Wavelets: An Elementary Introduction and Examples

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ABSTRACT

Wavelets have been found to be very useful in many scientific and engineering applications including computer graphics, scientific visualization, data compression and signal processing. Unfortunately, the two most well known monographs on wavelets by Chui and Daubechies are still unaccessible to a beginner due to the level of mathematical sophistication needed to go through these monographs. Moreover both these monographs introduce wavelets from a signal processing perspective. The objective here is to introduce wavelets via scaling functions using the theory of multiresolution analysis. The primary focus is to describe many examples of scaling functions and their corresponding wavelets without proofs. The two-scale reconstruction and decomposition relations are described in order to provide the reader with a quick working knowledge of wavelets. Examples of wavelets included in this monograph are Haar wavelets, B-spline wavelets and Daubechies wavelets on real line, and Legendre wavelets and flatlet wavelets on bounded intervals.

Keywords: wavelets, Daubechies, Haar, Legendre, B-spline, flatlet, introduction, examples.

Contents

1. Introduction

Wavelets have been very successful in many scientific and engineering fields. There are several reasons for their present success[2]. On the one hand, the concept of wavelets can be viewed as a synthesis of ideas which originated during the last twenty or thirty years in engineering, physics, and pure mathematics. As a consequence of these interdisciplinary origins, wavelets appeal to scientists and engineers of many different backgrounds. On the other hand, wavelets are a fairly simple mathematical tool with a great variety of possible applications. Already they have led to exciting applications in signal analysis and numerical analysis, and many other. This wide applicability also contributes to the interest they generate.

Unfortunately, the two most well known monographs on wavelets by Daubechies[2] and Chui[1] are still unaccessible to a beginner due to the level of mathematical sophistication needed to go through these monographs. Moreover both these monographs introduce wavelets from a signal processing perspective. The objective here is to introduce wavelets via scaling functions using the theory of multiresolution analysis. The primary focus is to describe many examples of scaling functions and their corresponding wavelets without proofs. The two-scale reconstruction and decomposition relations are described in order to provide the reader with a quick working knowledge of wavelets.

In this paper, some useful examples of wavelets on real line are presented. In order to make this monograph manageable, we have chosen certain wavelets, which we have found to be more useful in computer graphics and scientific visualization and reflects our bias. This monograph, although meant to be, self-contained and accessible to beginners, makes no claim to be comprehensive.

In chapter 2, we first describe wavelets on real line. Examples of Haar wavelet, a brief description of B-spline wavelets in general, linear, quadratic and cubic B-spline wavelets, and Daubechies wavelets are presented. In chapter 3, we describe multi-wavelets or wavelet-like basis functions on finite intervals for more useful examples, which include Legendre wavelets and flatlet wavelets. These wavelets are of more recent origin and very promising to various applications.

2. Wavelets on Real Line

This chapter first defines scaling functions, multiresolution analysis, wavelets and reconstruction and decomposition relations and algorithms. Then we present following examples of wavelets: Haar wavelets, B-spline wavelets and Daubechies wavelets. These related topics are discussed in great detail by Chui[1] and Daubechies[2].

2.1 Fundamentals

In this section, after presenting some elementary results from linear algebra, we review some basic definitions and useful properties of scaling functions, multiresolution analysis and wavelets. Then we discuss reconstruction and decomposition relations and algorithms.

2.1.1 Linear algebra

Definition A collection V of elements is a *vector space* over the real numbers R if:

1. for all $a, b \in R$, and for all $u, v \in V$, $au + bv \in V$. This provides closure under linear combination.
2. there exists a unique element $0 \in V$ such that for all $u \in V$, $0u = 0$ and for all $u \in V$, $0 + u = u$.

In general, if $u_1, u_2, \dots, u_n \in V$, then $a_1 u_1 + \cdots + a_n u_n \in V$.

Definition A subset V of a vector space U is said to be a *subspace* of U, written $V \subseteq U$, if V is a vector space.

Given a vector space U and a subspace V, there always exists a vector subspace W such that:

1. every element of U can be written as a linear combination of an element in V and an element in W, denoted by $U = V + W$.
2. representation in 1 is unique $\Longleftrightarrow V \cap W = \{0\}$. This is called the *direct-sum decomposition* of U and it is written as $U = V \dot{+} W$.

In general, W is not unique. In fact, given a vector space U and a subspace V, one can choose W so that $V \perp W$, i.e., every element in V is orthogonal to every element in W. It is called *orthogonal decomposition* of U and written as $U = V \oplus W$.

/* need to add some discussion on inner product */

2.1.2 Scaling functions

The basic constructions of wavelets using scaling functions is as follows: First, a *scaling function* ϕ is essentially a function $\phi(x)$ which can be written as a linear combination of $\phi(2x-k)$, which are $1/2$ scaled and $k/2$ translated version of $\phi(x)$. More precisely,

$$\phi(x) = \sum_{k=-\infty}^{\infty} p_k \phi(2x - k). \tag{2.1}$$

This is referred to as the *two-scale relation for the scaling functions* and the sequence $\{p_k\}$ is called the *two-scale sequence* of ϕ. We shall restrict our attention in this monograph to those scaling functions for which only finitely many p_k's are non-zero in the above relationship. These scaling functions have compact support.

Suppose we define a closed subspace V_0 be the linear span of the integer translates of ϕ, namely;

$$V_0 := \text{clos}_{L^2} \langle \phi(\cdot - k) : k \in Z \rangle \tag{2.2}$$

and consider

$$\phi_{j,k}(x) := \phi(2^j x - k), \quad j, k \in Z. \tag{2.3}$$

which are the scaled and translated version of $\phi(x)$. Now we define

$$V_j := \text{clos}_{L^2} \langle \phi_{j,k} : j, k \in Z \rangle, \tag{2.4}$$

Observe that because of the two-scale relationship (2.1), we have $V_0 \subset V_1$. In fact, the two-scale relation generates a nested sequence of subspaces

$$\begin{array}{c} \cdots V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \\ \longleftarrow \textit{Coarser} \quad \textit{Finer} \longrightarrow \end{array} \tag{2.5}$$

Furthermore, we would like that every "nice" function on real line R should be representable in terms of $\phi_{j,k}$ for some sufficiently large j or in other words,

$$\text{clos}_{L^2} \left(\bigcup_{j \in Z} V_j \right) = L^2(R) \tag{2.6}$$

These properties usually suffice to generate what is known as multiresolution analysis (MRA) defined below. Unfortunately there are pathological cases for which the conditions described above do not necessarily generate MRA.

Definition A function $\phi \in L^2(R)$ is said to generate a *multiresolution analysis (MRA)* if it generates a nested sequences of closed subspaces V_0 that satisfy

1. $\cdots V_{-1} \subset V_0 \subset V_1 \subset \cdots$ (containment)
2. $\text{clos}_{L^2} \left(\bigcup_{j \in Z} V_j \right) = L^2(R)$ (completeness)
3. $\bigcap_{j \in Z} V_j = \{0\}$ (unique)
4. $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}, \quad j \in Z$ (scaling property)

It is in general not true that any function ϕ satisfying the two-scale relation (2.1) and property (2.6) generate MRA with all the above desired properties. For pathological cases, see [1]. However in this monograph we shall restrict our attention to only those scaling functions which do generate MRA, that is they do satisfy those properties above.

Definition The family ϕ forms an *orthonormal basis* if

$$\langle \phi_{j,k}, \phi_{l,m} \rangle = \delta_{j,l} \cdot \delta_{k,m}, \quad j, k, l, m \in Z \tag{2.7}$$

2.1.3 Wavelets

Given a nested sequences of subspaces V_j, as in the containment property of MRA, there exists subspaces W_j, which are the orthogonal complements of V_j in V_{j+1}, that is,

$$V_{j+1} = V_j \oplus W_j, \quad j \in Z. \tag{2.8}$$

and

$$W_j \perp W_{j'} \quad \text{if } j \neq j' \tag{2.9}$$

Since the subspaces V_j are nested as (2.5), it follows that

$$V_J = V_j \oplus \bigoplus_{k=0}^{J-j-1} W_{j+k} \text{ for } j < J \tag{2.10}$$

where all these subspaces are orthogonal. By virtue of the properties 2 and 3 of MRA, this gives rise to an *orthogonal decomposition* of $L^2(R)$, namely

$$L^2(R) = \bigoplus_{j \in Z} W_j = \cdots \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus \cdots, \tag{2.11}$$

Furthermore, the W_j spaces inherit the scaling property 4 from the V_j;

$$f(x) \in W_j \Leftrightarrow f(2x) \in W_{j+1}, j \in Z \tag{2.12}$$

Given a scaling function ϕ in V_j, the basic tenet of MRA is that there exists another function ψ in W_0 called the *wavelet*, such that $\{\psi_{j,k} : k \in Z\}$ generates W_j, where

$$\psi_{j,k}(x) := \psi(2^j x - k), \quad j, k \in Z. \tag{2.13}$$

Since $V_1 = V_0 \oplus W_0$, $\psi(x) \in W_0$ can be written in terms of $\phi(2x - k)$, which forms a basis of V_1. Therefore, analogous to the two-scale relation for scaling function (2.1), that is, $\phi(x) \in V_0$ is written in term of $\phi(2x - k) \in V_1$, there exists the two-scale sequences $\{q_k\}$ such that

$$\psi(x) = \sum_{k=-\infty}^{\infty} q_k \phi(2x - k) \tag{2.14}$$

This relation (2.14) is called the *two-scale relation for wavelets.*

2.1.4 Reconstruction and decomposition relations

Both of the two-scale relations (2.1) and (2.14) together are called the *reconstruction relations.* On the other hand, since both $\phi(2x)$ and $\phi(2x-1)$ are in V_1 and $V_1 = V_0 \oplus W_0$, there are four sequences which we denote by $\{a_{-2k}\}$, $\{b_{-2k}\}$, $\{a_{1-2k}\}$, and $\{b_{1-2k}\}$, $k \in Z$, such that

$$\phi(2x) = \sum_k [a_{-2k}\phi(x-k) + b_{-2k}\psi(x-k)] \tag{2.15}$$

$$\phi(2x-1)=\sum_k\left[a_{1-2k}\phi(x-k)+b_{1-2k}\psi(x-k)\right]. \tag{2.16}$$

The two formulas (2.15) and (2.16) can be combined into a single formula:

$$\phi(2x-l)=\sum_k\left[a_{l-2k}\phi(x-k)+b_{l-2k}\psi(x-k)\right],\ l\in Z \tag{2.17}$$

which is called the *decomposition relation* for ϕ and ψ.

The two pairs of sequences $(\{p_k\},\{q_k\})$ and $(\{a_k\},\{b_k\})$ are used to formulate reconstruction and decomposition algorithms described below. $\{p_k\}$ and $\{q_k\}$ are called reconstruction sequences, while $\{a_k\}$ and $\{b_k\}$ are called decomposition sequences.

/* need to describe how to combine these two relations into a single formula and some comment on notation */

2.1.5 Reconstruction and decomposition algorithms

Let us consider the general structure of multiresolution analysis and wavelets as discussed in (2.10), where $\{V_j\}$ is generated by translates of some scaling function $\phi_j\in L^2(R)$ and $\{W_j\}$ is generated by translates of some wavelet $\psi_j\in L^2(R)$. In this case, by the property 2 of MRA, every function f in $L^2(R)$ can be approximated as closely as is desired by an $f_N\in V_N$, for some $N\in Z$. Since $V_j=V_{j-1}\dot{+}W_{j-1}$ for any $j\in Z$, f_N has a unique decomposition:

$$f_N=f_{N-1}+g_{N-1} \tag{2.18}$$

where $f_{N-1}\in V_{N-1}$ and $g_{N-1}\in W_{N-1}$. By repeating this process, we have

$$f_N=g_{N-1}+g_{N-2}+\cdots+g_{N-M}+f_{N-M} \tag{2.19}$$

where $f_j\in V_j$ and $g_j\in W_j$ for any j. The "decomposition" in (2.19) is called "wavelet decomposition". In the following, we will discuss an algorithmic approach for expressing f_N as a direct sum of its components $g_{N-1},\ldots,g_{N-M}$, and f_{N-M}, and recovering f_N from these components.

To describe decomposition and reconstruction algorithms, let us first note that both $f_j\in V_j$ and $g_j\in W_j$ have unique series representations:

$$\left\{\begin{array}{l} f_j(x)=\sum_k c_k^j\phi(2^jx-k),\\ \quad\text{with } c^j=\{c_k^j\}\in l^2; \end{array}\right. \tag{2.20}$$

and

$$\left\{\begin{array}{l} g_j(x)=\sum_k d_k^j\psi(2^jx-k),\\ \quad\text{with } d^j=\{d_k^j\}\in l^2. \end{array}\right. \tag{2.21}$$

In the following decomposition and reconstruction algorithms, the functions f_j and g_j are represented by the sequences c^j and d^j as defined in (2.20) and (2.21).

(i) Decomposition algorithm

By applying (2.17), (2.20) and (2.21), we have:

$$\begin{cases} c_k^{j-1} = \sum_l a_{l-2k} c_l^j; \\ d_k^{j-1} = \sum_l b_{l-2k} c_l^j. \end{cases} \tag{2.22}$$

$$\begin{array}{ccccccccc} & & d^{N-1} & & d^{N-2} & & & & d^{N-M} \\ & \nearrow & & \nearrow & & \nearrow & & \nearrow & \\ c^N & \rightarrow & c^{N-1} & \rightarrow & c^{N-2} & \rightarrow & \cdots & \rightarrow & c^{N-M} \end{array}$$

Figure 2.1: Wavelet decomposition

Observe that both c^{j-1} and d^{j-1} are obtained from c^j by "moving average" schemes, using the decomposition sequences as "weights", with the exception that these moving averages are sampled only at the even integers. This is called downsampling. Therefore, each of the arrows in Figure 2.1 indicates a moving average followed by a downsampling at the even indices.

/* need more explanation on moving average, weights and downsampling */

(ii) Reconstruction algorithm

By applying (2.1), (2.14), (2.20) and (2.21), we have:

$$c_k^j = \sum_l \left[p_{k-2l} c_l^{j-1} + q_{k-2l} d_l^{j-1} \right] \tag{2.23}$$

$$\begin{array}{ccccccccc} d^{N-M} & & d^{N-M+1} & & & & d^{N-1} & & \\ & \searrow & & \searrow & & \searrow & & \searrow & \\ c^{N-M} & \rightarrow & c^{N-M+1} & \rightarrow & \cdots & \rightarrow & c^{N-1} & \rightarrow & c^N \end{array}$$

Figure 2.2: Wavelet reconstruction

Here, c^j is obtained from c^{j-1} and d^{j-1} by two moving averages, using the reconstruction sequences as "weights", with the exception that an upsampling is required before the moving averages are performed. More precisely, the samples c_l^{j-1} and d_l^{j-1} are used at the even indices $m = 2l$ and zeros are used at the odd indices $m = 2l + 1$, when the (discrete) convolutions are taken with respect to $\{p_m\}$ and $\{q_m\}$.

/* need to add informal description of usefulness of wavelets */

2.2 Haar Wavelets

2.2.1 Scaling functions

Our first example is Haar scaling function and Haar wavelet which are very simple but useful to illustrate many nice properties of scaling functions and wavelets as well as to apply for practical use. Haar scaling function is defined by

$$\phi(x) = \begin{cases} 1 & \text{for } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.24}$$

and is shown in Figure 2.3.

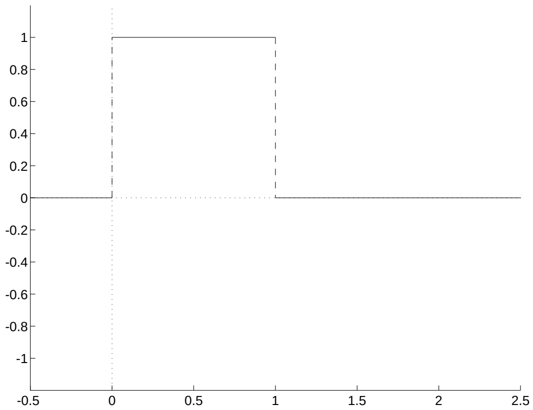

Figure 2.3: Haar Scaling Function $\phi(x)$

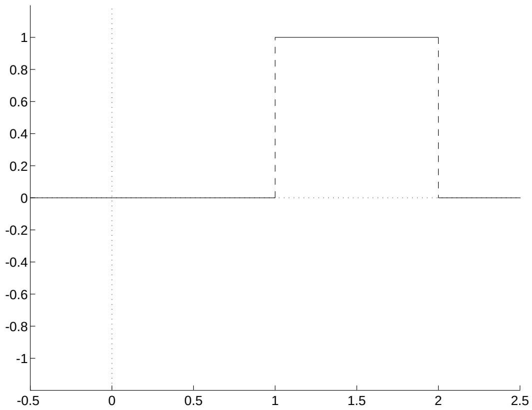

Figure 2.4: Haar Scaling Function $\phi(x-1)$

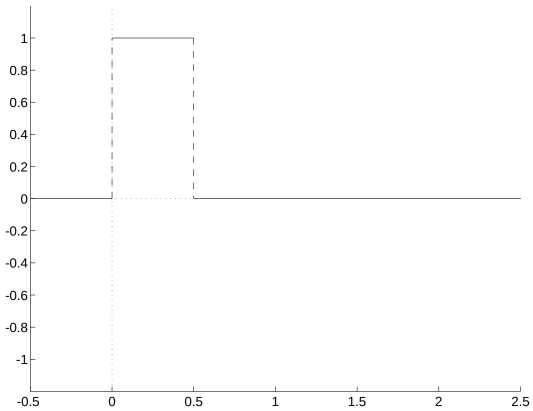

Figure 2.5: Haar Scaling Function $\phi(2x)$

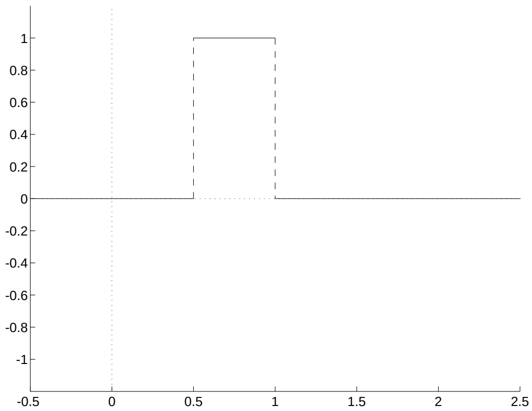

Figure 2.6: Haar Scaling Function $\phi(2x-1)$

Some examples of its translated and scaled versions are shown in Figures 2.4–2.6. The subspace V_0 is spanned by scaling functions $\phi(x-k)$ which are integer translates of piecewise constant functions on unit interval . The subspace V_1 is spanned by $\phi(2x-k)$ which are $k/2$

translates of piece-wise constant functions on 1/2 interval. In general, V_j is spanned by $k/2^j$ translates of piece-wise constant functions on $1/2^j$ interval. The two-scale relation (2.1) for Haar Scaling Function is

$$\begin{aligned} \phi(x) &= \sum_k p_k \phi(2x-k) \\ &= \phi(2x) + \phi(2x-1) \end{aligned} \tag{2.25}$$

Therefore, the two-scale sequences $\{p_k\}$ for Haar scaling function have non-zero values $p_0 = p_1 = 1$ and 0's for other p_j's. The two-scale relation (2.25) can be viewed graphically from Figures 2.3, 2.5 and 2.6, that is, the sum of two functions in Figures 2.5 and 2.6 yields the function in Figure 2.3.

