# POINTWISE CONVERGENCE OF WAVELET EXPANSIONS

SUSAN E. KELLY, MARK A. KON, AND LOUISE A. RAPHAEL

Abstract. In this note we announce that under general hypotheses, wavelet-type expansions (of functions in $L^p$, $1 \leq p \leq \infty$, in one or more dimensions) converge pointwise almost everywhere, and identify the Lebesgue set of a function as a set of full measure on which they converge. It is shown that unlike the Fourier summation kernel, wavelet summation kernels $P_j$ are bounded by radial decreasing $L^1$ convolution kernels. As a corollary it follows that best $L^2$ spline approximations on uniform meshes converge pointwise almost everywhere. Moreover, summation of wavelet expansions is partially insensitive to order of summation.

We also give necessary and sufficient conditions for given rates of convergence of wavelet expansions in the sup norm. Such expansions have order of convergence $s$ if and only if the basic wavelet $\psi$ is in the homogeneous Sobolev space $H_h^{-s-d/2}$. We also present equivalent necessary and sufficient conditions on the scaling function. The above results hold in one and in multiple dimensions.

## 1. Introduction

In this note we present several convergence results for wavelet and multiresolution-type expansions. It is natural to ask where such expansions converge (and whether they converge almost everywhere) and what are the rates of convergence. The answer is that under rather weak hypotheses, one- and multidimensional wavelet expansions converge pointwise almost everywhere and, more specifically, on the Lebesgue set of a function being expanded. We will also give exact rates of convergence in the supremum norm, in terms of Sobolev properties of the basic wavelet or of the scaling function.

Wavelets with local support in the time and frequency domains were defined by A. Grossman and J. Morlet [GM] in 1984 in order to analyze seismic data. The prototypes of wavelets, however, can be found in the work of A. Haar [Ha] and the modified Franklin systems of J.-O. Strömberg [St].

To identify the underlying structure and to generate interesting examples of orthonormal bases for $L^2(\mathbf{R})$, S. Mallat [Ma] and Y. Meyer [Me] developed the framework of multiresolution analysis. P. G. Lemarié and Y. Meyer [LM] constructed wavelets in $\mathcal{S}(\mathbf{R}^n)$, the space of rapidly decreasing smooth functions. J.-O. Strömberg [St] developed spline wavelets while looking for unconditional bases for Hardy spaces. G. Battle [Ba] and P. G. Lemarié [Le] developed these bases in the context of wavelet theory. Spline wavelets have exponential decay but only $C^N$ smoothness (for a finite $N$ depending on the order of the associated splines). I. Daubechies [Da1] constructed compactly supported wavelets with $C^N$ smoothness. The support of these wavelets increased with the smoothness; in general, to have $C^\infty$ smoothness, wavelets must have infinite support.

Y. Meyer [Me] was among the first to study convergence results for wavelet expansions; he was followed by G. Walter [Wa1, Wa2]. Meyer proved that under some regularity assumptions on the wavelets, wavelet expansions of continuous functions converge everywhere. In contrast to these results, the pointwise convergence results presented here give almost everywhere convergence (and convergence on the Lebesgue set) for expansions of general $L^p$ $(1 \leq p \leq \infty)$ functions. We assume rather mild bounds and no differentiability for the wavelet or the scaling function; our conditions allow inclusion of the families of so-called $r$-regular wavelets [Me], as well as some other wavelets.

These results parallel L. Carleson's [Ca] and R. A. Hunt's [Hu] theorems for Fourier series. One difference and slight advantage of wavelet expansions comes from the fact that almost everywhere convergence occurs on a simple set of full measure, namely the Lebesgue set, while almost everywhere convergence for Fourier series is established on a much more elaborate set of full measure.

We also give necessary and sufficient conditions for given pointwise (sup-norm) rates of convergence of wavelet or multiresolution expansions, in terms of Sobolev conditions on the basic wavelet or the scaling function. It has been shown previously by Mallat [Ma] and Meyer [Me] that the Sobolev class of a function is determined by the $L^2$ rates of convergence of its wavelet expansion. Necessary and sufficient conditions for $L^2$ rates of convergence which are analogous to our sup-norm conditions have been obtained by de Boor, DeVore, and Ron [BDR], who have also studied sup-norm convergence in more general situations [BR].

