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Signal Recovery and Wavelet Reproducing Kernels

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Abstract— **A class of signal recovery problems can be formulated as finding missing data at the finest scale of a discrete wavelet transform. A unique and stable recovery can be obtained by solving the regularized wavelet reproducing equation. We show that this approach has close relations to unconstrained and constrained least-squares techniques and derive a family of regularizing operators adapted to the degrading operator. Experimental results present restored images using regularizing operators of this type.**

Keywords— **Regularization, Wavelets, Reproducing Kernel.**

I. Introduction

A class of signal degradation problems can be modeled as convolving the signal against a linear shift-invariant lowpass filter. This kind of degradation can be called "blurring" after what happens in an image. The recovery of the original signal from the blurred one, or deblurring, in continuous-time amounts to finding a solution of a Fredholm integral (convolution type) equation of the first kind [10]. In one dimension, this amounts to trying to find the signal $x(t)$ from the degraded measurement

$$y(t) = \int_a^b h(t-s)x(s)\,ds, \tag{1}$$

which results from the blurring kernel $h(t)$.

In practice, we deal with a discretized equation and a finite, discrete observation of the blurred signal. Furthermore, the observation of the blurred signal is often contaminated by noise. Consequently, we solve for the original signal $\mathbf{x}$ from a vector equation,

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} \tag{2}$$

where $\mathbf{y}$ and $\mathbf{n}$ are $n \times 1$ vectors representing the observed signal and noise, respectively, $\mathbf{x}$ is an $m \times 1$ vector and $\mathbf{H}$ is an $n \times m$ Toeplitz matrix.

Solving Eq. (2) for $\mathbf{x}$ is a notoriously ill-posed problem in the sense that a unique solution may not even exist as $\mathbf{H}$ is often singular, and in that the stability of the solution from (2) is degraded for ill-conditioned $\mathbf{H}$ because the inversion amplifies noise by a factor proportional to the condition number of $\mathbf{H}$.

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A popular technique for treating (2) is by means of regularization. This amounts to incorporating some *a priori* knowledge and assumptions about signal and noise. The problem is then turned into a functional minimization subject to some constraints[11]:

$$\begin{gathered}\text{Min}\left\{\Gamma_\lambda = ||\mathbf{y} - \mathbf{H}\mathbf{x}||^2 + \lambda||\mathbf{C}\mathbf{x}||^2\right\} \\ \text{with respect to } \mathbf{x}.\end{gathered} \tag{3}$$

where $\mathbf{C}$ is a regularizing operator, and $\lambda > 0$ is a parameter to be determined by matching specified constraints. (3) is a constrained least-squares (CLS) minimization[1] and the solution must satisfy the normal equation,

$$(\mathbf{H}^*\mathbf{H} + \lambda\mathbf{C}^*\mathbf{C})\mathbf{x} = \mathbf{H}^*\mathbf{y} \tag{4}$$

where $\mathbf{H}^*$ and $\mathbf{C}^*$ denote the Hermitian transpose of $\mathbf{H}$ and $\mathbf{C}$, respectively.

In general, the regularizing operator $\mathbf{C}$ is chosen to be a highpass filter. Then the second term in the cost functional of (3) prescribes a penalty for excessive high frequency energy in the solution. The role of λ is to control the level of penalty and hence the degree of smoothness of the solution. Since the first term in the cost functional of (3) represents a measure of consistency of the solution to the observed data, there will be a compromise between consistency and smoothness of the solution. With λ properly adjusted and $|\mathbf{H}^*\mathbf{H} + \lambda\mathbf{C}^*\mathbf{C}| \neq 0$, the CLS minimization of (3) has a solution from (4),

$$\hat{\mathbf{x}}_{\mathbf{C}} = (\mathbf{H}^*\mathbf{H} + \lambda\mathbf{C}^*\mathbf{C})^{-1}\mathbf{H}^*\mathbf{y} \tag{5}$$

It is worth noting that when $\lambda \to 0$, (5) becomes the solution to an unconstrained least-squares (UCLS) minimization without any smoothing (ultra-rough); when $\lambda \to 1$, smoothing and deblurring are approximately balanced; when $\lambda \to \infty$, smoothing is dominant and (5) gives an ultra-smooth solution[6].