2.2.2 Wavelets

The Haar wavelet $\psi(x)$ corresponding to the Haar scaling function $\phi(x)$ is given by

$$\phi(x) = \begin{cases} 1 & \text{for } 0 \le x < \frac{1}{2} \\ -1 & \text{for } \frac{1}{2} \le x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.26}$$

and is shown in Figure 2.7. We can easily construct the two-scale relation for Haar wavelet as

$$\psi(x) = \phi(2x) - \phi(2x-1) \tag{2.27}$$

where the two-scale sequences $\{q_k\}$ in (2.14) are 0's except for $q_0 = 1$ and $q_1 = -1$. This two-scale relation can be viewed graphically from Figures 2.5, 2.6 and 2.7.

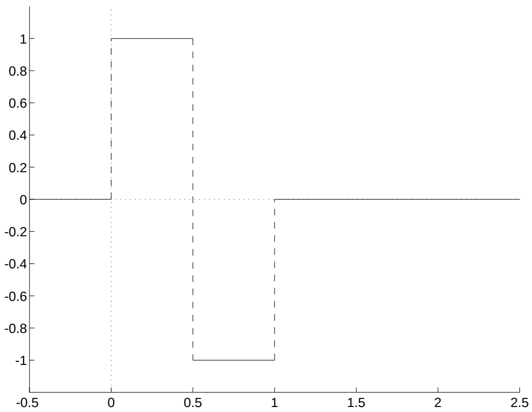

Figure 2.7: Haar Wavelet $\psi(x)$

2.2.3 Decomposition relations

The two two-scale relations (2.25) for Haar scaling functions express $\phi(x)$ in terms of $\phi(2x)$ and $\phi(2x-1)$. While, the two two-scale relations (2.27) for Haar wavelets express $\psi(x)$ also in terms of $\phi(2x)$ and $\phi(2x-1)$. Both of the two-scale relations together are called the reconstruction relations.

$$\begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \phi(2x) \\ \phi(2x-1) \end{bmatrix} \tag{2.28}$$

Whereas, the decomposition relations for Haar wavelets are the expression of $\phi(2x)$ and $\phi(2x-1)$ in terms of $\phi(x)$ and $\psi(x)$ which are the inverse of the reconstruction relations (2.28). Since the supports of both the scaling function and the wavelets are within the same interval, the decomposition relations are easily derived by just inverting the reconstruction relation as follows:

$$\begin{bmatrix} \phi(2x) \\ \phi(2x-1) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix} \tag{2.29}$$

Note that the non-zero elements of decomposition sequences $\{a_k\}$ and $\{b_k\}$ in (2.15) and (2.16) are $a_0 = \frac{1}{2}$, $b_0 = \frac{1}{2}$, $a_1 = \frac{1}{2}$ and $b_1 = -\frac{1}{2}$.

2.2.4 Properties

Let us observe some nice properties of Haar scaling functions and wavelets[5].

1. Orthogonality
   (a) scaling functions
   $$\langle \phi(x-j), \phi(x-k) \rangle = \delta_{j,k}, \quad j,k \in Z \tag{2.30}$$
   (b) wavelets
   $$\langle \psi(x-j), \psi(x-k) \rangle = \delta_{j,k}, \quad j,k \in Z \tag{2.31}$$
   (c) scaling function and wavelet
   $$\langle \psi(x-j), \psi(x-k) \rangle = 0, \quad j,k \in Z \tag{2.32}$$
2. Compact support
   $$\text{supp } \phi(x) = \text{supp } \psi(x) = [0,1] \tag{2.33}$$
3. Symmetry/Anti-symmetry
   (a) Haar scaling function is symmetric
   (b) Haar wavelet is anti-symmetric
4. Analytic expressions are available
5. Have vanishing moments
   $$\int_{-\infty}^{\infty} \psi_i(x)dx = 0 \tag{2.34}$$
6. Haar wavelets $\psi_{j,k}(x) := \psi(2^j x - k)$ form an orthogonal basis, i.e., $\langle \psi_{j,k}, \psi_{l,m} \rangle = \delta_{j,l} \cdot \delta_{k,m}$, $j,k,l,m \in Z$

2.3 General Order B-Spline Wavelets

The second set of examples that we discuss are B-spline wavelets. For these examples, scaling functions on real line are the m-th order (degree $m-1$) B-splines $N_m(x)$.

In this subsection, we first introduce general order B-splines $N_m(x)$ with a brief summary of some nice properties. Then the corresponding B-spline wavelets $\psi_m(x)$ are investigated. Some concrete examples of B-spline for small order m are illustrated in the following few sections.

2.3.1 Scaling functions

There are several ways to define B-splines. Typically, the m-th order B-splines N_m is defined recursively by convolution:

$$N_1(x) = \begin{cases} 1 & \text{for } 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.35}$$

$$\begin{aligned} N_m(x) &= \int_{-\infty}^{\infty} N_{m-1} N_1(t)\,dt \\ &= \int_0^1 N_{m-1}(x-t)\,dt \end{aligned} \tag{2.36}$$

Note that the 1st order B-spline $N_1(x)$ is the Haar scaling function. Reader may convince yourself of any equation discussed for general B-spline in this section by letting $m = 1$ and verifying if it results in the Haar case.

The two-scale relation (2.1) for B-spline scaling functions of general order m is written as

$$N_m(x) = \sum_{k=0}^{m} p_k N_m(2x-k) \tag{2.37}$$

where the two-scale sequence $\{p_k\}$ for B-spline scaling functions are given by:

$$p_k = 2^{-m+1} \begin{pmatrix} m \\ k \end{pmatrix}, \quad \text{for } 0 \le k < m \tag{2.38}$$

This is a remarkable property of B-splines, not often emphasized in computer graphics. For example, there is no mention of this property in the book, "Computer Science"[6]. Equation (2.38) for B-spline scaling functions is derived in Appendix A.

2.3.2 Properties of the scaling functions

B-splines have many nice properties, some of which are summarized below.[5]

1. Compact support

$$\text{supp } N_m = [0, m] \tag{2.39}$$

2. $N_m(x) > 0$ for $0 < x < m$
3. Partition of unity

$$\sum_{k=-\infty}^{\infty} N_m(x-k) = 1 \tag{2.40}$$

4. Symmetry

$$N_m(\frac{m}{2}+x) = N_m(\frac{m}{2}-x) \qquad x \in R \tag{2.41}$$

5. Recursive Property

$$N_m(x) = \frac{x}{m-1}N_{m-1}(x) + \frac{m-x}{m-1}N_{m-1}(x-1) \tag{2.42}$$

2.3.3 Wavelets

Since B-splines N_m are scaling functions, it follows from the general theory presented in Section 2.1 that the B-spline subspaces V_j form a nested sequence. Moreover, the complementary subspaces W_j in (2.8) are mutually orthogonal and any function in $L^2(R)$ can be represented as a linear sum of functions in W_j as described by (2.11).

In general, wavelets ψ_m in subspace W_j are not unique. For the moment, we focus on the unique wavelet $\psi_m \in W_0$ that has the smallest support. These will be referred to as B-spline wavelets.

The two-scale relation for B-spline wavelets for general order m is given by

$$\psi_m(x) = \sum_{k=0}^{3m-2} q_k N_m(2x-k) \tag{2.43}$$

where

$$q_k = (-1)^k 2^{1-m} \sum_{l=0}^{m} \left(\begin{array}{c} m \\ l \end{array} \right) N_{2m}(k+1-l) \tag{2.44}$$

For the derivation of the two-scale relation, refer to [1] section 6.2.

2.3.4 Properties of the wavelets

The B-spline wavelets have several nice properties[5][1], some of which we mention below:

1. There exists analytic expressions for B-spline wavelets as in (2.43) and (2.44) above.
2. Compact support:

$$\text{support of } \psi_m = [0, 2m-1] \quad \text{for } m = 1, 2, \ldots \tag{2.45}$$

which follows from (2.43) above.

3. They are symmetric for even m and anti-symmetric for odd m around center, namely:

$$\left\{ \begin{array}{ll} \psi_m(x) = \ \ \psi_m(2m-1-x), & \text{for even } m \\ \psi_m(x) = -\psi_m(2m-1-x), & \text{for odd } m \end{array} \right. \tag{2.46}$$

4. q_k are finite series; only $3m-1$ q_k's are non-zero as is evident by (2.43).
5. Exhibit complete oscillation property.

6. Have vanishing moments

$$\int_{-\infty}^{\infty} \psi_m(x) x^i dx = 0, \quad i = 0, 1, \ldots, m-1 \tag{2.47}$$

7. B-spline wavelets ψ_m's are orthogonal at different scale, that is

$$\langle \psi_m(2^{j_1}x - k_1), \psi_m(2^{j_2}x - k_2) \rangle = 0 \quad \text{if } j_1 \neq j_2. \tag{2.48}$$

However, different translates of ψ_m's are not orthogonal at the same scale, that is

$$\langle \psi_m(2^{j}x - k_1), \psi_m(2^{j}x - k_2) \rangle \neq 0 \quad \text{for } m \neq 1. \tag{2.49}$$

This property is referred to as semi-orthogonality.

2.3.5 Decomposition relations

The decomposition relation for m-th order B-spline is

$$\phi(2x - l) = \sum_k \left[a_{l-2k}\phi(x-k) + b_{l-2k}\psi(x-k) \right], \; l \in Z \tag{2.50}$$

where the decomposition sequences[3] $\{a_k\}$ and $\{b_k\}$ in (2.50) are given in the following forms:

$$a_k = \frac{1}{2}(-1)^{k+1} \sum_{l \in Z} q_{-k+2m-1-2l}\, c_{l,2m} \tag{2.51}$$

$$b_k = -\frac{1}{2}(-1)^{k+1} \sum_{l \in Z} p_{-k+2m-1-2l}\, c_{l,2m} \tag{2.52}$$

and $\{p_k\}$ and $\{q_k\}$ are two-scale sequences given by (2.38) and (2.44), and $\{c_{k,m}\}$ are coefficients in representing *Fundamental Cardinal Spline*[3] function $L_m(x)$ in terms of m-th order B-splines, $N_m(x)$, that is,

$$L_m(x) = \sum_{k=-\infty}^{\infty} c_{k,m} N_m(x + \frac{m}{2} - k) \tag{2.53}$$

Some discussions on Fundamental Cardinal Spline are found in Appendix B.

The coefficients $\{c_{k,m}\}$ can be determined by using the following relation:

$$\sum_k c_{k,m} z^k = \frac{1}{\sum_k N_m(k + \frac{m}{2}) z^k} \tag{2.54}$$

Typically, $N_m(k + \frac{m}{2})$ in (2.54) are easily computed using the properties of B-spline (2.42) and only finitely many of them are non-zero, as we will see in Table 2.1. The inversion of the series can then be used to compute the coefficients $\{c_{k,m}\}$. This process will be illustrated later in Linear and Quadratic B-spline sections.

Note that the decomposition sequences $\{a_k\}$, $\{b_k\}$ in (2.50) for B-splines are infinite. However the decay of the sequences are exponential so that many of them can be truncated with low errors.

2.4 Linear B-Spline Wavelets

We will present some examples of B-spline scaling functions and B-spline wavelets for some small values of m. As is already mentioned, the simplest example ($m = 1$) of B-spline scaling function and wavelet is nothing but the Haar scaling function and wavelet. Here, we start with $m = 2$ which is called a Linear B-spline.

2.4.1 Scaling functions

Linear B-Spline $N_2(x)$ is derived from the recurrence (2.35) and (2.36) as the case of $m = 2$ for general B-splines as follows and is shown in Figure 2.8.

$$N_2(x) = \phi(x) = \begin{cases} x & \text{for } 0 \le x < 1 \\ 2 - x & \text{for } 1 \le x < 2 \\ 0 & \text{otherwise} \end{cases} \tag{2.55}$$

Then the functions $\phi(2x - k)$ in V_1 subspace are expressed explicitly as

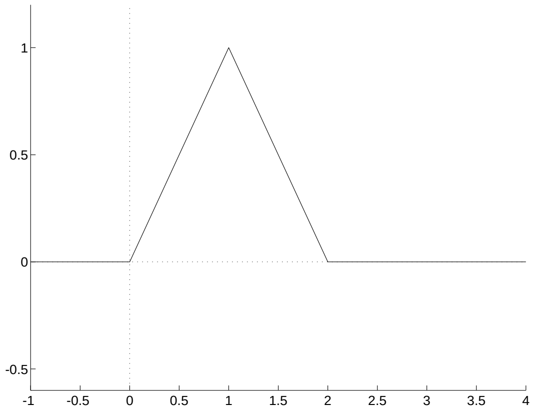

Figure 2.8: Linear B-Spline $N_2(x)$

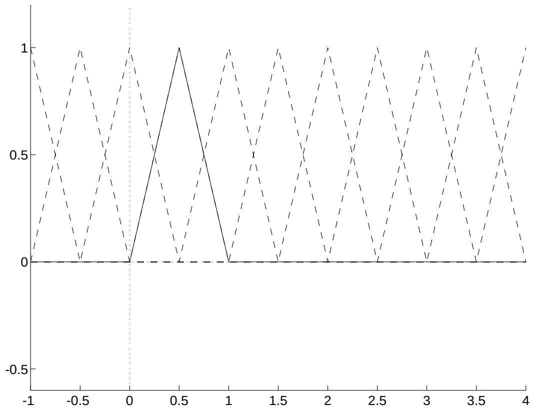

Figure 2.9: Linear B-Spline $N_2(2x - k)$

$$N_2(2x-k) = \phi(2x-k) = \begin{cases} 2x - k & \text{for } \frac{k}{2} \le x < \frac{k}{2} + \frac{1}{2} \\ 2 + k - 2x & \text{for } \frac{k}{2} + \frac{1}{2} \le x < \frac{k}{2} + 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.56}$$

Figure 2.9 shows the linear B-spline $N_2(2x - k)$ in V_1 subspace for $k \in Z$. The graph for $k = 0$ is drawn with solid lines.

Since the support of $\phi(x)$ is $[0, 2]$, its two-scale relation is in the form

$$\phi(x) = \sum_{k=0}^{2} p_k \phi(2x - k) \tag{2.57}$$

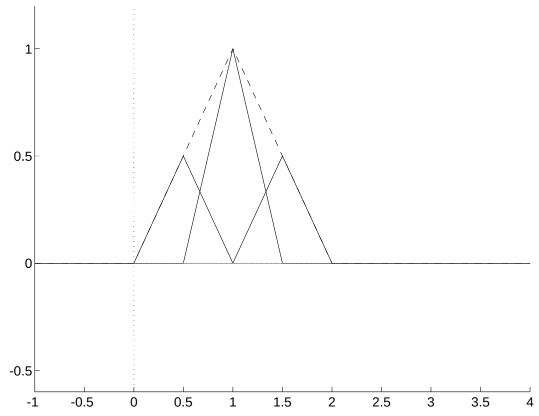

Figure 2.10: Two-scale relation for N_2

By substituting the expressions (2.55) and (2.56) for each 1/2 interval between $[0, 2]$ into (2.57), the coefficients $\{p_k\}$ are obtained and the two scale relation for Linear B-Spline is given by

$$\phi(x) = \frac{1}{2}\phi(2x) + \phi(2x-1) + \frac{1}{2}\phi(2x-2). \tag{2.58}$$

Figure 2.10 is helpful to get a geometrical understanding of the two-scale relation (2.58) where the scaling function $\phi(x)$ in the left hand side of (2.58) is drawn with dashed lines and three scaling functions $\phi(2x-k)$ for $k = 0, 1$ and 2 in the right hand side are drawn with solid lines with the magnitudes of the two-scale sequences.