Our results on convergence rates can be viewed as a sharpening in the context of wavelets of the well-known Strang-Fix [SF] conditions for convergence of multiscale expansions.

The results given here hold for multiresolution expansions in multiple dimension. The proofs, which will appear elsewhere [KKR], involve the kernels of the partial sums of such expansions and the above-mentioned result that such kernels are bounded by rescalings of $L^1$ radial functions. We should add that such bounds for wavelet expansions are nontrivial and arise from cancellations which occur in the sum representations of the partial sum kernels. Naive bounding of the summation kernels by using absolute values in their sum representations fails to yield the needed radial bounds for any class of wavelets. We remark that such bounds on the summation kernel can be obtained more easily by writing it using the orthonormal translates $\phi(x-k)$ of the scaling function, instead of the wavelets $\psi_{jk}$. However, in proving results for convergence of wavelet expansions (Theorem 2.1(iii)), we wish to avoid making any assumptions about radial bounds for the scaling function.

The pointwise and $L^p$ convergence results contained here were obtained independently by the first author and by the second two authors. Results on the Gibbs effect obtained by the first author will appear elsewhere.

To start, we define a multiresolution analysis on $L^2(\mathbf{R}^d)$ [Ma, Me].

**Definition 1.1.** A *multiresolution analysis* on $L^2(\mathbf{R^d})$, $d \geq 1$, is an increasing sequence $\{V_j\}_{j\in\mathbf{Z}}$,

$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots$$

of closed subspaces in $L^2(\mathbf{R}^d)$ where

$$\bigcap_{j\in\mathbf{Z}} V_j = \{0\}\,, \qquad \overline{\bigcup_{j\in\mathbf{Z}} V_j} = L^2(\mathbf{R}^d)\,,$$

and the spaces $V_j$ satisfy the following additional properties:
(i) For all $f \in L^2(\mathbf{R}^d)$, $j \in \mathbf{Z}$, and $k \in \mathbf{Z}^d$,

$$f(x) \in V_j \Longleftrightarrow f(2x) \in V_{j+1}$$

and

$$f(x) \in V_0 \Longleftrightarrow f(x-k) \in V_0 .$$

(ii) There exists a *scaling function* $\phi \in V_0$ such that $\{\phi_{jk}\}_{k\in\mathbf{Z}^d}$ is an orthonormal basis of $V_j$, where

$$\phi_{jk}(x) = 2^{jd/2}\phi(2^j x - k)\,, \tag{1}$$

for $x \in \mathbf{R}^d$, $j \in \mathbf{Z}$, and $k \in \mathbf{Z}^d$.

Associated with the $V_j$ spaces, we additionally define $W_j$ to be the orthogonal complement of $V_j$ in $V_{j+1}$, so that $V_{j+1} = V_j \oplus W_j$. Thus, $L^2(\mathbf{R}^d) = \overline{\sum \oplus W_j}$. We define $P_j$ and $Q_j = P_{j+1} - P_j$, respectively, to be the orthogonal projections onto the spaces $V_j$ and $W_j$, with kernels $P_j(x,y)$ and $Q_j(x,y)$.

Under the assumptions in the above definition and with some additional regularity, it can be proved [Me, Da2] that there then exists a set $\{\psi^\lambda\} \in W_0$, where $\lambda$ belongs to an index set $\Lambda$ of cardinality $2^d - 1$, such that $\{\psi^\lambda_{jk}\}_{k\in\mathbf{Z}^d,\lambda}$ is an orthonormal basis of $W_j$, and thus $\{\psi^\lambda_{jk}\}_{j\in\mathbf{Z},k\in\mathbf{Z}^d,\lambda}$ is a *wavelet basis* of $L^2(\mathbf{R}^d)$, where

$$\psi^\lambda_{jk}(x) = 2^{jd/2}\psi^\lambda(2^j x - k)\,, \tag{2}$$

for $x \in \mathbf{R}^d$, $j \in \mathbf{Z}$, and $k \in \mathbf{Z}^d$.

**Definition 1.2.** For $f \in L^p(\mathbf{R}^d)$ $(1 \le p \le \infty)$ we define the following related expansions:

(a) The sequence of projections $\{P_j f(x)\}_j$ will be called the *multiresolution expansion* of $f$.

