the method on the inviscid Burgers equation in Figure 6.1. The domain is periodic, $\Omega = (0, 1)$, the initial data is $u_0(x) = \sin(2\pi x)$. The computation is done with continuous piecewise linear finite elements and RK2 time stepping (the Heun and the midpoint method give similar results). The solution is shown at $T =$

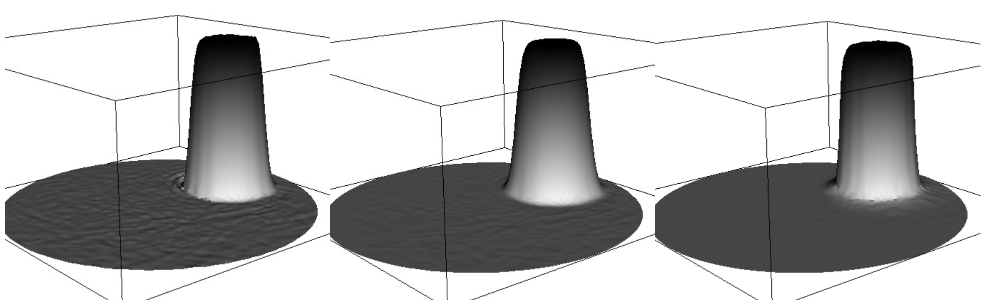


(a) $E(u) = u$ (b) $E(u) = \frac{1}{2}u^2$ (c) $E(u) = u$ (d) $E(u) = \frac{1}{2}u^2$

FIG. 6.1. *Burgers equation, $\mathbb{P}_1$ continuous finite elements, RK3, 50 elements.*

0.25. The displayed results have been obtained with the residual viscosity, $E(u) = u$, and the square entropy $E(u) = \frac{1}{2}u^2$. We observe that the method performs very well in both cases and the viscosity focuses in the shock (note that the viscosity field in displayed in log scale).

In some cases it may be beneficial to use nonlinear entropies like $E(u) = |u - c|^\gamma$, $\gamma > 1$. Although we have numerically observed that the method performs well with these entropies, we have not yet been able to prove stability. To motivate the use of higher-order entropies even for the linear transport equation, $\mathbf{f}(u) = \boldsymbol{\beta} u$, we show in Figure 6.2 numerical tests on the transport equation in the unit disk $\Omega = \{(x, y) \in \mathbb{R}^2, \sqrt{x^2 + y^2} < 1\}$ using the entropy viscosity method with three different entropies: $E(u) = u - \frac{1}{2}$ (Figure 6.2(a)), $E(u) = (u - \frac{1}{2})^2$ (Figure 6.2(b)), and $E(u) = (u - \frac{1}{2})^{30}$ (Figure 6.2(c)). The velocity field is a solid rotation of angular velocity 2π, i.e., $\boldsymbol{\beta} = 2\pi(-y, x)$. The initial field is $u_0(\mathbf{x}) = 1$ if $\mathbf{x}$ is in the disk of radius 0.5 centered at $(0.6, 0)$ and $u_0(\mathbf{x}) = 0$ otherwise. The space approximation is done on a mesh composed of 25901 $\mathbb{P}_2$ nodes ($h \approx 0.025$). The time stepping is done with the SSP RK3 method. The solution is computed at $T = 10$, i.e., after 10 revolutions. This example



(a) $E(u) = 2u - 1$ (b) $E(u) = (2u - 1)^2$ (c) $E(u) = (2u - 1)^{30}$

FIG. 6.2. *Tests on the linear transport equation with three different entropies.*

shows that the higher the nonlinearity in the entropy the better the performance of the method when applied to the linear transport equation with piecewise constant data (at least in the eyeball-norm).

Finally, we would like to emphasis once again that the choice of entropy viscosity to be used is problem-dependent. It may happen that for problems with non-convex fluxes more that one entropy may have to be used to construct the viscosity.

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