The proper choice of the regularizing operator $\mathbf{C}$ has been studied to some extent[5,6]. Basically, two types of regularizing operators have been proposed. Those of the first type are dependent on the signal and noise. For example, a parametric Wiener filter has a regularizing operator based on the second-order statistics of the true signal and noise[1]; Reeves and Mersereau[9] proposed another operator based on the generalized cross-validation using only the observed signal. Regularizing operators of the second type are independent of specific signals and noise. One of most popular operators of this type is the discrete Laplacian[1,11].

In contrast to the two types of regularizing operators mentioned above, there have been few attempts to construct regularizing operators that depend on the degrading operator $\mathbf{H}$. In this correspondence, we reformulate the signal recovery problem (3) as finding missing data at the finest scale of a discrete wavelet transform. The wavelet reproducing equation then plays a fundamental role in determining a unique and stable recovery. We show that the solution from the wavelet reproducing equation is equivalent to a least-squares solution, and the wavelet reproducing kernel defines a family of regularizing operators adapted to the degrading operator. We also present experimental results from restoring artificially distorted medical images with regularizing operators of this type.

II. Signal Recovery with Wavelet Reproducing Kernels

We now reformulate the signal recovery problem (2) within a wavelet framework and study conditions for a unique recovery to exist. Assume $\mathbf{H}$ in (2) is $\mathbf{H}[i,j] = h_{i-j}$, with sequence $\{h_n\}$ obtained by discretizing the bandlimited blurring kernel in (1). Further, assume that $\Delta = \sum h_n = 1$ by rescaling if necessary. ($\Delta \neq 0$ as h_n is lowpass). Let $H(\omega)$ be the Fourier transform of h_n. Since the continuous kernel is bandlimited, H satifies

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 \leq 1, \tag{6}$$

assuming h_n sampled sufficiently. Inasmuch as $H(0) = \sum h_n = 1$, (6) implies that $H(\pi) = 0$, so that H is a regular scaling filter which may be associated with a regular wavelet [3,8]. That is, one can construct a highpass filter G by requiring

$$|H(\omega)|^2 + |G(\omega)|^2 = 1. \tag{7}$$

For H and G satisfying these conditions, a discrete wavelet transform (DWT) of a sequence $\{s_n\} = S_1 f(n)$ may be computed recursively from scale 2^1 to 2^J in the Fourier domain by

$$\begin{cases} W_{2^j}F(\omega) = G(2^{j-1}\omega)\; S_{2^{j-1}}F(\omega) \\ S_{2^j}F(\omega) = H(2^{j-1}\omega)\; S_{2^{j-1}}F(\omega) \end{cases} \quad j = 1, 2, ..., J \tag{8}$$

where $W_{2^j}F(\omega)$ is the Fourier transform of the DWT coefficients of $\{s_n\}$ at scale 2^j, and $S_{2^j}F(\omega)$ denotes the Fourier transform of $\{s_n\}$ smoothed to scale 2^j. Here we have normalized scale by defining the input sequence $\{s_n\}$ to be at the finest scale, $2^0 = 1$.

The inverse DWT is given recursively by

$$S_{2^{j-1}}F(\omega) = \overline{G}(2^{j-1}\omega)\; W_{2^j}F(\omega) + \overline{H}(2^{j-1}\omega) S_{2^j}F(\omega), \quad j = J, J-1, ..., 1. \tag{9}$$

where $\overline{G}$ and $\overline{H}$ denote the conjugates of G and H. Substituting (9) into (8) gives the discrete wavelet reproducing equation:

$$\begin{pmatrix} W_{2^1}F(\omega) \\ \vdots \\ W_{2^J}F(\omega) \\ S_{2^J}F(\omega) \end{pmatrix} = \begin{pmatrix} K_{1,1}(\omega) & \cdots & K_{1,J+1}(\omega) \\ \vdots & \cdots & \vdots \\ K_{J,1}(\omega) & \cdots & K_{J,J+1}(\omega) \\ K_{J+1,1}(\omega) & \cdots & K_{J+1,J+1}(\omega) \end{pmatrix} \begin{pmatrix} W_{2^1}F(\omega) \\ \vdots \\ W_{2^J}F(\omega) \\ S_{2^J}F(\omega) \end{pmatrix} \tag{10}$$