(b) The *scaling expansion* of $f$ is defined to be

$$f \sim \sum_k b_k \phi(x-k) + \sum_{j\ge 0;k;\lambda} a_{jk}\psi^\lambda_{jk}(x)\,, \tag{3}$$

where the coefficients $a_{jk}$ and $b_k$ are the $L^2$ expansion coefficients of $f$.

(c) The *wavelet expansion* of $f$ is

$$f \sim \sum_{j;k;\lambda} a_{jk}\psi^\lambda_{jk}(x)\,, \tag{4}$$

where the coefficients $a_{jk}$ are the $L^2$ expansion coefficients of $f$.

We remark for part (a) of the above definition that it can be shown that the projections $P_j$ (defined by their integral kernels) extend to bounded operators on $L^p$, $1 \le p \le \infty$. The $L^2$ expansion coefficients in (b) and (c) (defined by integration against $f$) are defined and uniformly bounded for any $f \in L^p$, $1 \le p \le \infty$.

**Definition 1.3.** The point $x$ is a *Lebesgue point* of the measurable function $f(x)$ on $\mathbf{R}^d$ if $f$ is integrable in some neighborhood of $x$, and

$$\lim_{\epsilon\to 0}\frac{1}{V(B_\epsilon)}\int_{B_\epsilon}|f(x) - f(x+y)|\,dy = 0,$$

where $B_\epsilon$ denotes the ball of radius $\epsilon$ about the origin, and $V$ denotes volume.

Such points $p$ are essentially characterized by the fact that the average values of $f(x)$ around $p$ converge to the values of $f$ at these points, as averages are taken over smaller balls centered at $x$. Note that all continuity points are Lebesgue points, but the converse is not true.

**Definition 1.4.** A function $f(x)$ is in the class $\mathcal{RB}$ if it is absolutely bounded by an $L^1$ radial decreasing function $\eta(x)$, i.e., $\eta(x_1) = \eta(x_2)$ whenever $|x_1| = |x_2|$, $\eta(x_1) \leq \eta(x_2)$ when $|x_1| \geq |x_2|$, and $\eta(x) \in L^1(\mathbf{R}^d)$.

A function $f$ is *partially continuous* if there exists a set $A$ of vectors $a \in \mathbf{R}^d$ with positive measure such that $\lim_{\epsilon \to 0} \psi(x + \epsilon a) = \psi(x)$ for $a \in A$.

**Definition 1.5.** The *homogeneous Sobolev space* of order $s$ is defined by

$$H_h^s \equiv \left\{ f \in L^2(\mathbf{R}^d) : \; ||f||_{h,s} \equiv \sqrt{\int |\hat{f}(\xi)|^2 |\xi|^{2s} \, d\xi} < \infty \right\}. \tag{5}$$

The ordinary Sobolev space $H^s$ is defined as $H_h^s$ in (5), with replacement of $|\xi|^{2s}$ by $(|\xi|^{2s} + 1)$. Under the Fourier transform, the space $H_h^s$ is a dense subspace of the complete weighted $L^2$ space of *all* measurable $\hat{f}(\xi)$ with $||f||_{h,s} < \infty$. This dense subspace consists of those functions $\hat{f}(\xi)$ which are also in the regular unweighted $L^2$ space.

Convergence rates of wavelet expansions are sensitive to both the smoothness of the wavelet and the smoothness of the function being expanded. For a wavelet $\psi$ of given smoothness, the sensitivity to the Sobolev space of the function being expanded disappears when the function's Sobolev parameter is sufficiently large.

**Definition 1.6.** A family $\psi^\lambda$ of wavelets yields *pointwise order of approximation* (or *pointwise order of convergence*) $r$ in the space $H^s$ if for any function $f \in H^s$, the $j$th order wavelet approximation $P_j f$ satisfies

$$||P_j f - f||_\infty = O(2^{-jr}), \tag{6}$$

as $j$ tends to infinity. More generally, the wavelets $\psi^\lambda$ yield pointwise order of approximation (or convergence) $r$ if for any function $f$ which is sufficiently smooth (i.e., is in a Sobolev space of sufficiently large order $s$), (6) holds.

We will give several necessary and sufficient conditions (in terms of their Fourier transforms and membership in homogeneous Sobolev spaces) on the basic wavelet $\psi$ or the scaling function $\phi$ for given orders of convergence. In practice, sufficient smoothness for a function $f$ (in the sense of the above definition) will mean that $f$ is in $H^{s+d/2}$ or a higher Sobolev space.