2.4.2 Wavelets

Now, let us study Linear B-spline wavelets denoted by $\psi_2(x)$. By letting $m = 2$ in the two-scale relations for B-spline wavelets (2.43) and (2.44), we have

$$\psi_2(x) = \sum_{k=0}^{4} q_k N_2(2x-k) \tag{2.59}$$

where

$$\begin{aligned} q_k &= (-1)^k 2^{-1} \sum_{l=0}^{2} \begin{pmatrix} 2 \\ l \end{pmatrix} N_4(k+1-l) \\ &= \left(-\tfrac{1}{2}\right)^k \{N_4(k+1) + 2N_4(k) + N_4(k-1)\}. \end{aligned} \tag{2.60}$$

The term $N_4(k)$ in (2.60) is cubic B-spline which we have not explored yet, but the recursion relation (2.42) shown again here with the base case:

$$N_2(k) = \delta_{k,1}, \quad \text{for } k \in Z \tag{2.61}$$

$$N_m(k) = \frac{k}{m-1}N_{m-1}(k) + \frac{m-k}{m-1}N_{m-1}(k-1) \tag{2.62}$$

$N_m(k)$		k					
		0	1	2	3	4	5
m	2	0	1	0	$\cdots$		
	3	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\cdots$	
	4	0	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$	0	$\cdots$
	5	0	$\frac{1}{24}$	$\frac{11}{24}$	$\frac{11}{24}$	$\frac{1}{24}$	0
	6	0	$\frac{1}{120}$	$\frac{26}{120}$	$\frac{66}{120}$	$\frac{26}{120}$	$\frac{1}{120}$

Table 2.1: Non-zero $N_m(k)$ values for $m = 2, \ldots, 6$

is useful to compute $N_m(k)$ at some integer values. Non-zero values of $N_m(k)$ $k \in Z$ for some small m are summarized in Table 2.1[1]. Then the two-scale sequence $\{q_k\}$ for $\psi_2(x)$ is computed as follows:

$$\begin{aligned}
q_0 &= \ (1/2)\{N_4(1) + 2N_4(0) + N_4(-1)\} = (1/2)(1/6) = 1/12 \\
q_1 &= (-1/2)\{N_4(2) + 2N_4(1) + N_4(0)\} = (-1/2)(1) = -1/2 \\
q_2 &= \ (1/2)\{N_4(3) + 2N_4(2) + N_4(1)\} = (1/2)(5/3) = 5/6 \\
q_3 &= (-1/2)\{N_4(4) + 2N_4(3) + N_4(2)\} = (-1/2)(1) = -1/2 \\
q_4 &= \ (1/2)\{N_4(5) + 2N_4(4) + N_4(3)\} = (1/2)(1/6) = 1/12
\end{aligned}$$

Thus the two-scale relation for Linear B-spline wavelets is:

$$\psi_2(x) = \frac{1}{12}N_2(2x) - \frac{1}{2}N_2(2x-1) + \frac{5}{6}N_2(2x-2) - \frac{1}{2}N_2(2x-3) + \frac{1}{12}N_2(2x-4) \qquad (2.63)$$

where $N_2(2x-k)$ is given by (2.56). The Linear B-spline wavelet is shown in Figure 2.11 and its explicit expression is obtained as follows:

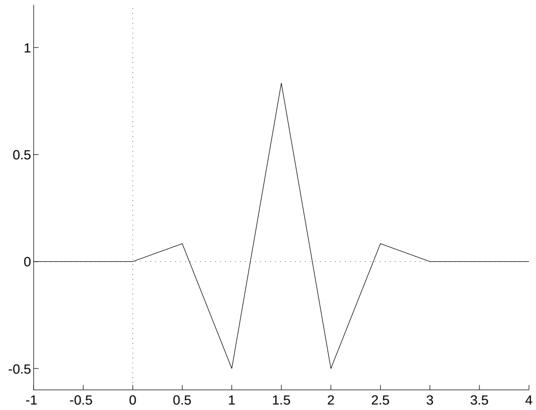

Figure 2.11: Linear B-spline wavelet $\psi_2(x)$

$$\psi_2(x) = \begin{cases} \frac{x}{6} & \text{for } 0 \le x < \frac{1}{2} \\ \frac{1}{6}(-7x+4) & \text{for } \frac{1}{2} \le x < 1 \\ \frac{1}{6}(16x-19) & \text{for } 1 \le x < \frac{3}{2} \\ \frac{1}{6}(-16x+29) & \text{for } \frac{3}{2} \le x < 2 \\ \frac{1}{6}(7x-17) & \text{for } 2 \le x < \frac{5}{2} \\ \frac{1}{6}(-x+3) & \text{for } \frac{5}{2} \le x < 3 \\ 0 & \text{otherwise} \end{cases} \tag{2.64}$$

2.4.3 Decomposition relations

The objective here is to establish the decomposition relation (2.17) for Linear B-spline. The decomposition sequences $\{a_k\}$ and $\{b_k\}$ given in (2.51) and (2.52) are written for $m = 2$ as

$$a_k = \frac{1}{2}(-1)^{k+1} \sum_{l \in Z} q_{-k+3-2l} c_{l,4} \tag{2.65}$$

$$b_k = -\frac{1}{2}(-1)^{k+1} \sum_{l \in Z} p_{-k+3-2l} c_{l,4} \tag{2.66}$$

We already know the two-scale sequences $\{p_k\}$ and $\{q_k\}$ for Linear B-spline ($m = 2$). Noting that only three $\{p_k\}$ and five $\{q_k\}$ are non-zero, i.e., $\{p_0, p_1, p_2\} = \{\frac{1}{2}, 1, \frac{1}{2}\}$ and $\{q_0, q_1, q_2, q_3, q_4\} = \{\frac{1}{12}, -\frac{1}{2}, \frac{5}{6}, -\frac{1}{2}, \frac{1}{12}\}$, the equations (2.51) and (2.52) are written more explicitly:

$$\begin{aligned} a_k &= \begin{cases} -\frac{1}{2}(q_1 c_{\frac{-k+2}{2},4} + q_3 c_{\frac{-k}{2},4}) & \text{for } k: \text{even} \\ \frac{1}{2}(q_0 c_{\frac{-k+3}{2},4} + q_2 c_{\frac{-k+1}{2},4} + q_4 c_{\frac{-k-1}{2},4}) & \text{for } k: \text{odd} \end{cases} \\ &= \begin{cases} \frac{1}{4}(c_{\frac{-k+2}{2},4} + c_{\frac{-k}{2},4}) & \text{for } k: \text{even} \\ \frac{1}{24}(c_{\frac{-k+3}{2},4} + 10 c_{\frac{-k+1}{2},4} + c_{\frac{-k-1}{2},4}) & \text{for } k: \text{odd} \end{cases} \end{aligned} \tag{2.67}$$

$$\begin{aligned} b_k &= \begin{cases} \frac{1}{2} p_1 c_{\frac{-k+2}{2},4} & \text{for } k: \text{even} \\ -\frac{1}{2}(p_0 c_{\frac{-k+3}{2},4} + p_2 c_{\frac{-k+1}{2},4}) & \text{for } k: \text{odd} \end{cases} \\ &= \begin{cases} \frac{1}{2} c_{\frac{-k+2}{2},4} & \text{for } k: \text{even} \\ -\frac{1}{4}(c_{\frac{-k+3}{2},4} + c_{\frac{-k+1}{2},4}) & \text{for } k: \text{odd} \end{cases} \end{aligned} \tag{2.68}$$

The only unknown variables $\{c_{l,4}\}$ are the coefficients for Cubic Fundamental Cardinal Spline $L_4(x)$. Let us determine $\{c_{l,4}\}$ using (2.54) for $m = 4$, that is,

$$\sum_k c_{k,4} z^k = \frac{1}{\sum_k N_4(k+2) z^k} \tag{2.69}$$

The non-zero values of $N_4(k)$, $k \in Z$ are provided in Table 2.1:

$$N_4(1) = \frac{1}{6}, \quad N_4(2) = \frac{2}{3}, \quad N_4(3) = \frac{1}{6}$$

Then the right hand side of (2.69) is computed as follows:

$$
\begin{aligned}
\sum_k c_{k,4} z^k &= \frac{1}{\frac{1}{6}z^{-1} + \frac{2}{3} + \frac{1}{6}z} \\
&= \frac{6z}{z^2 + 4z + 1} \\
&= \frac{6z}{(z-\alpha)(z-\beta)} \quad \text{where } \alpha = -2 + \sqrt{3}, \quad \beta = -2 - \sqrt{3} \\
&= \frac{6}{\alpha - \beta}\left(\frac{\alpha}{z-\alpha} - \frac{\beta}{z-\beta}\right) \\
&= \sqrt{3}\left\{\frac{(\frac{\alpha}{z})}{1-(\frac{\alpha}{z})} + \frac{1}{1-(\frac{z}{\beta})}\right\} \\
&= \sqrt{3}\left\{\sum_{n=1}^{\infty}(\frac{\alpha}{z})^n + \sum_{n=0}^{\infty}(\frac{z}{\beta})^n\right\}
\end{aligned}
$$

Since $\alpha\beta = 1$

$$
\begin{aligned}
&= \sqrt{3}\left\{\sum_{n=-\infty}^{-1}(\frac{z}{\alpha})^n + \sum_{n=0}^{\infty}(\alpha z)^n\right\} \\
&= \sqrt{3}\sum_{n=-\infty}^{\infty}\alpha^{|n|}z^n
\end{aligned}
$$

Therefore, $\{c_{l,4}\}$ is given by:

$$
c_{l,4} = \sqrt{3}\alpha^{|l|} = (-1)^l\sqrt{3}(2-\sqrt{3})^{|l|} \tag{2.70}
$$

Note that $c_{l,4}$ are non-zero for all l though it decays exponentially for large l. Therefore unlike the Haar wavelet, the decomposition relation for Linear B-spline consists of infinite terms of $\phi(x-k)$ and $\psi(x-k)$ in the right hand side of (2.17). Some $\{a_k\}$ and $\{b_k\}$ whose absolute values are greater than 0.02 are shown in Table 2.2.[1]

$\lvert k\rvert$	a_{k+1}	b_{k+2}
0	0.683012701829	0.866025403784
1	0.316987298108	− 0.316987298108
2	− 0.116025403784	− 0.232050807569
3	− 0.084936490539	0.084936490539
4	0.031088913246	0.062177826491
5	0.022758664048	− 0.022758664048
⋮	⋮	⋮

Table 2.2: Linear B-spline decomposition sequences

2.5 Quadratic B-Spline Wavelets

2.5.1 Scaling functions

Let us go on to the next example $N_3(x)$ given by

$$N_3(x) = \phi(x) = \begin{cases} \frac{1}{2}x^2 & \text{for } 0 \leq x < 1 \\ \frac{3}{4} - (x - \frac{3}{2})^2 & \text{for } 1 \leq x < 2 \\ \frac{1}{2}(x-3)^2 & \text{for } 2 \leq x < 3 \\ 0 & \text{otherwise} \end{cases} \tag{2.71}$$

and shown in Figure 2.12. We proceed in a way similar to the previous example. Functions

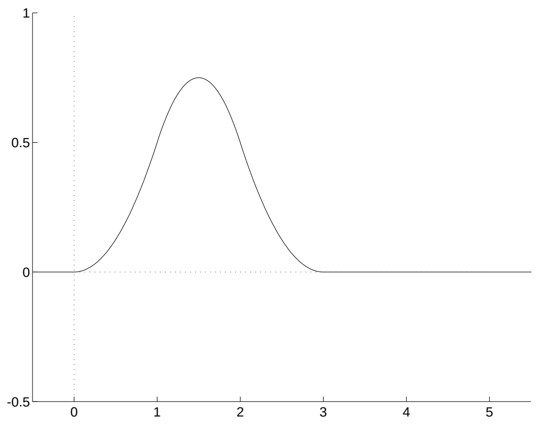

Figure 2.12: Quadratic B-Spline $N_3(x)$

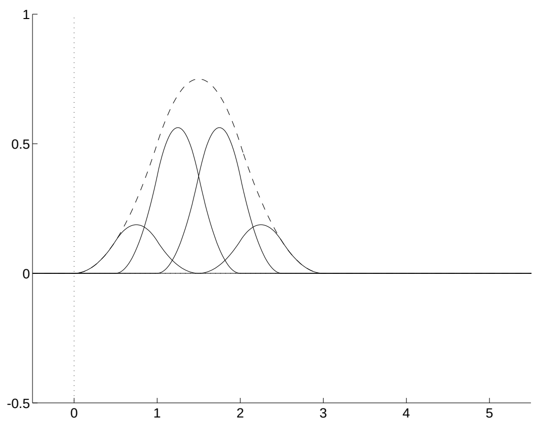

Figure 2.13: Two-scale relation for N_3

$\phi(2x-k)$ in V_1 space are expressed as

$$\phi(2x-k) = \begin{cases} \frac{1}{2}(2x-k)^2 & \text{for } \frac{k}{2} \leq x < \frac{k}{2} + \frac{1}{2} \\ \frac{3}{4} - (2x - k - \frac{3}{2})^2 & \text{for } \frac{k}{2} + \frac{1}{2} \leq x < \frac{k}{2} + 1 \\ \frac{1}{2}(2x-k-3)^2 & \text{for } \frac{k}{2} + 1 \leq x < \frac{k}{2} + \frac{3}{2} \\ 0 & \text{otherwise} \end{cases} \tag{2.72}$$

The two-scale relation for quadratic B-Spline $N_3(x)$ is

$$\phi(x) = \frac{1}{4}\phi(2x) + \frac{3}{4}\phi(2x-1) + \frac{3}{4}\phi(2x-2) + \frac{1}{4}\phi(2x-3) \tag{2.73}$$

Four of the scaling functions $\phi(2x-k)$ for $k = 0, 1, 2$ and 3 appeared in the right hand side of (2.73) with the magnitudes of the two-scale sequence are drawn with solid curves and $\phi(x)$ in the left hand side is drawn with dashed line in Figure 2.13.

2.5.2 Wavelets

To obtain the two-scale relations for quadratic B-spline wavelets, we can follow the same procedure as linear B-spline case illustrated in the previous section.

The two-scale sequence $\{q_k\}$ with non-zero values are computed using (2.44) with $m = 3$, using non-zero values of $N_6(k)$ provided in Table 2.1. The resulting quadratic B-spline wavelet is is shown in Figure 2.14 and the two-scale relation is given in (2.74).

$$\psi_3(x) = \frac{1}{480}N_3(2x) - \frac{29}{480}N_3(2x-1) + \frac{147}{480}N_3(2x-2) - \frac{303}{480}N_3(2x-3)$$
$$+ \frac{303}{480}N_3(2x-4) - \frac{147}{480}N_3(2x-5) + \frac{29}{480}N_3(2x-6) - \frac{1}{480}N_3(2x-7) \tag{2.74}$$

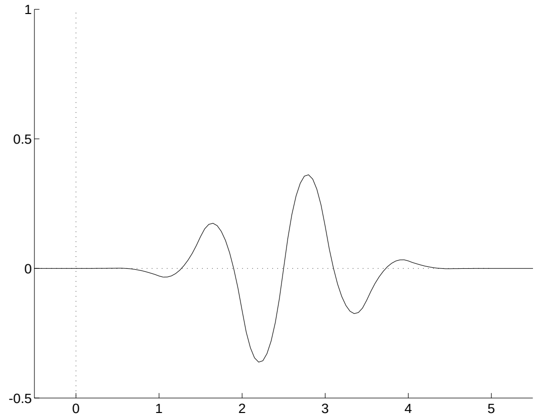

Figure 2.14: Quadratic B-spline wavelet $\psi_3(x)$

2.5.3 Decomposition relations

The decomposition relations for Quadratic B-spline can be obtained in a similar way to Linear Quadratic B-spline.

The decomposition sequences $\{a_k\}$ and $\{b_k\}$ in (2.51) for $m = 3$ are

$$a_k = \frac{1}{2}(-1)^{k+1} \sum_{l \in Z} q_{-k+5-2l} c_{l,6} \tag{2.75}$$

$$b_k = -\frac{1}{2}(-1)^{k+1} \sum_{l \in Z} p_{-k+5-2l} c_{l,6} \tag{2.76}$$

where the non-zero values of the two-scale sequences $\{p_k\}$ and $\{q_k\}$ for $m = 3$ are available as the coefficients of (2.73) and (2.74), respectively, that is, $\{p_0, p_1, p_2, p_3\} = \{\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4}\}$ and $\{q_0, q_1, q_2, q_3, q_4, q_5, q_6, q_7\} = \{\frac{1}{480}, -\frac{29}{480}, \frac{147}{480}, -\frac{303}{480}, \frac{303}{480}, -\frac{147}{480}, \frac{29}{480}, -\frac{1}{480}\}$. Therefore more explicit formulas for (2.75) and (2.76) are as follows:

$$a_k = \begin{cases} \frac{1}{960}(29c_{\frac{-k+4}{2},6} + 303c_{\frac{-k+2}{2},6} + 147c_{\frac{-k}{2},6} + c_{\frac{-k-2}{2},6}) & \text{for } k : \text{even} \\ \frac{1}{960}(c_{\frac{-k+5}{2},6} + 147c_{\frac{-k+3}{2},6} + 303c_{\frac{-k+1}{2},6} + 29c_{\frac{-k-1}{2},6}) & \text{for } k : \text{odd} \end{cases} \tag{2.77}$$

$$b_k = \begin{cases} \frac{1}{8}(3c_{\frac{-k+4}{2},6} + c_{\frac{-k+2}{2},6}) & \text{for } k : \text{even} \\ -\frac{1}{8}(c_{\frac{-k+5}{2},6} + 3c_{\frac{-k+3}{2},6}) & \text{for } k : \text{odd} \end{cases} \tag{2.78}$$

The closed form solution for $\{c_{l,6}\}$ is

$$c_{l,6} = \frac{120}{\alpha_1 - \alpha_2} \left(\frac{\beta_1^{|l|}}{\beta_1 - \beta_2} - \frac{\gamma_1^{|l|}}{\gamma_1 - \gamma_2} \right) \tag{2.79}$$

where

$$\alpha_1, \alpha_2 = -13 \pm \sqrt{105}, \quad \alpha_1 > \alpha_2$$

$$\beta_1, \beta_2 = \frac{1}{2}(\alpha_1 \pm \sqrt{\alpha_1^2 - 4}), \quad \beta_1 > \beta_2$$

$$\gamma_1, \gamma_2 = \frac{1}{2}(\alpha_2 \pm \sqrt{\alpha_2^2 - 4}), \quad \gamma_1 > \gamma_2$$

and its derivation is shown in Appendix C.