where the entries $K_{m,n}(\omega)$ are Fourier transforms of the reproducing kernels. Explicit expressions for these are given in [7]. In this paper, we are particularly interested in the reproducing kernel $K_{1,1}$. It can be shown that

$$K_{1,1}(\omega) = G(\omega)\overline{G}(\omega) = |G(\omega)|^2. \tag{11}$$

In the time domain the reproducing equation states that a collection of sequences, $\{\{\mathbf{g}_j(n)\} | 1 \leq j \leq J+1, n \in \mathbb{Z}\}$, is the DWT of a sequence if

$$\mathbf{g}_j(n) = \sum_{i=1}^{J+1} \mathbf{g}_i * k_{i,j}(n) \tag{12}$$

with $k_{i,j}(n)$ given by the inverse Fourier transform of $K_{i,j}$.

If the continuous degrading kernel is not bandlimited, the discretized kernel may not satisfy (6) and so (8) and (9) are not a DWT pair in the regular sense. However, they can be regarded as an octave-band filter bank transform pair and the reproducing equation (10) or (12) still holds.

Let us assume that we have a discretized degrading kernel satisfying (6). Consider the original signal recovery problem (2) with $\mathbf{n} = \mathbf{0}$. Define $\mathbf{x} = S_1\mathbf{x}$ to be at the finest scale and take the degrading operator $\mathbf{H}$ as the scaling filter. In view of (8), $\mathbf{y} = S_{2^1}\mathbf{x}$ is one part of the DWT for $J = 1$. The other part, $W_{2^1}\mathbf{x}$, has been lost as a result of blurring. By (8), $\mathbf{y} = S_{2^1}\mathbf{x}$ can be further decomposed to some scale 2^J for $J > 1$. The signal recovery problem can be then reformulated as:

$$\begin{aligned} \text{Given}: &\quad W_{2^2}\mathbf{x},\ W_{2^3}\mathbf{x}, ..., W_{2^J}\mathbf{x},\ S_{2^J}\mathbf{x}, \\ \text{Find}: &\quad W_{2^1}\mathbf{x}. \end{aligned} \tag{13}$$

With $W_{2^1}\mathbf{x}$ recovered as well as possible, an estimate of the original signal $\mathbf{x}$ can be synthesized by (9).

For notational simplicity, denote the DWT representation $\{W_{2^1}\mathbf{x}, W_{2^2}\mathbf{x}, ..., W_{2^J}\mathbf{x}, S_{2^J}\mathbf{x}\}$ by $\{\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_J, \mathbf{g}_{J+1}\}$. Rewrite the reproducing equation (12) in vector form with respect to the unknown $\mathbf{g}_1$,

$$\mathbf{g}_1 = \mathbf{K}_{1,1}\mathbf{g}_1 + \sum_{n=2}^{J+1} \mathbf{K}_{1,n}\mathbf{g}_n \tag{14}$$

where $\mathbf{K}_{1,n}$ is a Toeplitz matrix, $\mathbf{K}_{1,n}[i,j] = k_{1,n}(i-j)$.

(14) is the discrete form of the Fredholm integral equation of the second kind with a self-adjoint operator. In general, if 1 is not an eigenvalue of $\mathbf{K}_{1,1}$, (14) has a unique

solution[4]. In practice, however, $\mathbf{K}_{1,1}$ often has eigenvalues equal or close to 1 due to the lowpass nature of $\mathbf{H}$. We turn to solving a stabilized equation,

$$\mathbf{g}_1 = \alpha \mathbf{K}_{1,1}\mathbf{g}_1 + \sum_{n=2}^{J+1} \mathbf{K}_{1,n}\mathbf{g}_n \tag{15}$$

where α is a real number and is adjusted so that $\alpha\mathbf{K}_{1,1}$ does not have eigenvalues close to 1. It is possible to solve for $\mathbf{g}_1$ by iterating (15) with a tighter constraint on the choice ofα. Denote by $||\cdot||$ the L^2 norm of an operator. By the fixed point theorem[4], for $1 \leq ||\mathbf{K}_{1,1}|| \leq \infty$, there exists a sufficiently small real number such that $|\alpha| \cdot ||\mathbf{K}_{1,1}|| < 1$ and (15) has a unique solution. In this case, the unique solution can be obtained by iterating (15) with an initial value.