## 2. Pointwise convergence results

With the background given, our main results can now be presented.

**Theorem 2.1.** (i) *Assume only that the scaling function $\phi$ of a given multiresolution analysis is in $\mathcal{RB}$, i.e., that it is bounded by an $L^1$ radial decreasing function. Then for any $f \in L^p(\mathbf{R}^d)$ $(1 \leq p \leq \infty)$, its multiresolution expansion $\{P_j f\}$ converges to $f$ pointwise almost everywhere.*

(ii) *If $\phi, \psi^\lambda \in \mathcal{RB}$ for all $\lambda$, then also both the scaling (3) (if $1 \leq p \leq \infty$) and wavelet (4) (if $1 \leq p < \infty$) expansions of any $f \in L^p$ converge to $f$ pointwise almost everywhere. If further $\psi^\lambda$ and $\phi$ are (partially) continuous, then both expansions converge to $f$ on its Lebesgue set.*

(iii) *If we assume only* $\psi^\lambda(x)\ln(2+|x|) \in \mathcal{RB}$ *for all* $\lambda$, *then for* $f \in L^p$, *its wavelet (for* $1 \le p < \infty$*) and multiresolution (for* $1 \le p \le \infty$*) expansions converge to* $f$ *pointwise almost everywhere. If further* $\psi^\lambda$ *is (partially) continuous for all* $\lambda$, *then both the wavelet and multiresolution expansions converge to* $f$ *on its Lebesgue set.*

(iv) *The last two statements hold for any order of summation in which the range of the values of* $j$ *for which the sum over* $k$ *and* $\lambda$ *is partially complete always remains bounded.*

In statement (iv) above, the summation over $k$ and $\lambda$ is *partially complete* for a fixed $j$ if it contains some terms, but not all with the given value of $j$. By the *range of values* for which the sum is partially complete we mean the difference of the largest and smallest value of $j$ for which the sum is partially complete. Statement (iv) requires that this range always be smaller than some constant $M$.

The above result on convergence of multiresolution expansions applies to spline expansions as well. Given a uniform grid $K$ in $\mathbf{R}$, one might ask whether given a function $f \in L^2(\mathbf{R}^d)$, the best $L^2$ approximations $P_j f$ of $f$ (by splines of a fixed polynomial order $k$) converge to $f$ pointwise as the grid size goes to 0. The answer to this is affirmative.

**Corollary 2.2.** *For* $f \in L^p(\mathbf{R})$ $(1 \le p \le \infty)$, *the order* $k$ *best* $L^2$ *spline approximations* $P_j f$ *of* $f$ *converge to* $f$ *pointwise almost everywhere, and more specifically on the Lebesgue set of* $f$, *as the uniform mesh size goes to* 0.

The proof follows from the fact that best spline approximations are partial sums of multiresolution expansions, with some radially bounded ($\mathcal{RB}$) scaling function $\phi$. This result also extends to multidimensional splines. Technically, the best $L^2$ approximation of $f \in L^p$ only makes sense when $p = 2$, but it can be defined for functions in $L^p$ by continuous extension of the projections $P_j$ from $L^2$ to $L^p$.

The following proposition is a consequence of the proof of Theorem 2.1. It has been proved before under somewhat stronger hypotheses, yielding stronger conclusions in [Me].

**Proposition 2.3.** *Under the hypotheses of case* (i), *case* (ii), *or case* (iii) *of Theorem 2.1,* $L^p$ *convergence of the expansions also follows for* $1 \le p < \infty$.

Thus for wavelet series and more generally for one- and multidimensional multiresolution expansions, essentially all hoped for convergence properties hold, regardless of rates of convergence.