2.6 Cubic B-Spline Wavelets

2.6.1 Scaling functions

Cubic B-Spline $N_4(x)$ shown in Figure 2.15 is given by

$$N_4(x) = \phi(x) = \begin{cases} \frac{1}{6}x^3 & \text{for } 0 \le x < 1 \\ \frac{1}{6}(-3x^3 + 12x^2 - 12x + 4) & \text{for } 1 \le x < 2 \\ \frac{1}{2}(3x^3 - 24x^2 + 60x - 44) & \text{for } 2 \le x < 3 \\ \frac{1}{6}(4-x)^3 & \text{for } 3 \le x < 4 \\ 0 & \text{otherwise} \end{cases} \tag{2.80}$$

and its two-scale relation

$$\phi(x) = \frac{1}{8}\phi(2x) + \frac{4}{8}\phi(2x-1) + \frac{6}{8}\phi(2x-2) + \frac{4}{8}\phi(2x-3) + \frac{1}{8}\phi(2x-4) \tag{2.81}$$

is shown in Figure 2.16

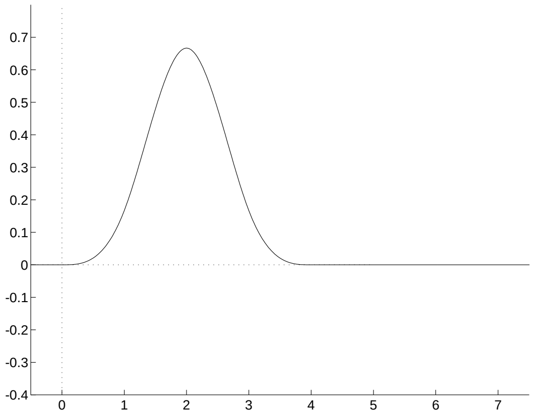

Figure 2.15: Cubic B-Spline $N_4(x)$

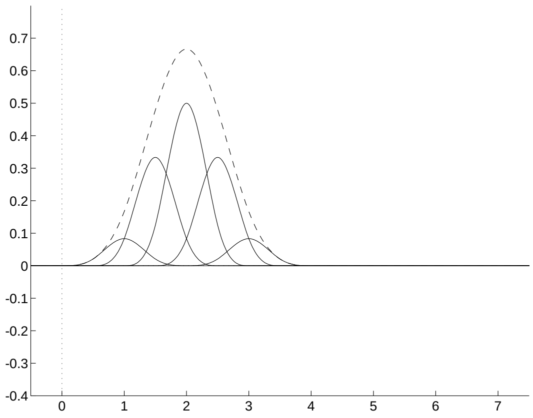

Figure 2.16: Two-scale relation for N_4

2.6.2 Wavelets

Two-scale relations for cubic B-spline wavelets can also be obtained in exactly the same way as linear and quadratic cases. We just provide the results of the two-scale relation and the graph for the cubic B-spline wavelet.

$$\psi_4(x) = \frac{1}{8!}N_4(2x) - \frac{124}{8!}N_4(2x-1) + \frac{1677}{8!}N_4(2x-2) - \frac{7904}{8!}N_4(2x-3) + \frac{18482}{8!}N_4(2x-4) - \frac{24264}{8!}N_4(2x-5)$$

$$+ \frac{18482}{8!}N_4(2x-6) - \frac{7904}{8!}N_4(2x-7) + \frac{1677}{8!}N_4(2x-8) - \frac{124}{8!}N_4(2x-9) + \frac{1}{8!}N_4(2x-10) \tag{2.82}$$

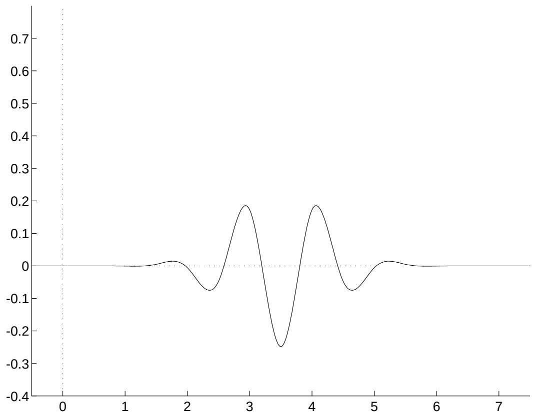

Figure 2.17: Cubic B-spline wavelet $\psi_4(x)$

2.6.3 Decomposition relations

The decomposition sequences for Cubic B-spline are:

$$a_k = \frac{1}{2}(-1)^{k+1} \sum_{l \in Z} q_{-k+7-2l} c_{l,8} \tag{2.83}$$

$$b_k = -\frac{1}{2}(-1)^{k+1} \sum_{l \in Z} p_{-k+7-2l} c_{l,8} \tag{2.84}$$

The closed form solution for $\{c_{l,8}\}$ is not easy to obtain analytically. The numerical results of the sequences $\{a_k\}$ and $\{b_k\}$ for Cubic B-spline for some small $|k|$ are shown in Table 2.3.

$\lvert k \rvert$	a_{k+2}	b_{k+5}
0	0.893162856314	− 1.475394519892
1	0.400680825467	0.468422596633
2	− 0.282211870811	0.742097698477
3	− 0.232924626134	− 0.345770890775
4	0.129083571218	− 0.389745580800
5	0.126457446356	0.196794277304
⋮	⋮	⋮

Table 2.3: Cubic B-spline decomposition sequences

2.7 Daubechies Wavelets

2.7.1 Scaling functions

Another example of compactly supported wavelets defined on real line is Daubechies wavelets. Daubechies scaling function ϕ_3^D is defined by the following two-scale relation:

$$\begin{aligned}\phi_3^D(x) &= \sum_{k=0}^{3} p_k \phi(2x-k) \\ &= \frac{1+\sqrt{3}}{4}\phi_3^D(2x) + \frac{3+\sqrt{3}}{4}\phi_3^D(2x-1) + \frac{3-\sqrt{3}}{4}\phi_3^D(2x-2) + \frac{1-\sqrt{3}}{4}\phi_3^D(2x-3),\end{aligned} \tag{2.85}$$

that is, non-zero values of the two-scale sequence $\{p_k\}$ are: $\{p_0, p_1, p_2, p_3\} = \{\frac{1+\sqrt{3}}{4}, \frac{3+\sqrt{3}}{4}, \frac{3-\sqrt{3}}{4}, \frac{1-\sqrt{3}}{4}\}$. Note that the coefficients $\{p_k\}$ have properties $p_0 + p_2 = p_1 + p_3 = 1$. In general, two-scale sequence $\{p_k\}$ for any scaling functions has the property:

$$\sum_k p_{2k} = \sum_k p_{2k+1} = 1 \tag{2.86}$$

There is no closed form for $\phi_3^D(x)$. However the display algorithm[1] can be used to draw the graph of $\phi_3^D(x)$.

2.7.2 Wavelets

The two-scale relation for the Daubechies wavelets is in the form:

$$\psi(x)_3^D = \sum_k q_k \phi_3^D(2x-k) \tag{2.87}$$

where

$$q_k = (-1)^k \bar{p}_{-k+1} \tag{2.88}$$

and $\bar{p}_{-k+1}$ is the complex conjugate of p_{-k+1}. Note that since the two-scale coefficients $\{p_k\}$ are all real for $\phi_3^D(x)$, we simply have $\bar{p}_{-k+1} = p_{-k+1}$. Therefore the non-zero values of the two-scale sequence $\{q_k\}$ are: $\{q_{-2}, q_{-1}, q_0, q_1\} = \{p_3, -p_2, p_1, -p_0\} = \{\frac{1-\sqrt{3}}{4}, -\frac{3-\sqrt{3}}{4}, \frac{3+\sqrt{3}}{4}, -\frac{1+\sqrt{3}}{4}\}$, and the explicit formula for (2.87) is:

$$\begin{aligned}\psi_3^D(x) &= \sum_{k=-2}^{1} q_k \phi_3^D(2x-k) \\ &= \frac{1-\sqrt{3}}{4}\phi_3^D(2x+2) - \frac{3-\sqrt{3}}{4}\phi_3^D(2x+1) + \frac{3+\sqrt{3}}{4}\phi_3^D(2x) - \frac{1+\sqrt{3}}{4}\phi_3^D(2x-1)\end{aligned} \tag{2.89}$$

The special feature of the Daubechies wavelet family is the presence of orthogonality in combination with compact support. To achieve this, we must put up with asymmetry and low regularity.

2.7.3 Decomposition relations

The decomposition relations for Daubechies Wavelets are rather simple and given by:

$$\phi_3^D(2x-l) = \sum_k \left[a_{l-2k}\phi_3^D(x-k) + b_{l-2k}\psi_3^D(x-k)\right],\ l \in Z \tag{2.90}$$

where

$$a_{l-2k} = \frac{1}{2}\bar{p}_{l-2k} \tag{2.91}$$

$$b_{l-2k} = \frac{1}{2}\bar{q}_{l-2k} \tag{2.92}$$

Since only four $\{p_k\}$ and four $\{q_k\}$ are non-zero, the decomposition relation (2.90) is written in the following more explicit formula:

$$\phi_3^D(2x-l) = \begin{cases} \frac{1}{2}p_2\phi_3^D(x-\frac{l-2}{2}) + \frac{1}{2}p_0\phi_3^D(x-\frac{l}{2}) + \frac{1}{2}q_0\psi_3^D(x-\frac{l}{2}) + \frac{1}{2}q_{-2}\psi_3^D(x-\frac{l+2}{2}) & \text{for } l : \text{even} \\ \frac{1}{2}p_3\phi_3^D(x-\frac{l-3}{2}) + \frac{1}{2}p_1\phi_3^D(x-\frac{l-1}{2}) + \frac{1}{2}q_1\psi_3^D(x-\frac{l-1}{2}) + \frac{1}{2}q_{-1}\psi_3^D(x-\frac{l+1}{2}) & \text{for } l : \text{odd} \end{cases} \tag{2.93}$$

Furthermore,

$$\begin{aligned} \phi_3^D(2x) &= \tfrac{3-\sqrt{3}}{8}\phi_3^D(x+1) + \tfrac{1+\sqrt{3}}{8}\phi_3^D(x) + \tfrac{3+\sqrt{3}}{8}\psi_3^D(x) + \tfrac{1-\sqrt{3}}{8}\psi_3^D(x-1) \\ \phi_3^D(2x-1) &= \tfrac{1-\sqrt{3}}{8}\phi_3^D(x+1) + \tfrac{3+\sqrt{3}}{8}\psi_3^D(x) - \tfrac{1+\sqrt{3}}{8}\psi_3^D(x) - \tfrac{3-\sqrt{3}}{8}\psi_3^D(x-1) \end{aligned} \tag{2.94}$$

2.8 Summary

We have presented some useful examples of wavelets on real line, which include Haar wavelets, B-spline wavelets (general order, Linear, Quadratic and Cubic B-spline wavelets) and Daubechies wavelets. For each example, scaling function is defined and its corresponding wavelet is introduced with brief summaries of their properties. Reconstruction and decomposition relations for each example are also explored.

The Haar wavelet is the simplest examples among all families of wavelets. In fact, it has been used for many applications. However, one of the drawback is its discontinuity. It consists of only box functions and cannot reproduce even linear functions smoothly.

On the other hand, B-spline wavelets have higher continuity. They are more suitable for representing any continuous function. However, a shortcoming is the large amount of its computations. As we mentioned, the decomposition relation consists of infinite terms, which means that the computations of the wavelet coefficients are much more expensive than Haar wavelets.

3. Wavelets on a Bounded Interval

Now we construct wavelets on a bounded interval. To this extent, we start with a finite set of linearly independent functions $\phi_0, \ldots, \phi_{m-1}$ defined on the interval and let V_0 be the vector space spanned by these functions. Observe that in contrast to one scaling function ϕ, we start with a finite set of scaling functions.

3.1 Linear Legendre Wavelets

Our first concrete examples of wavelets on a bounded interval are Linear Legendre bases.

3.1.1 Scaling functions

We define a pair of Linear Legendre scaling functions $\phi_0(x) and \phi_1(x)$ on a bounded interval $[0,1]$ as

$$\begin{cases} \phi_0(x) = 1 & \text{for } 0 \le x < 1 \\ \phi_1(x) = \sqrt{3}(2x-1) & \text{for } 0 \le x < 1 \end{cases} \tag{3.1}$$

and they are shown in Figure 3.1. The integer translates of $\phi_0(x)$ and $\phi_1(x)$ together span a subspace V_0. The 1/2 scaled version of $\phi_0(x)$ and $\phi_1(x)$ whose translates span the subspace V_1 such that $V_0 \subset V_1$ are given by

$$\begin{cases} \phi_0(2x-k) = 1 & \text{for } \frac{k}{2} \le x < \frac{k+1}{2} \\ \phi_1(2x-k) = \sqrt{3}(4x-2k-1) & \text{for } \frac{k}{2} \le x < \frac{k+1}{2}. \end{cases} \tag{3.2}$$

Figure 3.2 shows $\phi_i(2x-k)$ for $k=0$ and 1.

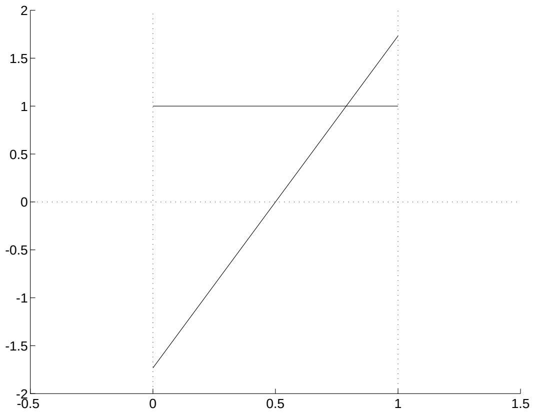

Figure 3.1: Linear Legendre scaling functions $\phi_0(x)$ and $\phi_1(x)$ defined on $[0,1]$

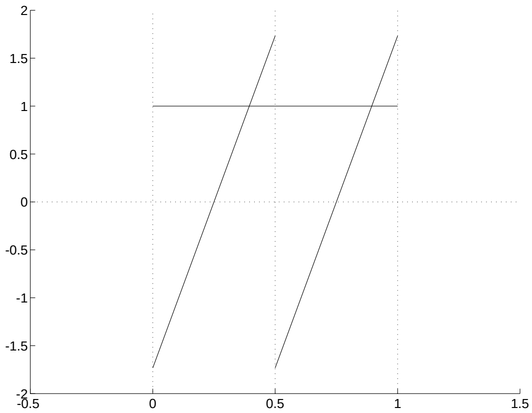

Figure 3.2: Dilates and translates; $\phi_0(2x)$, $\phi_1(2x)$, $\phi_0(2x-1)$ and $\phi_1(2x-1)$

First, let us study the two-scale relations for Linear Legendre scaling functions. The two-scale relations should express each of the scaling functions $\phi_0(x)$ and $\phi_1(x)$ in V_0 in terms of the four scaling functions $\phi_i(2x)$ and $\phi_i(2x-1)$ for $i=0,1$ in V_1, namely, introducing two-scale sequence $\{p\}$, the two-scale relations are in the following form:

$$\begin{bmatrix} \phi_0(x) \\ \phi_1(x) \end{bmatrix} = \begin{bmatrix} p_{0,0} & p_{0,1} & p_{0,2} & p_{0,3} \\ p_{1,0} & p_{1,1} & p_{1,2} & p_{1,3} \end{bmatrix} \begin{bmatrix} \phi_0(2x) \\ \phi_1(2x) \\ \phi_0(2x-1) \\ \phi_1(2x-1) \end{bmatrix} \tag{3.3}$$

Substituting the explicit formulae (3.1) and (3.2) into the two-scale relations (3.3), the two-scale sequence $\{p\}$ is determined uniquely and we obtain the two-scale relations for Linear Legendre scaling functions as follows

$$\begin{bmatrix} \phi_0(x) \\ \phi_1(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \phi_0(2x) \\ \phi_1(2x) \\ \phi_0(2x-1) \\ \phi_1(2x-1) \end{bmatrix} \tag{3.4}$$

Second, we present some properties of the Linear Legendre scaling functions. The pair of the scaling functions ϕ_0 and ϕ_1 are orthogonal each other, i.e.,

$$\int_{-\infty}^{\infty} \phi_0(x-k)\phi_1(x-l)dx = 0, \quad \text{for } k, l \in Z \tag{3.5}$$

This orthogonality property implies that ϕ_1 has a vanishing moment with constant

$$\int_{-\infty}^{\infty} \phi_1(x)dx = 0. \tag{3.6}$$

Note that the magnitude of the coefficients in (3.1) have been determined due to the normalization properties:

$$\int_{-\infty}^{\infty} \{\phi_i(x)\}^2 \, dx = 1 \tag{3.7}$$

The equations (3.5) and (3.7) together can be combined into the following orthonormal property:

$$\int_{-\infty}^{\infty} \phi_i(x-k)\phi_j(x-l)dx = \delta_{i,j}\delta_{k,l}, \quad \text{for } i, j = 0, 1 \text{ and } k, l \in Z. \tag{3.8}$$

In addition to these properties, $\phi_0(x)$, $\phi_1(x)$ has symmetry, anti-symmetry property, respectively, as is obvious from Figure 3.1.

3.1.2 Wavelets

In this subsection, we illustrate the constructions of Linear Legendre wavelets as well as just give their formulae.

Let $\psi_0(x)$ and $\psi_1(x)$ be wavelets defined on $[0, 1]$ which are corresponding to Linear Legendre scaling functions. Using the two-scale relations for wavelets, $\psi_0(x)$ and $\psi_1(x)$ should be expressed in terms of $\phi_i(2x)$ and $\phi_i(2x-1)$ for $i = 0, 1$. Similar to the previous case of the scaling functions, introducing the two-scale sequence $\{q\}$, the two-scale relations for wavelets are in the following form:

$$\begin{bmatrix} \psi_0(x) \\ \psi_1(x) \end{bmatrix} = \begin{bmatrix} q_{0,0} & q_{0,1} & q_{0,2} & q_{0,3} \\ q_{1,0} & q_{1,1} & q_{1,2} & q_{1,3} \end{bmatrix} \begin{bmatrix} \phi_0(2x) \\ \phi_1(2x) \\ \phi_0(2x-1) \\ \phi_1(2x-1) \end{bmatrix} \tag{3.9}$$

Our objective is to obtain the two-scale sequences $\{q\}$. Since we have 8 unknowns, we need 8 independent conditions to determine the two-scale relations. There are many possibilities in choosing the conditions to be used. They result in different wavelets which are all valid but have different properties. Here, we use orthonormal conditions and vanishing moments conditions, as is illustrated below.

We first impose a orthonormal condition on the pair of the wavelets $\psi_0(x)$ and $\psi_1(x)$, i.e.,

$$\int_{-\infty}^{\infty} \psi_i(x-k)\psi_j(x-l)dx = \delta_{i,j}\delta_{k,l}, \quad \text{for } i,j = 0,1 \text{ and } k,l \in Z \tag{3.10}$$

which is analogious to (3.8) for the scaling functions. Noting that $\psi(x)$ is zero outside $[0,1]$, the orthonormal condition (3.10) can be decomposed into 3 explicit equations; 1 for orthogonality $\psi_0(x)\perp\psi_1(x)$ and 2 for normality of $\psi_0(x)$ and $\psi_1(x)$, which are summarized in Table 3.1. Now we need 5 more conditions to determine 8 unknowns uniquely. We use as many as 5 vanishing moment conditions; 2 for $\psi_0(x)$ and 3 for $\psi_1(x)$ shown also in Table 3.1. Note that the translate factors are omitted for simplicity as any function described here vanishes outside $[0,1]$.