III. Relations to Least-Squares and Regularization Techniques

A. The Equivalent Least-Squares Solutions

Recovering the original signal $\mathbf{x}$ from (14) or (15) has close relations to the least-squares and regularization techniques. We now show that the solution to (15) is equivalent to a CLS solution and that to (14) is equivalent to a UCLS solution. Let us rearrange (15) to the form,

$$(\mathbf{I} - \alpha\mathbf{K}_{1,1})\,\mathbf{g}_1 = \sum_{n=2}^{J+1} \mathbf{K}_{1,n}\mathbf{g}_n \tag{16}$$

Changing variable to $\lambda = 1 - \alpha$, setting $J = 1$ and substituting for $\mathbf{K}_{1,1}$ and $\mathbf{K}_{1,2}$ in (16), and using the identity $\mathbf{g}_2 = S_{2^1}\mathbf{x} = \mathbf{y}$, (16) can be rewritten,

$$(\mathbf{H}\mathbf{H}^* + \lambda\mathbf{G}\mathbf{G}^*)\,\mathbf{g}_1 = \mathbf{G}\mathbf{H}^*\mathbf{y} \tag{17}$$

If $\mathbf{G}\mathbf{H}^* = \mathbf{H}\mathbf{G}^*$, (17) is the normal equation for the CLS minimization,

$$\mathrm{Min}\left\{\Gamma_\lambda = ||\mathbf{H}^*\mathbf{g}_1 - \mathbf{G}^*\mathbf{y}||^2 + \lambda||\mathbf{G}^*\mathbf{g}_1||^2\right\} \quad \text{with respect to } \mathbf{g}_1. \tag{18}$$

With λ properly adjusted and $|\mathbf{H}\mathbf{H}^* + \lambda\mathbf{G}\mathbf{G}^*| \neq 0$, the CLS solution for $\mathbf{g}_1$ is readily obtained from (17),

$$\hat{\mathbf{g}}_1 = (\mathbf{H}\mathbf{H}^* + \lambda\mathbf{G}\mathbf{G}^*)^{-1}\,\mathbf{G}\mathbf{H}^*\mathbf{y} \tag{19}$$

The solution for $\mathbf{x}$ is synthesized by the inverse DWT given by (9) for $J = 1$. That is,

$$\hat{\mathbf{x}}_{\mathbf{G}} = \mathbf{G}^*\hat{\mathbf{g}}_1 + \mathbf{H}^*\mathbf{y} = \left\{\mathbf{G}^*(\mathbf{H}\mathbf{H}^* + \lambda\mathbf{G}\mathbf{G}^*)^{-1}\,\mathbf{G} + \mathbf{I}\right\}\mathbf{H}^*\mathbf{y} \tag{20}$$

When $\lambda = 0$ or $\alpha = 1$, (17) becomes the normal equation of a UCLS minimization, which is equivalent to (14). Consequently, (19) becomes a UCLS solution for $\mathbf{g}_1$ and (20) gives the corresponding solution for $\mathbf{x}$. Finally, we give the Fourier-domain form of (20) for easy implementation.

$$\hat{X}_G(\omega) = \left[\frac{|G(\omega)|^2}{|H(\omega)|^2 + \lambda|G(\omega)|^2} + 1\right]\overline{H}(\omega)Y(\omega) \tag{21}$$

B. Remarks

1) The CLS minimization of (18) can be rewritten in terms of $\mathbf{x}$:

$$\mathrm{Min}\left\{\tilde{\Gamma}_\lambda = ||\mathbf{H}^*\mathbf{G}\mathbf{x} - \mathbf{G}^*\mathbf{y}||^2 + \lambda||\mathbf{G}^*\mathbf{G}\mathbf{x}||^2\right\} \quad \text{with respect to } \mathbf{x}. \tag{22}$$