The basis for Theorem 2.1 is the bound on the kernel of the projection $P_j$ onto the scaling space $V_j$. It can be shown that under any of the hypotheses in Theorem 2.1, the kernel $P_m(x, y)$ has the form

$$P_m(x,y) = \sum_{j<m;\, k\in\mathbf{Z}^d;\, \lambda} \psi^\lambda_{jk}(x)\overline{\psi^\lambda_{jk}(y)} = \sum_{k\in\mathbf{Z}} \phi_{mk}(x)\overline{\phi_{mk}}(y)$$

for $x, y \in \mathbf{R}^d$, with convergence of both sums on the right occurring pointwise, uniformly on subsets a positive distance away from the diagonal $D = \{(x, y) : x = y\}$. The kernel converges to a delta distribution $\delta(x - y)$ in the following sense:

**Theorem 2.4.** *Under the assumption that* $\phi \in \mathcal{RB}$ *or that* $\psi^\lambda(x)\ln(2+|x|) \in \mathcal{RB}$ *for all* $\lambda$, *the kernels* $P_j(x, y)$ *of the projections onto* $V_j$ *satisfy the convolution* *bound*

$$|P_j(x,y)| \le C2^{jd}H(2^j(|x-y|)), \tag{7}$$

*where* $H(|\cdot|) \in \mathcal{RB}$, *i.e.,* $H(|\cdot|)$ *is a radial decreasing* $L^1$ *function.*

In one dimension, precise bounds can be obtained for kernels of specific wavelets. Two examples are illustrated in the result below.

**Theorem 2.5.** *In* $\mathbf{R}^1$, *let* $P_j(x,y) = \sum_{k\in\mathbf{Z}} \phi_{jk}(x)\phi_{jk}(y)$ *be the summation kernel, generated by the scaling function* $\phi \in L^2(\mathbf{R})$.

(a) *If* $\phi$ *has exponential decay, i.e.,* $\phi(x) \le Ce^{-a|x|}$ *for some positive* $a$, *then*

$$|P_j(x,y)| \le C2^j e^{-a2^j|x-y|/2}.$$

(b) *If* $\phi$ *has algebraic decay,* $\phi(x) \le \frac{C_N}{(1+|x|)^N}$ *for some* $N > 1$, *then*

$$|P_j(x,y)| \le C_N \frac{2^j}{(1+2^j|x-y|)^N} \le C_N 2^j$$

*for* $N > 1$.

As a corollary, if a scaling function $\phi$ has rapid decay (faster than any polynomial), then $P_j(x,y)$ is bounded by a scaled convolution kernel (of the form on the right side of (7)) which has rapid decay. Spline wavelets (see [St, Ba, Le]) satisfy the conditions in part (a), and wavelets constructed by P. G. Lemarié and Y. Meyer [LM] are an example of wavelets of rapid decay.

## 3. Rates of convergence

We will now give necessary and sufficient conditions on the basic wavelet $\psi^\lambda$ and on the scaling function $\phi$, for given supremum-norm rates of convergence of wavelet expansions. The conditions on $\psi^\lambda$ (here given in terms of membership in homogeneous Sobolev spaces) can be translated into differentiability and then moment conditions on $\psi^\lambda$ (see the introduction).

**Theorem 3.1.** *Given a multiresolution analysis with either* (i) *a scaling function* $\phi \in \mathcal{RB}$, (ii) *basic wavelets satisfying* $\psi^\lambda \ln(2+|x|) \in \mathcal{RB}$ *for each* $\lambda$, *or* (iii) *a kernel for the basic projection* $P$ *satisfying* $|P(x,y)| \le H(|x-y|)$ *with* $H \in \mathcal{RB}$, *the following conditions* ((a) *to* (e$'$)) *are equivalent*:

(a) *The multiresolution expansion* (*see Definition* 1.2) *yields pointwise order of approximation* $s$.

(a$'$) *The multiresolution expansion yields pointwise order of approximation* $s$ *in every Sobolev space* $H^r$ *for* $r \ge s + d/2$.

(b) *The projection* $I - P_j$: $H_h^{s+d/2} \to L^\infty$ *is a bounded operator, where* $I$ *is the identity and* $d$ *denotes dimension.*

*If there exists a family* $\{\psi^\lambda\}_\lambda \subset \mathcal{RB}$ *of basic wavelets corresponding to* $\{P_j\}$, *then*:

(c) *For every family of basic wavelets* $\{\psi^\lambda\}_\lambda \subset \mathcal{RB}$ *corresponding to* $\{P_j\}$, *and for each* $\lambda$, $\psi^\lambda \in H_h^{-s-d/2}$.

(d) *For every family of basic wavelets* $\{\psi^\lambda\}_\lambda \subset \mathcal{RB}$ *corresponding to* $\{P_j\}$, *and for each* $\lambda$,

$$\int_{|\xi|<\epsilon} |\psi^\lambda(\xi)|^2 |\xi|^{-2(s+d/2)}\, d\xi < \infty \tag{8}$$

*for some (or for all)* $\epsilon > 0$.