	ψ_0	ψ_1
vanishing moments	$\int_{-\infty}^{\infty} \psi_0(x)dx = 0$	$\int_{-\infty}^{\infty} \psi_1(x)dx = 0$
	$\int_{-\infty}^{\infty} \psi_0(x)xdx = 0$	$\int_{-\infty}^{\infty} \psi_1(x)xdx = 0$
orthogonality	$\int_{-\infty}^{\infty} \psi_0(x)\psi_1(x)dx = 0$	$\int_{-\infty}^{\infty} \psi_1(x)x^2dx = 0$
normality	$\int_{-\infty}^{\infty} \{\psi_0(x)\}^2\,dx = 1$	$\int_{-\infty}^{\infty} \{\psi_1(x)\}^2\,dx = 1$

Table 3.1: Conditions to be used to determine the Linear Legendre Wavelets

We can first solve for the coefficients $\{q_{1,j}\}$ for $\psi_1(x)$ using a system of 4 equations shown in the right column of Table 3.1. After some calculations, the solutions for $\{q_{1,j}\}$ are obtained as $\{q_{1,0}, q_{1,1}, q_{1,2}, q_{1,3}\} = \pm\{\frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{1}{2}, \frac{\sqrt{3}}{2}\}$. Once the solution for $\{q_{1,j}\}$ are obtained, we can proceed for $\{q_{0,j}\}$ using another system of 4 equations in the left column of Table 3.1 which yields the solution for $\{q_{0,j}\}$ as $\{q_{0,0}, q_{0,1}, q_{0,2}, q_{0,3}\} = \pm\{0,-1,0,1\}$.

Thus we have obtained the two-scale relation for Linear Legendre wavelets as follows:

$$\begin{bmatrix} \psi_0(x) \\ \psi_1(x) \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \phi_0(2x) \\ \phi_1(2x) \\ \phi_0(2x-1) \\ \phi_1(2x-1) \end{bmatrix} \tag{3.11}$$

The explicit formulae for $\psi_0(x)$ and $\psi_1(x)$ are:

$$\psi_0(x) = \pm \begin{cases} -\sqrt{3}(4x-1) & \text{for } 0 \le x < \frac{1}{2} \\ +\sqrt{3}(4x-3) & \text{for } \frac{1}{2} \le x < 1 \end{cases} \tag{3.12}$$

$$\psi_1(x) = \pm \begin{cases} 6x-1 & \text{for } 0 \le x < \frac{1}{2} \\ 6x-5 & \text{for } \frac{1}{2} \le x < 1 \end{cases} \tag{3.13}$$

The graphs of $\psi_0(x)$, $\psi_1(x)$ are shown in Figure 3.3, 3.4, respectively, where the positive signs in (3.12) and (3.13) are chosen. The choice of the negative signs are also valid solutions which are the symmetric images along x axis.

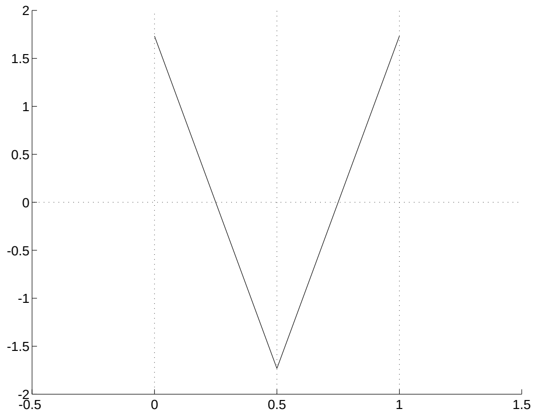

Figure 3.3: Linear Legendre wavelet $\psi_0(x)$

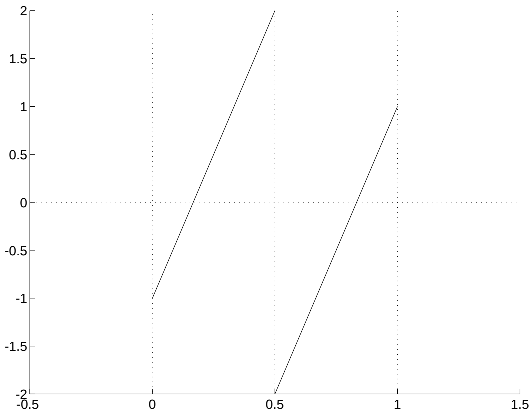

Figure 3.4: Linear Legendre wavelet $\psi_1(x)$

Now let us explore some properties of the wavelets ψ_0 and ψ_1. We already know that the wavelets have all the properties shown in Table 3.1 since we have used them to construct the wavelets. The vanishing moment properties imply that any of the wavelet $\psi_j(x)$ is orthogonal to any of the scaling function $\phi_i(x)$, that is,

$$\int_{-\infty}^{\infty} \phi_i(x-k)\psi_j(x-l)dx = 0, \quad \text{for } i,j = 0,1 \text{ and } k,l \in Z \tag{3.14}$$

In addition to these properties, we have symmetry property for ψ_0 and anti-symmetry property for ψ_1:

$$\psi_0(x+\frac{1}{2}) = \psi_0(-x+\frac{1}{2}) \tag{3.15}$$

$$\psi_1(x+\frac{1}{2}) = -\psi_1(-x+\frac{1}{2}) \tag{3.16}$$

Note that using either the symmetry condition (3.15) on ψ_0 or the anti-symmetry condition (3.16) on ψ_1 instead of the highest vanishing moment condition with x^2 on $\psi_1(x)$ results in the same wavelets.

3.1.3 Decomposition relations

Let us rewrite the two-scale relations for both scaling functions (3.4) and wavelets (3.11) in the single matrix form

$$\begin{bmatrix} \phi_0(x) \\ \phi_1(x) \\ \psi_0(x) \\ \psi_1(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & -1 & 0 & 1 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \phi_0(2x) \\ \phi_1(2x) \\ \phi_0(2x-1) \\ \phi_1(2x-1) \end{bmatrix} \tag{3.17}$$

These two-scale relations together are called the reconstruction relation, where the bases $\phi_0(x)$, $\phi_1(x)$ in V_0 and $\psi_0(x)$, $\psi_1(x)$ in W_0 subspaces are expressed in terms of 4 bases $\phi_0(2x)$, $\phi_1(2x)$, $\phi_0(2x-1)$ and $\phi_1(2x-1)$ in V_1 subspace.

The decomposition relation is simply the inverse of the reconstruction relation (3.17). The square matrix in (3.17) is an orthogonal matrix with constant magnitude since the bases $\{\phi_0(x)$, $\phi_1(x)$, $\psi_0(x)$, $\psi_1(x)\}$ are orthogonal each other and so are the bases $\{\phi_0(2x)$, $\phi_1(2x)$, $\phi_0(2x-1)$, $\phi_1(2x-1)\}$. Thanks to the nice property $M^{-1} = M^T$ for any orthogonal matrix M, the decomposition relation is easily obtained as follows:

$$\begin{bmatrix} \phi_0(2x) \\ \phi_1(2x) \\ \phi_0(2x-1) \\ \phi_1(2x-1) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -1 & \frac{\sqrt{3}}{2} \\ 1 & \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 1 & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \phi_0(x) \\ \phi_1(x) \\ \psi_0(x) \\ \psi_1(x) \end{bmatrix} \tag{3.18}$$

3.2 Quadratic Legendre Wavelets

In this section, we will study Quadratic Legendre wavelets as the next examples of wavelets on a bounded interval.

In the previous section, we started our examples with the Linear Legendre wavelets. However, the most simplest example in Legendre family is the one just with the constant function $\phi_0(x)$. We do not discuss this case in this chapter because it is reduced to the Haar wavelet which we explored thoroughly in the previous chapter.

3.2.1 Scaling functions

The Quadratic scaling functions consist of three basis functions shown in (3.19),

$$\begin{cases} \phi_0(x) = 1 & \text{for } 0 \le x < 1 \\ \phi_1(x) = \sqrt{3}(2x-1) & \text{for } 0 \le x < 1 \\ \phi_2(x) = \sqrt{5}(6x^2 - 6x + 1) & \text{for } 0 \le x < 1 \end{cases} \tag{3.19}$$

which are made of the pair of Linear Legendre scaling functions $\phi_0(x)$ and $\phi_1(x)$ and an additional quadratic function $\phi_2(x)$. The set of three scaling functions are shown in Figure 3.5.

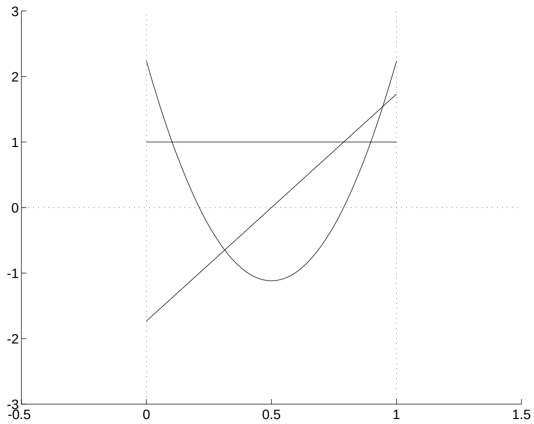

Figure 3.5: Quadratic Legendre scaling functions $\phi_0(x)$, $\phi_1(x)$ and $\phi_2(x)$

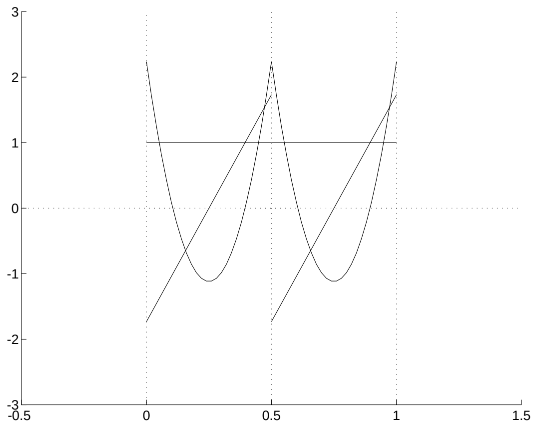

Figure 3.6: Dilates and translates; $\phi_j(2x)$ and $\phi_j(2x-1)$ for $j = 0, 1, 2$

Their translates $\phi_0(x-k)$, $\phi_1(x-k)$ and $\phi_2(x-k)$, $k \in Z$ span a subspace V_0. The 1/2 scaled version, $\phi_0(2x)$, $\phi_1(2x)$ and $\phi_2(2x)$ are given by

$$\begin{cases} \phi_0(2x-k) = 1 & \text{for } \frac{k}{2} \le x < \frac{k+1}{2} \\ \phi_1(2x-k) = \sqrt{3}(4x-2k-1) & \text{for } \frac{k}{2} \le x < \frac{k+1}{2} \\ \phi_2(2x-k) = \sqrt{5}\{24x^2 - 12(2k+1)x + (6k^2+6k+1)\} & \text{for } \frac{k}{2} \le x < \frac{k+1}{2} \end{cases} \tag{3.20}$$

which span a subspace V_1 such that $V_0 \subset V_1$. Figure 3.6 shows $\phi_i(2x-k)$ for $k=0,1$ and $i=0,1,2$.

Let us follow the same procedure when we obtained the two-scaling relation for the Linear Legendre scaling functions. The two-scale relations should express each of the three scaling functions $\phi_i(x)$ for $i=0,1,2$ in terms of the six scaling functions $\phi_i(2x)$ and $\phi_i(2x-1)$ for $i=0,1,2$ namely,

$$\begin{bmatrix} \phi_0(x) \\ \phi_1(x) \\ \phi_2(x) \end{bmatrix} = \begin{bmatrix} p_{0,0} & p_{0,1} & p_{0,2} & p_{0,3} & p_{0,4} & p_{0,5} \\ p_{1,0} & p_{1,1} & p_{1,2} & p_{1,3} & p_{1,4} & p_{1,5} \\ p_{2,0} & p_{2,1} & p_{2,2} & p_{2,3} & p_{2,4} & p_{2,5} \end{bmatrix} \begin{bmatrix} \phi_0(2x) \\ \phi_1(2x) \\ \phi_2(2x) \\ \phi_0(2x-1) \\ \phi_1(2x-1) \\ \phi_2(2x-1) \end{bmatrix} \tag{3.21}$$

Substituting the explicit formulae (3.19) and (3.20) for scaling functions $\phi_i(x)$ and $\phi_i(2x-k)$ into the two-scale relations (3.21), the two-scale sequence $\{p\}$ in the two-scale relations for Quadratic Legendre scaling functions are determined uniquely as follows:

$$\begin{bmatrix} \phi_0(x) \\ \phi_1(x) \\ \phi_2(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{\sqrt{15}}{4} & \frac{1}{4} & 0 & \frac{\sqrt{15}}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \phi_0(2x) \\ \phi_1(2x) \\ \phi_2(2x) \\ \phi_0(2x-1) \\ \phi_1(2x-1) \\ \phi_2(2x-1) \end{bmatrix} \tag{3.22}$$

It is not surprising but is interesting to see that the two-scale relations for Quadratic Legendre functions (3.22) are superset of the ones for Linear Legendre functions (3.4).

Let us review some properties of Quadratic Legendre scaling functions. Like (3.8) for Linear Legendre, the definition of the quadratic function $\phi_2(x)$ in (3.19) was determined using the following orthonormal conditions where their translates are omitted.

$$\int_{-\infty}^{\infty} \phi_i(x)\phi_j(x)dx = \delta_{i,j}, \quad i,j=0,1,2 \tag{3.23}$$

In other words, all the three scaling fuctions $\phi_0(x)$, $\phi_1(x)$ and $\phi_2(x)$ are orthogonal each other and are normalized. This orthonormal property implies that $\phi_2(x)$ has a vanishing moment with constant and x, namely,

$$\int_{-\infty}^{\infty} \phi_2(x)x^j dx = 0 \quad \text{for } j=0,1 \tag{3.24}$$

Not to be mentioned, the function $\phi_2(x)$ has symmetry property.

3.2.2 Wavelets

Now, let us focus on the Quadratic Legendre wavelets. We will obtain their two-scale relations in a similar way as the Linear case.

Let us introduce the two-scale sequence $\{q\}$ for the wavelets and express two-scale relations for the wavelets in the form:

$$\begin{bmatrix} \psi_0(x) \\ \psi_1(x) \\ \psi_2(x) \end{bmatrix} = \begin{bmatrix} q_{0,0} & q_{0,1} & q_{0,2} & q_{0,3} & q_{0,4} & q_{0,5} \\ q_{1,0} & q_{1,1} & q_{1,2} & q_{1,3} & q_{1,4} & q_{1,5} \\ q_{2,0} & q_{2,1} & q_{2,2} & q_{2,3} & q_{2,4} & q_{2,5} \end{bmatrix} \begin{bmatrix} \phi_0(2x) \\ \phi_1(2x) \\ \phi_2(2x) \\ \phi_0(2x-1) \\ \phi_1(2x-1) \\ \phi_2(2x-1) \end{bmatrix} \tag{3.25}$$

As we have 18 unknowns $q_{i,j}$ for $i = 0, 1, 2$, $j = 1, \ldots, 6$, we need 18 independent conditions to determine $q_{i,j}$. Similar to the Linear Legendre case, we use the following 18 conditions; 12 vanishing moment conditions, 3 orthogonality conditions and 3 normality conditions. as described below:

1. vanishing moments

$$\int_{-\infty}^{\infty} \psi_i(x) x^j dx = 0, \quad \text{for } i = 0, 1, 2 \text{ and } j = 0, 1, \ldots, i+2 \tag{3.26}$$

2. orthogonality

$$\int_{-\infty}^{\infty} \psi_i(x)\psi_j(x) dx = 0, \quad \text{for } i \neq j, \quad i, j = 0, 1, 2 \tag{3.27}$$

3. normality

$$\int_{-\infty}^{\infty} \{\psi_i(x)\}^2 dx = 1, \quad \text{for } i = 0, 1, 2 \tag{3.28}$$

Note that the 6 conditions for both orthogonality (3.27) and normality (3.28) can be combined into the following orthonormality conditions:

$$\int_{-\infty}^{\infty} \psi_i(x)\psi_j(x) dx = \delta_{i,j}, \quad \text{for } i, j = 0, 1, 2 \tag{3.29}$$

The explicit expressions for the 18 conditions are summarized in Table 3.2. Observe that the 6 conditions in the last column contains only $\psi_2(x)$ in their formulae, which implies that the 6 unknowns $\{q_{2,j}\}$ can be solved using a system of these 6 equations. Once the relation for $\psi_2(x)$ is obtained, we can proceed for $\psi_1(x)$ using another 6 conditions in the middle column which contain both $\psi_1(x)$ and $\psi_2(x)$. Finally, $\psi_0(x)$ are solved using the rest of 6 conditions in the first column.

The solution of the two-scale relation for the Quadratic Legendre wavelets are shown in (3.30). Note that one of the two possible signs has been chosen for each $\psi(x)$ for simplicity. For this reason, multiplying -1 to all the 6 elements in any row of the matrix in equation (3.30) yields the valid solutions as well.

$$\begin{bmatrix} \psi_0(x) \\ \psi_1(x) \\ \psi_2(x) \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{5}}{3} & -\frac{1}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{5}}{3} \\ 0 & \frac{1}{4} & \frac{\sqrt{15}}{4} & 0 & -\frac{1}{4} & \frac{\sqrt{15}}{4} \\ -\frac{\sqrt{5}}{6} & -\frac{\sqrt{15}}{6} & -\frac{2}{3} & \frac{\sqrt{5}}{6} & -\frac{\sqrt{15}}{6} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \phi_0(2x) \\ \phi_1(2x) \\ \phi_2(2x) \\ \phi_0(2x-1) \\ \phi_1(2x-1) \\ \phi_2(2x-1) \end{bmatrix} \tag{3.30}$$

	ψ_0	ψ_1	ψ_2
vanishing moments	$\int_{-\infty}^{\infty} \psi_0(x)dx = 0$	$\int_{-\infty}^{\infty} \psi_1(x)dx = 0$	$\int_{-\infty}^{\infty} \psi_2(x)dx = 0$
	$\int_{-\infty}^{\infty} \psi_0(x)xdx = 0$	$\int_{-\infty}^{\infty} \psi_1(x)xdx = 0$	$\int_{-\infty}^{\infty} \psi_2(x)xdx = 0$
	$\int_{-\infty}^{\infty} \psi_0(x)x^2dx = 0$	$\int_{-\infty}^{\infty} \psi_1(x)x^2dx = 0$	$\int_{-\infty}^{\infty} \psi_2(x)x^2dx = 0$
orthogonality	$\int_{-\infty}^{\infty} \psi_0(x)\psi_1(x)dx = 0$	$\int_{-\infty}^{\infty} \psi_1(x)x^3dx = 0$	$\int_{-\infty}^{\infty} \psi_2(x)x^3dx = 0$
	$\int_{-\infty}^{\infty} \psi_0(x)\psi_2(x)dx = 0$	$\int_{-\infty}^{\infty} \psi_1(x)\psi_2(x)dx = 0$	$\int_{-\infty}^{\infty} \psi_2(x)x^4dx = 0$
normality	$\int_{-\infty}^{\infty} \{\psi_0(x)\}^2\, dx = 1$	$\int_{-\infty}^{\infty} \{\psi_1(x)\}^2\, dx = 1$	$\int_{-\infty}^{\infty} \{\psi_2(x)\}^2\, dx = 1$

Table 3.2: Conditions to be used to determine the Quadratic Legendre Wavelets

The explicit formulae for the Quadratic Legendre wavelets $\psi_0(x)$, $\psi_1(x)$ and $\psi_2(x)$ are:

$$\psi_0(x) = \pm \begin{cases} -\frac{1}{3}(120x^2 - 72x + 7) & \text{for } 0 \leq x < \frac{1}{2} \\ +\frac{1}{3}(120x^2 - 168x + 55) & \text{for } \frac{1}{2} \leq x < 1 \end{cases} \tag{3.31}$$

$$\psi_1(x) = \pm \begin{cases} \sqrt{3}(30x^2 - 14x + 1) & \text{for } 0 \leq x < \frac{1}{2} \\ \sqrt{3}(30x^2 - 46x + 17) & \text{for } \frac{1}{2} \leq x < 1 \end{cases} \tag{3.32}$$

$$\psi_2(x) = \pm \begin{cases} -\frac{\sqrt{5}}{3}(48x^2 - 18x + 1) & \text{for } 0 \leq x < \frac{1}{2} \\ +\frac{\sqrt{5}}{3}(48x^2 - 78 + 31) & \text{for } \frac{1}{2} \leq x < 1 \end{cases} \tag{3.33}$$

The graphs of $\psi_0(x)$, $\psi_1(x)$ and $\psi_2(x)$ with the choice of the positive sign are shown in Figure 3.7, 3.8 and 3.9, respectively.