This indicates that $\mathbf{G}^*\mathbf{G}$ is a regularizing operator. From (11), the wavelet reproducing kernel is $\mathbf{K}_{1,1} = \mathbf{G}^*\mathbf{G}$. If $\mathbf{G}$ is circulant, $\mathbf{G}^*\mathbf{G} = \mathbf{G}\mathbf{G}^*$; the wavelet reproducing kernel is exactly the regularizing operator in (22). In practice, a circulant $\mathbf{G}$ is often implied in the computation of a DWT by imposing periodicity on the finite data. An observation from the Fourier domain also shows that the wavelet reproducing kernel $K_{1,1}(\omega) = G(\omega)\overline{G}(\omega) = \overline{G}(\omega)G(\omega) = |G(\omega)|^2$ is the regularizing operator in (22). For simplicity, we shall call (20) a wavelet regularized solution of $\mathbf{x}$. It is interesting to compare (18) or (22) with (3). Since $\mathbf{H}$ and $\mathbf{G}$ are a pair of lowpass and highpass filters, the signals $\mathbf{g}_1 = W_{2^1}\mathbf{x} = \mathbf{G}\mathbf{x}$, $\mathbf{y} = S_{2^1}\mathbf{x} = \mathbf{H}\mathbf{x}$ can be viewed respectively as "high-band" and "low-band" components of $\mathbf{x}$. Then the first term in (18) is a "mid-band" consistency measure of solution $\mathbf{x}$ to the observation $\mathbf{y}$, while the first term in (3) represents a "low-band" consistency measure. Nevertheless, given (20), the "low-band" consistency of wavelet regularized solution can be easily computed[7].

2) The role of λ in (18) is similar to that in (3). When $\lambda \to 0$, both (18) and (3) reduce to unconstrained least-squares minimization problems. When $\lambda \to 1$, the smoothing and deblurring terms are balanced in both (18) and (3). When $\lambda \to \infty$, smoothing is dominant, forcing $\hat{\mathbf{g}}_1 \to 0$ in (19) and $\hat{\mathbf{x}}_{\mathbf{C}} \to 0$ in (5). However, the final solution of $\mathbf{x}$ given by (20) does not vanish for $\lambda \to \infty$; rather

$$\lim_{\lambda\to\infty} \hat{\mathbf{x}}_{\mathbf{G}} = \mathbf{H}^*\mathbf{y} = \mathbf{H}^*\mathbf{H}\mathbf{x} + \mathbf{H}^*\mathbf{n}, \tag{23}$$

determining a limit of smoothness for the recovered signal. This reflects the conceptual difference of the wavelet approach from an "inverse" approach. In the reformulated signal recovery problem (13), we do not seek for an inversion of the blurring process. Instead, we look for the information lost in the blurring process and resynthesize it with blurred signal by (9). Therefore, the worst situation in terms of deblurring happens when $\lambda \to \infty$ and the solution is given by (23). In this limiting case, the phase distortion due to the degrading operator is corrected and noise is restricted to the same bandwidth as that of the blurred original signal. We lose no more bandwidth than the blurred original signal. However, the frequency components inside the band are a bit more attenuated. One sees the two sides of the wavelet regularized signal recovery revealed by (23). On one side, this scheme prevents over regularized signal recovery; on the other side it does not turn to a pure smoothing filter as λ becomes large. Thus wavelet regularization is perhaps not as useful as standard regularization if one wishes simply to smooth noise.

3) The condition $\mathbf{G}\mathbf{H}^* = \mathbf{H}\mathbf{G}^*$ is satisfied with $\mathbf{H}$ and $\mathbf{G}$ circulant and self-adjoint. For example, the circulant

condition is implied in the computation of DWT when periodicity is imposed on the finite data, and many degrading kernels result in real symmetric $\mathbf{H}$ and $\mathbf{G}$. When $\mathbf{GH}^* \neq \mathbf{HG}^*$, (17) is not a normal equation of (18). However, the interpretation in the frequency domain still applies because the spectral moduli of $\mathbf{GH}^*$ and $\mathbf{HG}^*$ are identical.

4) The 1-D wavelet regularization described in Sections II and III-A can be generalized to two dimensions. The extensions of 1-D results will be straightforward when the 2-D degrading operator is considered separable in two dimensions. Detailed derivations may be found in [7].