(d′) *For some family of basic wavelets* $\{\psi^\lambda\}_\lambda \subset \mathcal{RB}$ *corresponding to* $\{P_j\}$, *equation (8) holds.*

*If in addition there exists a scaling function* $\phi \in \mathcal{RB}$ *corresponding to some family of basic wavelets as above, then*:

(e) *For every such scaling function,*

$$\int_{|\xi|<\epsilon} \left((2\pi)^d |\hat{\phi}(\xi)|^2 - 1\right) |\xi|^{-2(s+d/2)}\, d\xi < \infty \tag{9}$$

*for some (or all)* $\epsilon > 0$.

(e′) *For some such scaling function* $\phi$, *equation (9) holds.*

The above conditions on the scaling function can also be given in modified form in the case where the scaling function $\phi$ has nonorthonormal integer translates. Necessary and sufficient conditions for convergence rates of spline and other nonorthonormal expansions can then be obtained directly from the same arguments as above.

## 4. Acknowledgments

The authors wish to thank Guido Weiss for bringing us together. The first author would like to acknowledge that this work was done under the direction of Richard Rochberg and Mitchell Taibleson, and she thanks them for their guidance. The last two authors wish to thank Ingrid Daubechies, Bjorn Jawerth, Stephane Mallat, and Robert Strichartz for useful comments.

## References

[Ba] G. Battle, *A block spin construction of ondelettes. Part* I: *Lemarié functions*, Comm. Math. Phys. **110** (1987), 601–615.

[BDR] C. de Boor, R. DeVore, and A. Ron, *Approximation from shift-invariant subspaces of* $L^2(\mathbf{R}^{\mathbf{d}})$, preprint.

[BR] C. de Boor and A. Ron, *Fourier analysis of the approximation power of principal shift-invariant subspaces*, preprint.

[Ca] L. Carleson, *On convergence and growth of partial sums of Fourier series*, Acta Math. **116** (1966), 135–157.

[Da1] I. Daubechies, *Orthonormal bases of compactly supported wavelets*, Comm. Pure Appl. Math. **41** (1988), 909–996.

[Da2] ———, *Ten lectures on wavelets*, SIAM, Philadelphia, PA, 1992.

[GM] A. Grossman and J. Morlet, *Decomposition of Hardy functions into square-integrable wavelets of constant shape,* SIAM J. Math. Anal. **15** (1984) 723–736.

[Ha] A. Haar, *Zur Theorie der orthogonalen Funktionensysteme*, Math. Ann. **69** (1910), 336.

[Hu] R. A. Hunt, *On the convergence of Fourier series*, Proceedings of the Conference on Orthogonal Expansions and Their Continuous Analogues (Southern Illinois University), 1968, pp. 235–255.

[Le] P. G. Lemarié, *Ondelettes á localisation exponentielle*, J. Math. Pures Appl. (to appear).

[LM] P. G. Lemarié and Y. Meyer, *Ondelettes et bases Hilbertennes*, Rev. Math. Iberamericana **2** (1986), 1–18.

[KKR] S. Kelly, M. Kon, and L. Raphael, *Pointwise convergence of wavelet expansions* (to appear).

[Ma] S. Mallat, *Multiresolution approximation and wavelets*, Trans. Amer. Math. Soc. **315** (1989), 69–88.

[Me] Y. Meyer, *Ondelettes*, Hermann, Paris, 1990.

[SF] G. Strang and G. Fix, *A Fourier analysis of the finite element variational method*, Constructive Aspects of Functional Analysis, Edizioni Cremonese, Rome, 1973.

[St] J.-O. Strömberg, *A modified Franklin system and higher-order spline system on* $\mathbf{R}^n$ *as unconditional bases for Hardy spaces*, Conference on Harmonic Analysis in Honor of Antoni Zygmund, Vol. II (W. Beckner et al., eds.), Wadsworth Math. Ser., Wadsworth, Belmont, CA, 1981, pp. 475–494.

[Wa1] G. Walter, *Approximation of the delta function by wavelets*, preprint, 1992.

[Wa2] ______, *Pointwise convergence of wavelet expansions*, preprint, 1992.