3.2.3 Decomposition relations

From the both two-scale relations (3.22) and (3.30) for the scaling functions and wavelets, the reconstruction relations are given by:

$$\begin{bmatrix} \phi_0(x) \\ \phi_1(x) \\ \phi_2(x) \\ \psi_0(x) \\ \psi_1(x) \\ \psi_2(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{\sqrt{15}}{4} & \frac{1}{4} & 0 & \frac{\sqrt{15}}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{5}}{3} & -\frac{1}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{5}}{3} \\ 0 & \frac{1}{4} & \frac{\sqrt{15}}{4} & 0 & -\frac{1}{4} & \frac{\sqrt{15}}{4} \\ -\frac{\sqrt{5}}{6} & -\frac{\sqrt{15}}{6} & -\frac{2}{3} & \frac{\sqrt{5}}{6} & -\frac{\sqrt{15}}{6} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \phi_0(2x) \\ \phi_1(2x) \\ \phi_2(2x) \\ \phi_0(2x-1) \\ \phi_1(2x-1) \\ \phi_2(2x-1) \end{bmatrix} \tag{3.34}$$

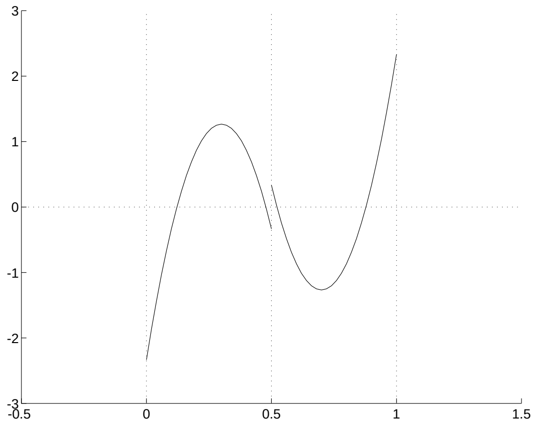

Figure 3.7: Quadratic Legendre wavelet $\psi_0(x)$

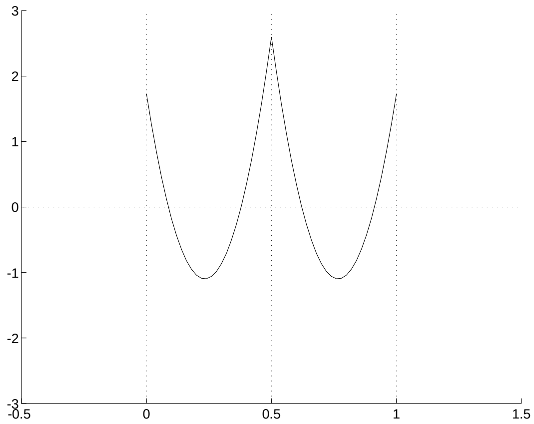

Figure 3.8: Quadratic Legendre wavelet $\psi_1(x)$

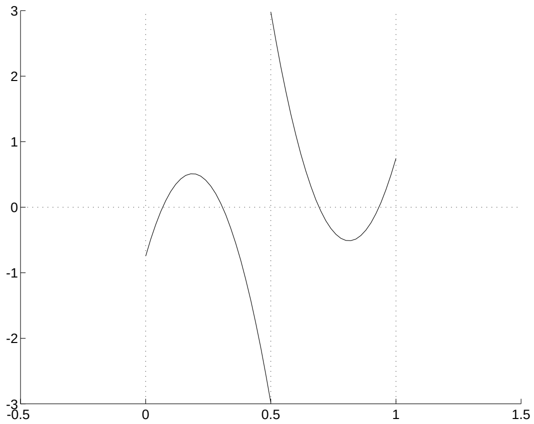

Figure 3.9: Quadratic Legendre wavelet $\psi_2(x)$

The decomposition relation is obtained as (3.35) by simply inverting the reconstruction relation (3.34). The square matrix containing two-scaling sequences $\{p\}$ and $\{q\}$ in (3.34) is again an orthogonal matrix with constant.

$$\begin{bmatrix} \phi_0(2x) \\ \phi_1(2x) \\ \phi_2(2x) \\ \phi_0(2x-1) \\ \phi_1(2x-1) \\ \phi_2(2x-1) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -\frac{\sqrt{3}}{2} & 0 & \frac{1}{3} & 0 & -\frac{\sqrt{5}}{6} \\ 0 & \frac{1}{2} & -\frac{\sqrt{15}}{4} & \frac{\sqrt{3}}{3} & \frac{1}{4} & -\frac{\sqrt{15}}{6} \\ 0 & 0 & \frac{1}{4} & -\frac{\sqrt{5}}{3} & \frac{\sqrt{15}}{4} & -\frac{2}{3} \\ 1 & \frac{\sqrt{3}}{2} & 0 & -\frac{1}{3} & 0 & \frac{\sqrt{5}}{6} \\ 0 & \frac{1}{2} & \frac{\sqrt{15}}{4} & \frac{\sqrt{3}}{3} & -\frac{1}{4} & -\frac{\sqrt{15}}{6} \\ 0 & 0 & \frac{1}{4} & \frac{\sqrt{5}}{3} & \frac{\sqrt{15}}{4} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \phi_0(x) \\ \phi_1(x) \\ \phi_2(x) \\ \psi_0(x) \\ \psi_1(x) \\ \psi_2(x) \end{bmatrix} \tag{3.35}$$

3.3 General Order Legendre Wavelets

To construct higher order Legendre wavelets, we can follow the same procedures as described in the previous two sections. In this section, we will summarize the definitions of the general order scaling functions and their properties. And then we will give procedures to obtain the solutions of their corresponding wavelets for general order. The solution for order 4 is given as an example.

3.3.1 Scaling functions

The m-th order Legendre scaling functions are the set of $m+1$ functions $\phi_0(x), \ldots, \phi_m(x)$ where $\phi_i(x)$ is a polynomial of i-th order and all ϕ's form orthonormal basis, that is, for $i = 0, 1, \ldots, m$,

$$\phi_i(x) = \sum_{k=0}^{i} a_{ik} x^k, \quad \text{for } i = 0, 1, \ldots, m \tag{3.36}$$

where

$$\int_{-\infty}^{\infty} \phi_i(x)\phi_k(x)dx = \delta_{i,k}, \quad \text{for } i, k = 0, 1, \ldots, m-1 \tag{3.37}$$

The scaling functions $\phi_i(x)$ for i is even, odd have symmetry, anti-symmetry properties, respectively.

The two-scale relations for Legendre scaling functions of order m are in the form:

$$\phi_i(x) = \sum_{j=0}^{m} p_{i,j}\phi_j(2x) + \sum_{j=0}^{m} p_{i,m+j+1}\phi_j(2x-1), \quad \text{for } i = 0, 1, \ldots, m \tag{3.38}$$

The coefficients $\{p\}$ are determined uniquely by substituting the equations (3.36) into (3.38). We would like to mention two remarks on the two-scale relations.

1. Since $\phi_i(x)$ is a i-th order polynomial, the right hand side of (3.38) has at most i-th order scaling functions. Therefore, $p_{i,j} = p_{i,m+j+1} = 0$ for $i < j$.
2. The two-scale relations for the Legendre scaling function of the order n which is lower than m is a subset of the first n two-scale relations for ϕ_i for $i = 0, 1, \ldots, n$ from m-th order two-scale relations.

3.3.2 Wavelets

The two-scale relations for the m-th order Legendre wavelets are in the form:

$$\psi_i(x) = \sum_{j=0}^{m} q_{i,j}\phi_j(2x) + \sum_{j=0}^{m} q_{i,m+j+1}\phi_j(2x-1) \tag{3.39}$$

As we have $2(m+1)^2$ unknown coefficients $\{q\}$ in (3.39), use the following $2m(m+1)$ vanishing moment conditions (3.40) and $2(m+1)$ orthonormal conditions (3.41) to determine them.

1. vanishing moments

$$\int_{-\infty}^{\infty} \psi_i(x) x^j dx = 0, \quad \text{for } i = 0, 1, \ldots, m \text{ and } j = 0, 1, \ldots, i+m \tag{3.40}$$

2. orthonormality

$$\int_{-\infty}^{\infty} \psi_i(x) \psi_j(x) dx = \delta_{i,j}, \quad \text{for } i, j = 0, 1, \ldots, m \tag{3.41}$$

3.3.3 The result of Cubic Legendre wavelets $m = 3$

The Cubic Legendre scaling functions consist of the four functions in (3.42) and are shown in Figure 3.10.

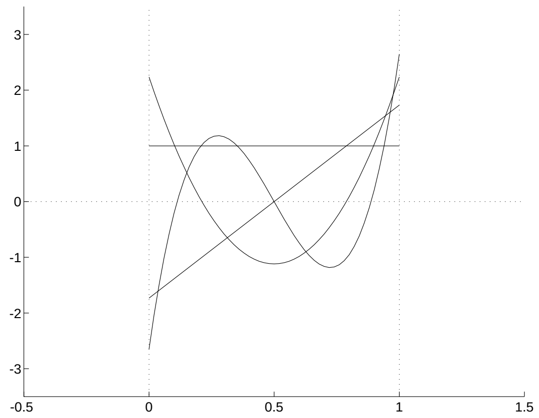

Figure 3.10: Cubic Legendre scaling functions $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$ and $\phi_3(x)$

$$\begin{cases} \phi_0(x) = 1 & \text{for } 0 \le x < 1 \\ \phi_1(x) = \sqrt{3}(2x-1) & \text{for } 0 \le x < 1 \\ \phi_2(x) = \sqrt{5}(6x^2 - 6x + 1) & \text{for } 0 \le x < 1 \\ \phi_3(x) = \sqrt{7}(20x^3 - 30x^2 + 12x - 1) & \text{for } 0 \le x < 1 \end{cases} \tag{3.42}$$

The closed form solution to the Cubic Legendre wavelets $\psi_0(x)$, $\psi_1(x)$, $\psi_2(x)$ and $\psi_3(x)$ are in (3.43) – (3.43) which are determined using the conditions (3.40) and (3.40). They are shown in Figure 3.11 – 3.14.

$$\psi_0(x) = \pm \begin{cases} -\sqrt{\frac{15}{17}}(224x^3 - 216x^2 + 56x - 3) & \text{for } 0 \le x < \frac{1}{2} \\ +\sqrt{\frac{15}{17}}(224x^3 - 456x^2 + 296x - 61) & \text{for } \frac{1}{2} \le x < 1 \end{cases} \tag{3.43}$$

$$\psi_1(x) = \pm \begin{cases} \sqrt{\frac{1}{21}}(1680x^3 - 1320x^2 + 270x - 11) & \text{for } 0 \le x < \frac{1}{2} \\ \sqrt{\frac{1}{21}}(1680x^3 - 3720x^2 + 2670x - 619) & \text{for } \frac{1}{2} \le x < 1 \end{cases} \tag{3.44}$$

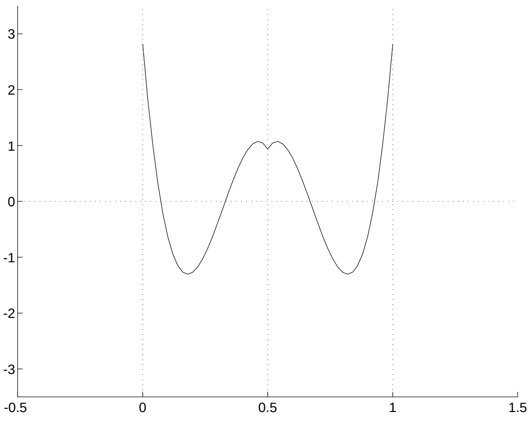

Figure 3.11: Cubic Legendre wavelet $\psi_0(x)$

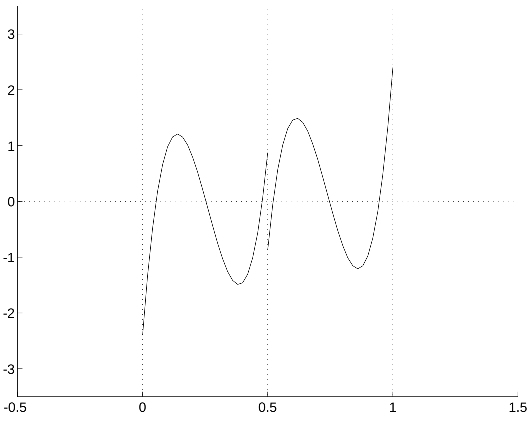

Figure 3.12: Cubic Legendre wavelet $\psi_1(x)$

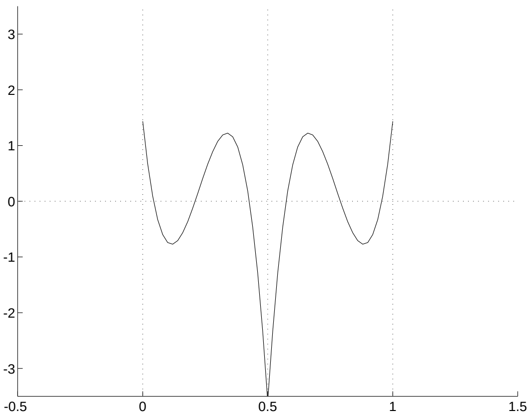

Figure 3.13: Cubic Legendre wavelet $\psi_2(x)$

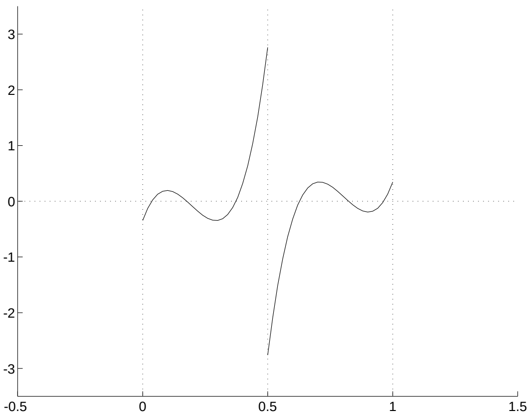

Figure 3.14: Cubic Legendre wavelet $\psi_3(x)$

$$\psi_2(x) = \pm \begin{cases} -\sqrt{\frac{35}{17}}(256x^3 - 174x^2 + 30x - 1) & \text{for } 0 \le x < \frac{1}{2} \\ +\sqrt{\frac{35}{17}}(256x^3 - 594x^2 + 450x - 111) & \text{for } \frac{1}{2} \le x < 1 \end{cases} \tag{3.45}$$

$$\psi_3(x) = \pm \begin{cases} \sqrt{\frac{5}{42}}(420x^3 - 246x^2 + 36x - 1) & \text{for } 0 \le x < \frac{1}{2} \\ \sqrt{\frac{5}{42}}(420x^3 - 1014x^2 + 804x - 209) & \text{for } \frac{1}{2} \le x < 1 \end{cases} \tag{3.46}$$

3.4 First Order Flatlet Wavelets

The m-th order Flatlet[8] basis functions consist of a set of $m+1$ piecewise unit constant functions defined on a bounded interval $[0,1]$.

The simplest example ($m=0$) for the Flatlet family is identical to the Haar wavelets which we presented in the previous Chapter. In this sectoin we start our example with first order Flatlet scaling functions ($m=1$) and their wavelets.

3.4.1 Scaling functions

The first order Flatlet basis functions are given as a pair of constant functions $\phi_0(x)$ and $\phi_1(x)$ on a bounded interval $[0,1]$ as

$$\begin{cases} \phi_0(x) = 1 & \text{for } 0 \le x < \frac{1}{2} \\ \phi_1(x) = 1 & \text{for } \frac{1}{2} \le x < 1 \end{cases} \tag{3.47}$$

and they are shown in Figure 3.15. The integer translates of $\phi_0(x)$ and $\phi_1(x)$ together span a subspace V_0. The 1/2 scaled version of $\phi_0(x)$ and $\phi_1(x)$ whose translates span the subspace V_1 such that $V_0 \subset V_1$ are given by

$$\begin{cases} \phi_0(2x-k) = 1 & \text{for } \frac{k}{2} \le x < \frac{k}{2} + \frac{1}{4} \\ \phi_1(2x-k) = 1 & \text{for } \frac{k}{2} + \frac{1}{4} \le x < \frac{k}{2} + \frac{1}{2}. \end{cases} \tag{3.48}$$

Figure 3.16 shows $\phi_0(2x)$ and $\phi_1(2x)$.