5) Recently, Bruneau *et al*[2] took a synthetic approach to regularized image restoration with biorthogonal wavelets. In a discrete biorthogonal wavelet transform, two pairs of discrete filters, $\{H_1, G_1\}$ and $\{H_2, G_2\}$, are used for decomposition and reconstruction. These correspond to dual wavelet bases and satisfy

$$H_1(\omega)H_2(\omega) + G_1(\omega)G_2(\omega) = 1. \tag{24}$$

In [2], H_1 is taken as the discretized degrading operator, next, G_1, H_2 and G_2 are constructed by (24) and biorthogonal relations between these filters. G_1 is used as a regularizing operator for the CLS minimization in (3). This formulation can be rewritten in accordance with the symbols used in (18) as follows:

$$\begin{aligned} &\text{Min}\left\{\Gamma_\lambda = ||\mathbf{H}_1\mathbf{G}_2\mathbf{g}_1 - \mathbf{G}_1\mathbf{G}_2\mathbf{y}||^2 + \lambda||\mathbf{g}_1||^2\right\} \\ &\qquad\qquad \text{with respect to } \mathbf{g}_1. \end{aligned} \tag{25}$$

(25) is similar to (18) in that the first term in (25) is also a "mid-band" consistency measure and the second term is a measure of high frequency energy of the solution $\mathbf{x}$. However, unlike (18), (25) does not relate to recovering $\mathbf{g}_1$ from the wavelet reproducing equation. The solution to (25) is not equivalent to that of (15).

IV. Wavelet-Regularized Image Restoration: Examples

We now present experimental results of the wavelet regularized image restoration with two examples. We synthetically blurred two MR images, Figures $1(a)$ and $2(a)$ with two often encountered degrading kernels: Fejér and Gaussian[7]. Noise was added to the blurred images and the blurred signal-to-noise ratio (BSNR) was controlled at 25 dB. The blurred and noisy images were then restored by using the 2-D version of (21) and the minimum mean square error (MMSE) criterion. For comparison, we also did MSE restoration by using (5) with a discrete 2-D Laplacian as the regularizing operator. The restored images are shown in Figures $1(c-d)$ and $2(c-d)$; numerical results are summarized in Table 1.

A Fejér kernel has a triangular spectrum. It is the point-spread function (PSF) of a diffraction-limited incoherent imaging system with a rectangular aperture. An appropriately sampled Fejér kernel satisfies (6) and is therefore associated with a scaling function and wavelets. A Gaussian kernel is a model often used to approximate a wide range of degradation such as blurring caused by atmospheric turbulence. Unlike the Fejér kernel, a Gaussian is not bandlimited and there are no scaling functions or wavelets in the regular sense to be associated with it. Nevertheless, our computational framework still applies. The difference is in terminology: the appropriate interpretation in the case of the Gaussian will be in terms of octave-band filter banks, instead of wavelets. However, we would still use the term "wavelet" for simplicity.

The numerical results in Table 1 seem to indicate that the wavelet regularized solutions have MMSEs very close to those of Laplacian regularized solutions. The apparent difference seems to be that the wavelet regularization always attain the MMSE at a smaller λ. However, this is not conclusive and further investigation with a wide range of degrading operators and images is warranted.

VI. Conclusions

In a class of signal recovery problems the degrading operators are, or can be modeled as, shift-invariant and band-limited lowpass filters. These problems can be reformulated as finding missing data at the finest scale of a discrete wavelet transform. The wavelet reproducing equation plays a fundamental role in determining a unique and stable recovery of the signal. The relation between this approach and the least-squares and regularization techniques was studied and the equivalent solutions derived. It was shown that the wavelet reproducing kernel defines a family of regularizing operators which are adapted to the degrading operators. Experimental results from image restoration show that this family of regularizing operators have a very similar performance to that of the Laplacian operator in terms of the MMSE criterion.

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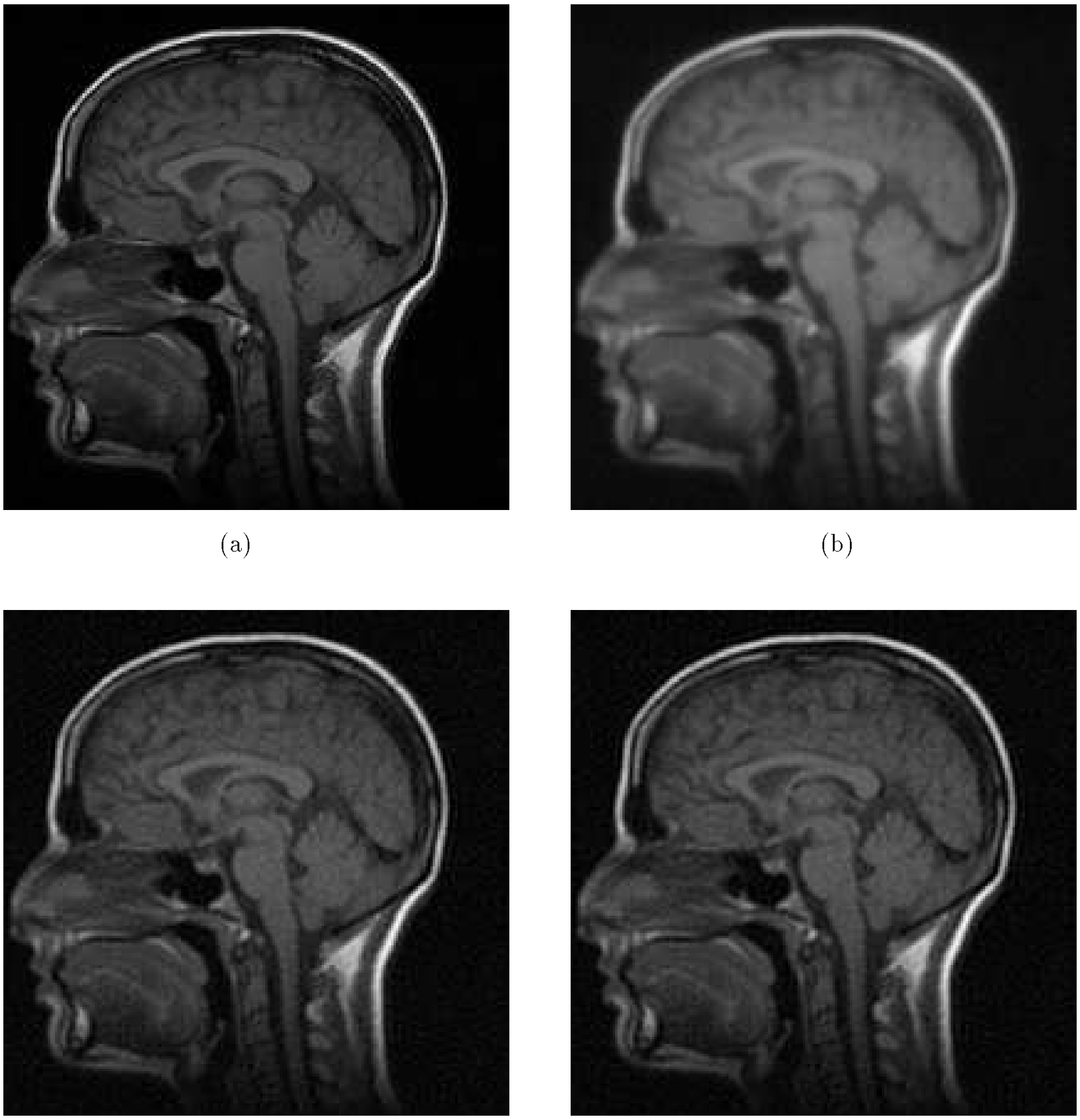
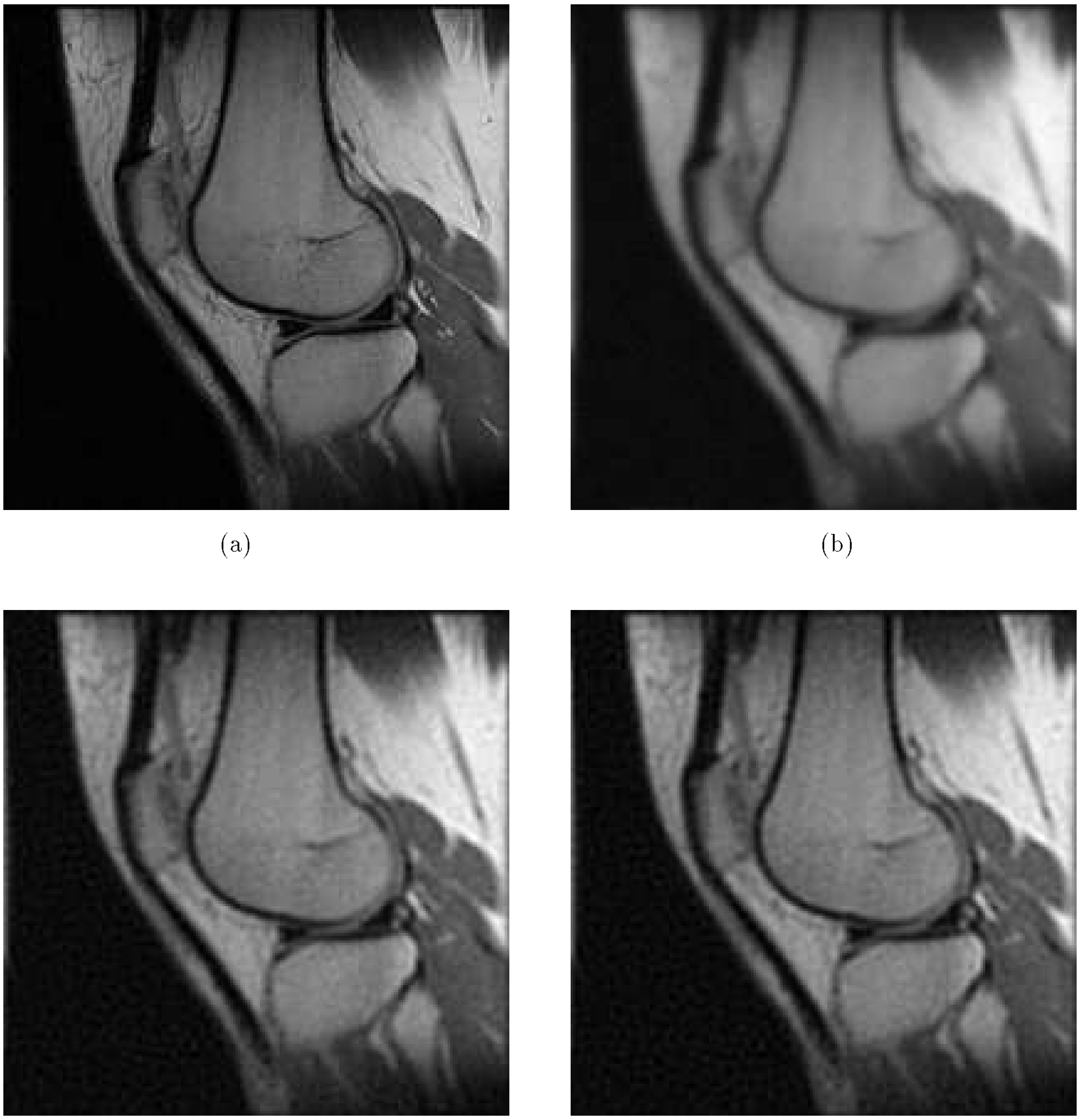


(a) (b) (c) (d)

Figure 1: Restoration of a noisy MR head image. (a) The original head image; (b) blurred by a Fejér kernel, $\omega_c = 0.5\pi$, BSNR = $25db$; (c) restored with wavelet regularization; (d) restored with Laplacian regularization. (See Table 1 for numerical results).



(a) (b)

(c) (d)

Figure 2: Restoration of a noisy MR knee image. (a) The original knee image; (b) blurred by a Gaussian kernel, $\sigma = 2$, BSNR = $25db$; (c) restored with wavelet regularization; (d) restored with Laplacian regularization. (See Table 1 for numerical results).

Table 1: Experimental results from restoring blurred MR images

Image	MMSE	λ
Fig. 1(a): The head image	–	–
Fig. 1(b): Blurred by a Fejér kernel	2753.02	–
Fig. 1(c): Wavelet-regularied restoration	1178.09	0.010
Fig. 1(d): Laplacian-regularized restoration	1165.33	0.050
Fig. 2(a): The knee image	–	–
Fig. 2(b): Blurred by a Gaussian kernel	1368.98	–
Fig. 2(c): Wavelet-regularized restoration	892.559	0.015
Fig. 2(d): Laplacian-regularized restoration	887.148	0.19