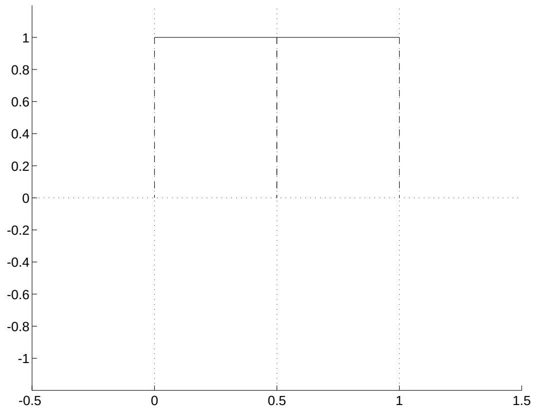

Figure 3.15: The first order Flatlet scaling functions $\phi_0(x)$ and $\phi_1(x)$ defined on $[0,1]$

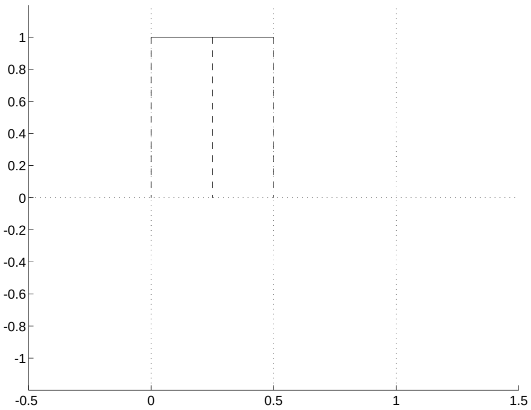

Figure 3.16: 1/2 dilates $\phi_0(2x)$ and $\phi_1(2x)$

First, let us study the two-scale relations for the first order Flatlet scaling functions. The two-scale relations should express each of the scaling functions $\phi_0(x)$ and $\phi_1(x)$ in V_0 in terms of the four scaling functions $\phi_i(2x)$ and $\phi_i(2x-1)$ for $i=0,1$ in V_1, namely, introducing two-scale sequence $\{p\}$, the two-scale relations are in the same form as (3.3). Substituting the explicit formulae (3.47) and (3.48) into the two-scale relations (3.3), the two-scale sequence $\{p\}$ is determined uniquely as follows:

$$\begin{bmatrix} \phi_0(x) \\ \phi_1(x) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \phi_0(2x) \\ \phi_1(2x) \\ \phi_0(2x-1) \\ \phi_1(2x-1) \end{bmatrix} \tag{3.49}$$

Now, let us explore some properties of the first order Flatlet scaling functions. The pair of the scaling functions ϕ_0 and ϕ_1 are orthogonal each other like the Linear Legendre scaling functions,

$$\int_{-\infty}^{\infty} \phi_0(x-k)\phi_1(x-l)dx = 0, \quad \text{for } k, l \in Z \tag{3.50}$$

Note that ϕ_0 and ϕ_1 are not normalized to 1 for our definitions.

$$\int_{-\infty}^{\infty} \phi_i(x)dx = \int_{-\infty}^{\infty} \{\phi_i(x)\}^2 dx = \frac{1}{2} \tag{3.51}$$

3.4.2 Wavelets

In this subsection, we illustrate the constructions of the first order Flatlet wavelets in the same way as the Linear Legendre wavelets.

Let $\psi_0(x)$ and $\psi_1(x)$ be wavelets defined on $[0,1]$ which are corresponding to the first order Flatlet scaling functions. Introducing two-scale sequence $\{q\}$, the two-scale relations for wavelets are in the same form as (3.9). Let us impose the same conditions as we used for the Linear Legendre wavelets shown in Table 3.1 except that the wavelets are normalized to $\frac{1}{2}$ instead of 1.

By solving two systems of 4 equations using 8 conditions in Table 3.1, the solutions for $\{q_{1,j}\}$ and $\{q_{0,j}\}$ are obtained as $\{q_{1,0}, q_{1,1}, q_{1,2}, q_{1,3}\} = \pm\frac{1}{\sqrt{10}}\{1, -3, 3, -1\}$ and $\{q_{0,0}, q_{0,1}, q_{0,2}, q_{0,3}\} = \pm\frac{1}{\sqrt{2}}\{1, -1, -1, 1\}$. Thus we have obtained the two-scale relation for the first order Flatlet wavelets as follows:

$$\begin{bmatrix} \psi_0(x) \\ \psi_1(x) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} & \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} \phi_0(2x) \\ \phi_1(2x) \\ \phi_0(2x-1) \\ \phi_1(2x-1) \end{bmatrix} \tag{3.52}$$

The graphs of the wavelets $\psi_0(x)$, $\psi_1(x)$ are shown in Figure 3.17, 3.18, respectively, The choice of the negative signs again are also valid solutions which are the symmetric images along x axis.

Now let us explore some properties of the wavelets ψ_0 and ψ_1. The wavelets already enjoy all the properties shown in Table 3.1. Unlike the Linear Legendre case, all the scaling functions are not orthogonal to all the wavelts. In fact,

$$\int_{-\infty}^{\infty} \phi_i(x-k)\psi_0(x-l)dx = 0, \quad \text{for } i = 0, 1 \text{ and } k, l \in Z \tag{3.53}$$

while

$$\int_{-\infty}^{\infty} \phi_i(x-k)\psi_1(x-l)dx \neq 0, \quad \text{for } i = 0, 1 \text{ and } k, l \in Z. \tag{3.54}$$

As is observed from Figures 3.17 and 3.18, ψ_0 is symmetry and ψ_1 is anti-symmetry.

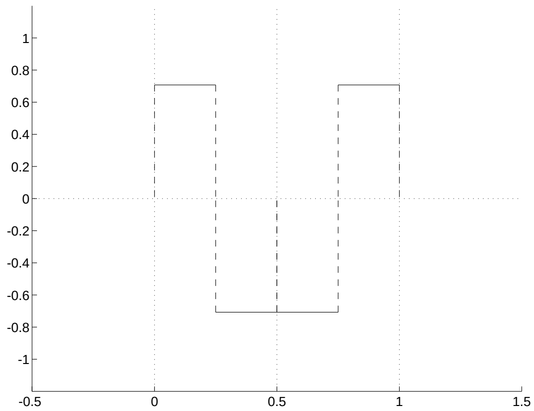

Figure 3.17: Flatlet wavelet $\psi_0(x)$

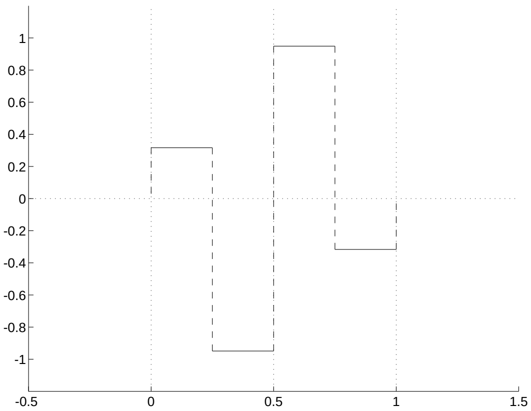

Figure 3.18: Flatlet wavelet $\psi_1(x)$

3.4.3 Decomposition relations

Let us rewrite the two-scale relations for both scaling functions (3.49) and wavelets (3.52) in the matrix form

$$\begin{bmatrix} \phi_0(x) \\ \phi_1(x) \\ \psi_0(x) \\ \psi_1(x) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} & \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} \phi_0(2x) \\ \phi_1(2x) \\ \phi_0(2x-1) \\ \phi_1(2x-1) \end{bmatrix} \tag{3.55}$$

which is called the reconstruction relation. Unlike Linear Legendre case, the square matrix in (3.55) is not an orthogonal matrix since $\psi_1(x)$ is not orthogonal to $\phi_i(x)$.

The decomposition relation is obtained by calculating the inverse of (3.55) and is as follows.

$$\begin{bmatrix} \phi_0(2x) \\ \phi_1(2x) \\ \phi_0(2x-1) \\ \phi_1(2x-1) \end{bmatrix} = \begin{bmatrix} \frac{5}{8} & -\frac{1}{8} & \frac{\sqrt{2}}{4} & \frac{\sqrt{10}}{8} \\ \frac{3}{8} & \frac{1}{8} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{10}}{8} \\ \frac{1}{8} & \frac{3}{8} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{10}}{8} \\ -\frac{1}{8} & \frac{5}{8} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{10}}{8} \end{bmatrix} \begin{bmatrix} \phi_0(x) \\ \phi_1(x) \\ \psi_0(x) \\ \psi_1(x) \end{bmatrix} \tag{3.56}$$

3.5 Second Order Flatlet Wavelets

We continue the examples of the Flatlet family. In this section, we will present the second order Flatlet wavelets.

3.5.1 Scaling functions

The second order Flatlet scaling functions consist of three piecewise basis functions defined in $[0, 1]$ shown in Figure 3.19.

$$\begin{cases} \phi_0(x) = 1 & \text{for } 0 \le x < \frac{1}{3} \\ \phi_1(x) = 1 & \text{for } \frac{1}{3} \le x < \frac{2}{3} \\ \phi_2(x) = 1 & \text{for } \frac{2}{3} \le x < 1 \end{cases} \tag{3.57}$$

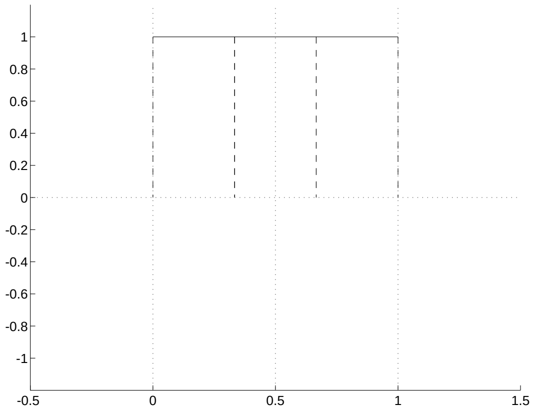

Figure 3.19: Second order Flatlet scaling functions $\phi_0(x)$, $\phi_1(x)$ and $\phi_2(x)$

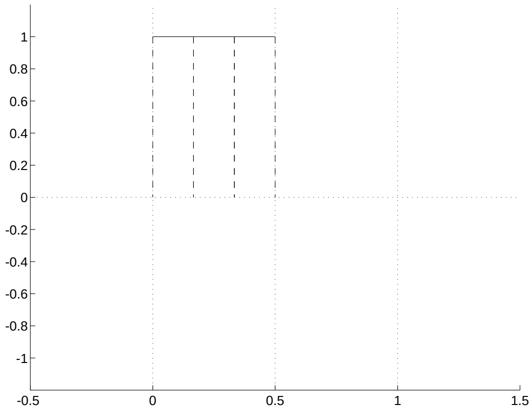

Figure 3.20: Dilates; $\phi_0(2x)$, $\phi_1(2x)$ and $\phi_2(2x)$

Their translates $\phi_0(x-k)$, $\phi_1(x-k)$ and $\phi_2(x-k)$, $k \in Z$ span a subspace V_0. The 1/2 scaled version, $\phi_0(2x)$, $\phi_1(2x)$ and $\phi_2(2x)$ are given by

$$\begin{cases} \phi_0(2x-k) = 1 & \text{for } \frac{k}{2} \le x < \frac{k}{2} + \frac{1}{6} \\ \phi_1(2x-k) = 1 & \text{for } \frac{k}{2} + \frac{1}{6} \le x < \frac{k}{2} + \frac{2}{6} \\ \phi_2(2x-k) = 1 & \text{for } \frac{k}{2} + \frac{2}{6} \le x < \frac{k}{2} + \frac{3}{6} \end{cases} \tag{3.58}$$

which span a subspace V_1 such that $V_0 \subset V_1$. Figure 3.20 shows $\phi_i(2x)$ for $i = 0, 1, 2$.

The two-scale relations for the second order Flatlet scaling functions are in the same form as (3.21). The two-scale sequence $\{p\}$ are easily determined and the two scale relation for the second order Flatlet scaling functions are as follows:

$$\begin{bmatrix} \phi_0(x) \\ \phi_1(x) \\ \phi_2(x) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \phi_0(2x) \\ \phi_1(2x) \\ \phi_2(2x) \\ \phi_0(2x-1) \\ \phi_1(2x-1) \\ \phi_2(2x-1) \end{bmatrix} \tag{3.59}$$

All the three scaling fuctions $\phi_0(x)$, $\phi_1(x)$ and $\phi_2(x)$ are orthogonal each other like other scaling functions on a bounded interval we have seen so far.

$$\int_{-\infty}^{\infty} \phi_i(x-k)\phi_j(x-l)dx = 0, \quad \text{for } i, j = 0, 1, 2, \quad i \neq j, \quad \text{for } k, l \in Z \tag{3.60}$$

Note that the magnitudes of ϕ_0 and ϕ_1 are not 1 but $\frac{1}{3}$, that is,

$$\int_{-\infty}^{\infty} \phi_i(x)dx = \int_{-\infty}^{\infty} \{\phi_i(x)\}^2 \, dx = \frac{1}{3} \tag{3.61}$$

3.5.2 Wavelets

Let us construct the second order Flatlet wavelets in the same way as we discussed in previous sections. The two-scale relations for the wavelets are in the same form as the Quadratic Legendre wavelets (3.25).

We also use the same 18 conditions as used for Quadratic Legendre wavelets shown in Table 3.2 except that the wavelets are normalized to $\frac{1}{3}$ instead of 1.

The solution of the two-scale relation for the Quadratic Legendre wavelets are shown in (3.62). Note that one of the two possible signs has been chosen for each $\psi(x)$ for simplicity. For this reason, multiplying -1 to all the 6 elements in any row of the matrix in equation (3.62) yields the valid solutions as well.

$$\begin{bmatrix} \psi_0(x) \\ \psi_1(x) \\ \psi_2(x) \end{bmatrix} = \begin{bmatrix} \frac{5}{3\sqrt{10}} & -\frac{7}{3\sqrt{10}} & -\frac{4}{3\sqrt{10}} & \frac{4}{3\sqrt{10}} & \frac{7}{3\sqrt{10}} & -\frac{5}{3\sqrt{10}} \\ \frac{1}{\sqrt{14}} & -\frac{3}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{2}{\sqrt{14}} & -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{14}} \\ \frac{1}{3\sqrt{14}} & -\frac{5}{3\sqrt{14}} & \frac{10}{3\sqrt{14}} & -\frac{10}{3\sqrt{14}} & \frac{5}{3\sqrt{14}} & -\frac{1}{3\sqrt{14}} \end{bmatrix} \begin{bmatrix} \phi_0(2x) \\ \phi_1(2x) \\ \phi_2(2x) \\ \phi_0(2x-1) \\ \phi_1(2x-1) \\ \phi_2(2x-1) \end{bmatrix} \tag{3.62}$$

The graphs for $\psi_0(x)$, $\psi_1(x)$ and $\psi_2(x)$ are shown in Figure 3.21, 3.22 and 3.23, respectively. Note that any scaling function $\phi(x)$ and $\psi(x)$ is not orthogonal except that $\psi_0(x)\perp\phi_1(x)$ and $\psi_2(x)\perp\phi_1(x)$.

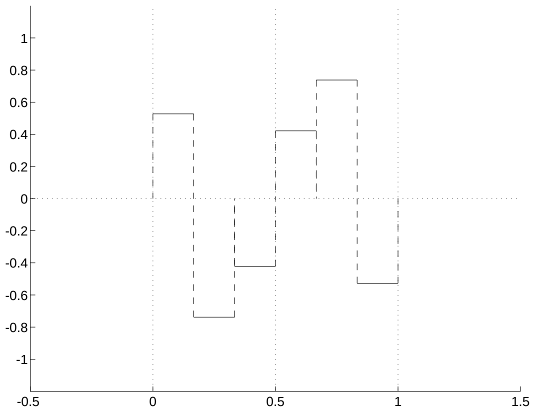

Figure 3.21: Second order Flatlet wavelet $\psi_0(x)$

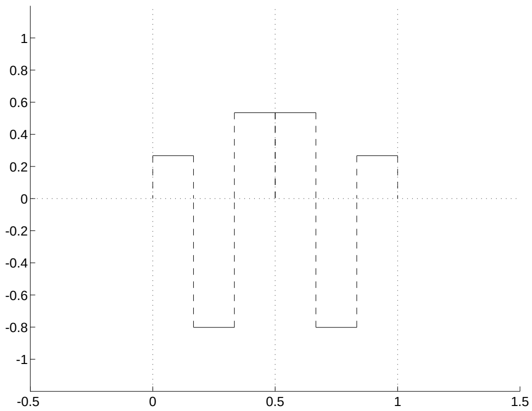

Figure 3.22: Second order Flatlet wavelet $\psi_1(x)$

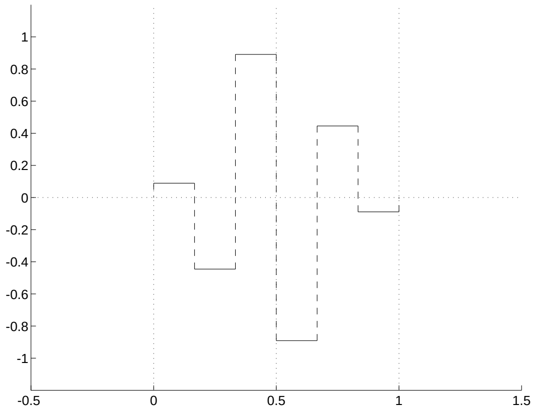

Figure 3.23: Second order Flatlet wavelet $\psi_2(x)$

3.5.3 Decomposition relations

From the both two-scale relations (3.59) and (3.62) for the scaling functions and wavelets, the reconstruction relations are given by:

$$\begin{bmatrix} \phi_0(x) \\ \phi_1(x) \\ \phi_2(x) \\ \psi_0(x) \\ \psi_1(x) \\ \psi_2(x) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ \frac{5}{3\sqrt{10}} & -\frac{7}{3\sqrt{10}} & -\frac{4}{3\sqrt{10}} & \frac{4}{3\sqrt{10}} & \frac{7}{3\sqrt{10}} & -\frac{5}{3\sqrt{10}} \\ \frac{1}{\sqrt{14}} & -\frac{3}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{2}{\sqrt{14}} & -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{14}} \\ \frac{1}{3\sqrt{14}} & -\frac{5}{3\sqrt{14}} & \frac{10}{3\sqrt{14}} & -\frac{10}{3\sqrt{14}} & \frac{5}{3\sqrt{14}} & -\frac{1}{3\sqrt{14}} \end{bmatrix} \begin{bmatrix} \phi_0(2x) \\ \phi_1(2x) \\ \phi_2(2x) \\ \phi_0(2x-1) \\ \phi_1(2x-1) \\ \phi_2(2x-1) \end{bmatrix} \tag{3.63}$$

The decomposition relation is simply the inverse of the reconstruction relation (3.63) and is shown in (3.64). Note that unlike the Quadratic Legendre case, the square matrix in (3.63) is not an orthogonal matrix.

$$\begin{bmatrix} \phi_0(2x) \\ \phi_1(2x) \\ \phi_2(2x) \\ \phi_0(2x-1) \\ \phi_1(2x-1) \\ \phi_2(2x-1) \end{bmatrix} = \begin{bmatrix} \frac{11}{16} & -\frac{1}{4} & \frac{1}{16} & \frac{5\sqrt{10}}{48} & \frac{\sqrt{14}}{8} & \frac{\sqrt{14}}{24} \\ \frac{5}{16} & \frac{1}{4} & -\frac{1}{16} & -\frac{5\sqrt{10}}{48} & -\frac{\sqrt{14}}{8} & -\frac{\sqrt{14}}{24} \\ \frac{1}{16} & 1 & -\frac{1}{16} & -\frac{\sqrt{10}}{16} & 0 & \frac{\sqrt{14}}{8} \\ -\frac{1}{16} & 1 & \frac{1}{16} & \frac{\sqrt{10}}{16} & 0 & -\frac{\sqrt{14}}{8} \\ -\frac{1}{16} & \frac{1}{4} & \frac{5}{16} & \frac{5\sqrt{10}}{48} & -\frac{\sqrt{14}}{8} & \frac{\sqrt{14}}{24} \\ \frac{1}{16} & -\frac{1}{4} & \frac{11}{16} & -\frac{5\sqrt{10}}{48} & \frac{\sqrt{14}}{8} & -\frac{\sqrt{14}}{24} \end{bmatrix} \begin{bmatrix} \phi_0(x) \\ \phi_1(x) \\ \phi_2(x) \\ \psi_0(x) \\ \psi_1(x) \\ \psi_2(x) \end{bmatrix} \tag{3.64}$$

3.6 General Order Flatlet Wavelets

To construct higher order Flatlet wavelets, we can follow the same procedures as described in the previous two sections. In this section, we will summarize the definitions of the general order Flatlet scaling functions and their properties. And then we will give procedures to obtain the solutions to their corresponding wavelets.

3.6.1 Scaling functions

The m-th order Flatlet scaling functions are defined as a set of $m+1$ unit constant functions $\phi_0(x), \ldots, \phi_m(x)$ divided equally into $m+1$ intervals on $[0,1]$:

$$\begin{cases} \phi_0(x) = 1 & \text{for } 0 \leq x < \frac{1}{m+1} \\ \quad\vdots & \\ \phi_i(x) = 1 & \text{for } \frac{i}{m+1} \leq x < \frac{i+1}{m+1} \\ \quad\vdots & \\ \phi_m(x) = 1 & \text{for } \frac{m}{m+1} \leq x < 1 \end{cases} \tag{3.65}$$

Because of the simplicity of the Flatlet scaling functions, the two-scale relations for general order Flatlet scaling functions are easily obtained in the following simple form:

$$\begin{bmatrix} \phi_0(x) \\ \vdots \\ \phi_m(x) \end{bmatrix} = \begin{bmatrix} 1 & 1 & & & & 0 \\ & & 1 & 1 & & \\ & & & & \ddots & \\ 0 & & & & 1 & 1 \end{bmatrix} \begin{bmatrix} \phi_0(2x) \\ \vdots \\ \phi_m(2x) \\ \phi_0(2x-1) \\ \vdots \\ \phi_m(2x-1) \end{bmatrix} \tag{3.66}$$

where the size of the matrix containig 0s and 1s is $(m+1) \times 2(m+1)$. The explicit formula for the two-scale relation for $\phi_i(x)$ for general order m is given as:

$$\phi_i(x) = \begin{cases} \phi_{2i}(2x) + \phi_{2i+1}(2x) & \text{if } \ 2i \leq m-1 \\ \phi_m(2x) + \phi_0(2x-1) & \text{if } \ 2i = m \\ \phi_{2i-m-1}(2x-1) + \phi_{2i-m}(2x-1) & \text{if } \ 2i \geq m+1 \end{cases} \tag{3.67}$$

3.6.2 Wavelets

The two-scale relations for the m-th order Legendre wavelets are in the form:

$$\psi_i(x) = \sum_{j=0}^{m} q_{i,j}\phi_j(2x) + \sum_{j=0}^{m} q_{i,m+j+1}\phi_j(2x-1) \tag{3.68}$$

which is the same as (3.39). We can construct the Flatlet wavelets using the same $2(m+1)^2$ conditions as we used for the Legendre wavelets; $2m(m+1)$ vanishing moment conditoins (3.40) and $2(m+1)$ orthonormal conditions (3.41).

Thanks to the simplicity of the Flatlet scaling functions, the vanishing moment condition (3.40) for general case is computed as follows:

$$\int_{-\infty}^{\infty} \psi_i(x) x^j dx = 0 \Longleftrightarrow \sum_{k=0}^{2(m+1)} \left\{ (k+1)^{j+1} - k^{j+1} \right\} q_{i,k} = 0 \tag{3.69}$$

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Appendix A. Two-scale relation for m-th order B-spline scaling functions

The objective here is to derive an explicit formula of two-scale sequence $\{p_{m,k}\}$ in the two-scale relation for B-splines $N_m(x)$ of order m (degree $m-1$):

$$N_m(x) = \sum_{k=-\infty}^{\infty} p_{m,k} N_m(2x-k). \tag{A.1}$$

Fourier Transformation is useful to get a closed solution for $p_{m,k}$. First, let $\hat{N}_m(w)$ be the Fourier Transformation of $N_m(x)$, namely,

$$\hat{N}_m(w) = \int_{-\infty}^{\infty} e^{-iwx} N_m(x)dx \tag{A.2}$$

For $m = 1$, using the definition of $N_1(x)$ in Equation (2.35) we have

$$\hat{N}_1(w) = \frac{1-e^{-iw}}{iw} \tag{A.3}$$

Noting the convolution property in Fourier domain:

$$(\hat{f} * \hat{g})(w) = \hat{f}(w)\hat{g}(w) \tag{A.4}$$

the Fourier Transformation of $N_m(x)$ that is recursively defined by convolution in Equation (2.36) can be obtained as follows:

$$\hat{N}_m(w) = \left(\frac{1-e^{-iw}}{iw}\right)^m. \tag{A.5}$$

Taking the Fourier Transformation on both sides of (A.1), we obtain the following equivalent formulation:

$$\hat{N}_m(w) = \frac{1}{2}\left(\sum_{k=-\infty}^{\infty} p_{m,k} e^{-\frac{ikw}{2}}\right)\hat{N}_m(\frac{w}{2}) \tag{A.6}$$

Substituting (A.5) into (A.6) yields,

$$\left(\frac{1-e^{-iw}}{iw}\right)^m = \frac{1}{2}\left(\sum_{k=-\infty}^{\infty} p_{m,k} e^{-\frac{ikw}{2}}\right)\left(\frac{1-e^{-iw/2}}{iw/2}\right)^m$$

Therefore,

$$\begin{aligned}\frac{1}{2}\left(\sum_{k=-\infty}^{\infty} p_{m,k} e^{-\frac{ikw}{2}}\right) &= \left(\frac{1-e^{-iw}}{iw}\right)^m \left(\frac{iw/2}{1-e^{-iw/2}}\right)^m \\ &= \left(\frac{1+e^{-iw/2}}{2}\right)^m\end{aligned}$$

$$= 2^{-m} \sum_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array} \right) e^{-\frac{ikw}{2}}$$

Comparing both sides of the above equations, $p_{m,k}$ is obtained as

$$p_{m,k} = \begin{cases} 2^{-m+1} \left(\begin{array}{c} m \\ k \end{array} \right) & \text{for } 0 \leq k < m \\ 0 & \text{otherwise} \end{cases} \tag{A.7}$$

Consequently, the precise formulation of (A.1) is given by

$$N_m(x) = \sum_{k=0}^{m} 2^{-m+1} \left(\begin{array}{c} m \\ k \end{array} \right) N_m(2x - k) \tag{A.8}$$

which is the two-scale relation for B-Splines of general order m.

Appendix B. Fundamental Cardinal Splines

As is stated in subsection 2.3.5, the m-th order Fundamental Cardinal Spline (FCS) function $L_m(x)$ is defined as:

$$L_m(x) = \sum_{k=-\infty}^{\infty} c_{k,m} N_m(x + \frac{m}{2} - k) \tag{B.1}$$

such that

$$L_m(j) = \delta_{j,0}, \quad j \in Z \tag{B.2}$$

that is, FCS is defined as series formed by integer translates of m-th order B-spline functions. Note that the term $\frac{m}{2}$ translates the B-spline so that it is symmetric along 0.

From the definition above, FCS takes the value 1 at 0 and is 0 at all other integers. Thanks to this property, FCS is useful to interpolate any data given at each integer interval. Suppose $\{f(k)\}$, $k \in Z$ are given, $\sum_k f(k)L_m(x-k)$ interpolates $\{f(k)\}$ using m-th order fundamental cardinal splines which are defined in terms of m-th order B-spline functions.

The derivation of the coefficients $\{c_{k,m}\}$ for $m = 4$ was shown in subsection 2.4.3. The closed form solutions for $\{c_{k,m}\}$, $m = 2, 3, 4$ are listed here:

$$\begin{aligned} c_{k,2} &= \delta_{k,0} \\ c_{k,3} &= (-1)^k \sqrt{2}(3 - 2\sqrt{2})^{|k|} \\ c_{k,4} &= (-1)^k \sqrt{3}(2 - \sqrt{3})^{|k|} \end{aligned} \tag{B.3}$$

Another way of viewing $\{c_{k,m}\}$ is to form the following bi-infinite system of equations:

$$\sum_{k=-\infty}^{\infty} c_{k,m} N_m(\frac{m}{2} + j - k) = \delta_{j,0}, \quad j \in Z \tag{B.4}$$

that is written explicitly for $m = 4$ as:

$$\begin{bmatrix} \ddots & & & & & 0 \\ & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & & \\ & & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & \\ 0 & & & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \tag{B.5}$$

In contrast to B-spline $N_m(x)$ which has compact support, we see that the coefficient sequence $\{c_{k,m}\}$ is not finite for each $m \geq 3$, so that FCS $L_m(x)$ does not vanish identically outside any compact set. However, the coefficients $\{c_{k,m}\}$ decays to zero exponentially fast as $k \to \infty$. This implies that the FCS $L_m(x)$ also decays to zero at the same rate as $x \to \pm\infty$.

The Quadratic ($m = 3$) and Cubic ($m = 4$) FCS are shown in Figure B.1 and B.2, respectively. The Linear FCS is merely identical to Linear B-spline function.

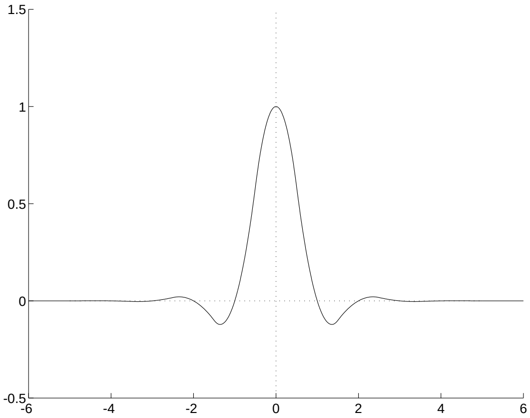

Figure B.1: Quadratic Fundamental Cardinal Spline $L_3(x)$

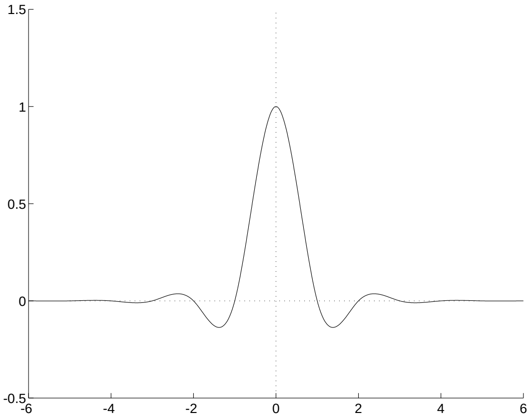

Figure B.2: Cubic Fundamental Cardinal Spline $L_4(x)$

Appendix C. Derivation of $\{c_{l,6}\}$ for the Decomposition Relation of Quadratic B-spline

We show how to derive $\{c_{l,6}\}$ for the Decomposition relation of Quadratic B-spline. The non-zero values of $N_6(k)$, $k \in Z$ are provided in Table 2.1:

$$N_6(1) = \tfrac{1}{120}, \quad N_6(2) = \tfrac{26}{120}, \quad N_6(3) = \tfrac{66}{120}, \quad N_6(4) = \tfrac{26}{120}, \quad N_6(5) = \tfrac{1}{120}$$

Then $\{c_{l,6}\}$ is computed as follows:

$$\begin{aligned}
\sum_k c_{k,6} z^k &= \frac{1}{\sum_k N_6(k+3) z^k} \\
&= \frac{1}{\frac{1}{120}z^{-2} + \frac{26}{120}z^{-1} + \frac{66}{120} + \frac{26}{120}z + \frac{1}{120}z^2} \\
&= \frac{120}{(z^2 + z^{-2}) + 26(z + z^{-1}) + 66} \\
&= \frac{120}{(z + z^{-1})^2 + 26(z + z^{-1}) + 64} \\
&= \frac{120}{y^2 + 26y + 64}, \quad \text{where } y = z + z^{-1} \\
&= \frac{120}{(y-\alpha_1)(y-\alpha_2)}, \quad \text{where } \alpha_1, \alpha_2 = -13 \pm \sqrt{105}, \quad \alpha_1 > \alpha_2 \\
&= \frac{120}{\alpha_1 - \alpha_2}\left(\frac{1}{y-\alpha_1} - \frac{1}{y-\alpha_2}\right) \\
&= \frac{120}{\alpha_1 - \alpha_2}\left(\frac{z}{z^2 - \alpha_1 z + 1} - \frac{z}{z^2 - \alpha_2 z + 1}\right) \\
&= \frac{120}{\alpha_1 - \alpha_2}\left\{\frac{z}{(z-\beta_1)(z-\beta_2)} - \frac{z}{(z-\gamma_1)(z-\gamma_2)}\right\}, \\
&\quad \text{where } \beta_1, \beta_2 = \tfrac{1}{2}(\alpha_1 \pm \sqrt{\alpha_1^2 - 4}), \quad \beta_1 > \beta_2 \\
&\quad \text{and } \gamma_1, \gamma_2 = \tfrac{1}{2}(\alpha_2 \pm \sqrt{\alpha_2^2 - 4}), \quad \gamma_1 > \gamma_2 \\
&= \frac{120}{\alpha_1 - \alpha_2}\left\{\frac{1}{\beta_1 - \beta_2}\left(\frac{\beta_1}{z-\beta_1} - \frac{\beta_2}{z-\beta_2}\right) - \frac{1}{\gamma_1 - \gamma_2}\left(\frac{\gamma_1}{z-\gamma_1} - \frac{\gamma_2}{z-\gamma_2}\right)\right\}
\end{aligned}$$

$$\begin{aligned}
&= \frac{120}{\alpha_1 - \alpha_2}\left\{\frac{1}{\beta_1 - \beta_2}\left(\frac{(\frac{\beta_1}{z})}{1 - (\frac{\beta_1}{z})} + \frac{1}{1 - (\frac{z}{\beta_2})}\right) - \frac{1}{\gamma_1 - \gamma_2}\left(\frac{(\frac{\gamma_1}{z})}{1 - (\frac{\gamma_1}{z})} + \frac{1}{1 - (\frac{z}{\gamma_2})}\right)\right\} \\
&= \frac{120}{\alpha_1 - \alpha_2}\left\{\frac{1}{\beta_1 - \beta_2}\left(\sum_{n=1}^{\infty}(\frac{\beta_1}{z})^n + \sum_{n=0}^{\infty}(\frac{z}{\beta_2})^n\right) - \frac{1}{\gamma_1 - \gamma_2}\left(\sum_{n=1}^{\infty}(\frac{\gamma_1}{z})^n + \sum_{n=0}^{\infty}(\frac{z}{\gamma_2})^n\right)\right\}
\end{aligned}$$

Since $\beta_1\beta_2 = \gamma_1\gamma_2 = 1$

$$\begin{aligned}
&= \frac{120}{\alpha_1 - \alpha_2}\left\{\frac{1}{\beta_1 - \beta_2}\left(\sum_{n=-\infty}^{-1}(\frac{z}{\beta_1})^n + \sum_{n=0}^{\infty}(\beta_1 z)^n\right) - \frac{1}{\gamma_1 - \gamma_2}\left(\sum_{n=-\infty}^{-1}(\frac{z}{\gamma_1})^n + \sum_{n=0}^{\infty}(\gamma_1 z)^n\right)\right\} \\
&= \frac{120}{\alpha_1 - \alpha_2}\left\{\frac{1}{\beta_1 - \beta_2}\sum_{n=-\infty}^{\infty}\beta_1^{|n|} z^n - \frac{1}{\gamma_1 - \gamma_2}\sum_{n=-\infty}^{\infty}\gamma_1^{|n|} z^n\right\} \\
&= \frac{120}{\alpha_1 - \alpha_2}\sum_{n=-\infty}^{\infty}\left(\frac{\beta_1^{|l|}}{\beta_1 - \beta_2} - \frac{\gamma_1^{|l|}}{\gamma_1 - \gamma_2}\right) z^n
\end{aligned}$$

Therefore, the closed form solution for $c_{l,6}$ is:

$$c_{l,6} = \frac{120}{\alpha_1 - \alpha_2} \left(\frac{\beta_1^{|l|}}{\beta_1 - \beta_2} - \frac{\gamma_1^{|l|}}{\gamma_1 - \gamma_2} \right) \tag{C.1}$$

where

$$\alpha_1, \alpha_2 = -13 \pm \sqrt{105}, \quad \alpha_1 > \alpha_2$$
$$\beta_1, \beta_2 = \frac{1}{2}(\alpha_1 \pm \sqrt{\alpha_1^2 - 4}), \quad \beta_1 > \beta_2$$
$$\gamma_1, \gamma_2 = \frac{1}{2}(\alpha_2 \pm \sqrt{\alpha_2^2 - 4}), \quad \gamma_1 > \gamma_2$$