611: Electromagnetic Theory

Homework 10

(**1a**) Making use of the result that

$$\nabla^2 \frac{1}{|\vec r - \vec r\,'|} = -4\pi\, \delta^3(\vec r - \vec r\,')\,, \tag{1}$$

show by directly acting with $\Box = \nabla^2 - \dfrac{\partial^2}{\partial t^2}$ on the right-hand side of the retarded potential formula

$$\phi(\vec r, t) = \int \frac{\rho(\vec r\,', t - |\vec r - \vec r\,'|)}{|\vec r - \vec r\,'|}\, d^3\vec r\,' \tag{2}$$

that $\phi(\vec r, t)$ obeys $\Box\phi(\vec r, t) = -4\pi\, \rho(\vec r, t)$.

[Present all the steps in the calculation. Don't make the common mistake of thinking that Leibnitz' rule $\vec\nabla(fg) = (\vec\nabla f)\, g + f\, \vec\nabla g$ means that $\nabla^2(fg)$ is equal to $(\nabla^2 f)\, g + f\, \nabla^2 g$! There is an additional cross-term too!]

(**1b**) Similarly, show that the advanced potential, given by

$$\phi_{\rm adv}(\vec r, t) = \int \frac{\rho(\vec r\,', t + |\vec r - \vec r\,'|)}{|\vec r - \vec r\,'|}\, d^3\vec r\,'\,, \tag{3}$$

also obeys $\Box\phi_{\rm adv}(\vec r, t) = -4\pi\, \rho(\vec r, t)$.

(**2a**) Calculate the magnetic field that follows from the Liénard-Wiechert potentials for an accelerating point charge (they are given in eqns (7.34) in the online lecture notes).

(**2b**) Show that $\vec B$ can be written as $\vec B = \dfrac{\vec R \times \vec E}{R}$, where $\vec E$ was obtained in eqn (7.62) in the online lecture notes.

(You may use without proof any results such as eqns (7.54), (7.55), (7.58), (7.59) that are given in the lecture notes.)

(**3a**) Starting from the expression in eqn (2) above for the retarded potential, consider a source $\rho(\vec r, t) = \lambda\, \delta(x)\delta(y)\delta(t)$, where λ is a constant. (This is an infinite line source along the z axis, which occurs instaneously at $t = 0$.) Show that the retarded potential due to this source is given by

$$\phi(x, y, z, t) = \frac{2\lambda\, \theta(t - R)}{\sqrt{t^2 - R^2}}\,, \qquad \text{where} \quad R = \sqrt{x^2 + y^2}\,, \tag{4}$$

and where θ is the standard Heaviside theta function, i.e. $\theta(x)$ is 1 if x is positive and zero if x is negative.

(**3b**) Now instead plug the charge density $\rho(\vec{r},t) = e\,\delta^3(\vec{r}\,)\,\delta(t)$, describing an instantaneous *point charge* source, into eqn (2), showing that this gives

$$\phi(\vec{r},t) = \frac{e\,\delta(t-r)}{r}\,. \qquad (5)$$

Comments: The result in part (3b) is revisiting what we already saw in the discussion in the lectures; namely, that in four spacetime dimensions the effect of turning on an instananeous "blip" of point charge at the origin propagates outwards as a sharp spherical blip in the potential, expanding at the speed of light, with no lingering "tail" remaining afterwards.

The result in part (3a) is exactly equivalent to solving the problem of a time-dependent charge in 3 spacetime dimensions. (An infinite line of charge along the entire z axis in 3 spatial dimensions (x,y,z) is equivalent to a point charge in the 2 spatial dimensions (x,y).) The result in part (3a) then shows that in 3 spacetime dimensions, after an instantaneous blip of point charge at the origin occurs (at $t=0$), then at a distance R nothing happens until t becomes equal to R (when the θ-function turns on). (That is, nothing is seen at R until a light-travel time after the original blip occurred.) By contrast to 4 spacetime dimensions, however, the potential remains non-vanishing thereafter, only gradually decaying away to zero as t goes to infinity.

Analogous qualitative behaviours are seen in all higher dimensions; that is to say, in *even* spacetime dimensions, the blip of charge gives rise to an instananeous blip of non-vanishing potential a light-travel time later. In *odd* spacetime dimensions, the blip of charge gives rise to a potential that turns on a light-travel time later, and then decays away gradually after that.

[**Terminology**: *Before the Covid pandemic, the term "spike" was widely understood in common parlance to mean a function that rises quickly and immediately falls again, and this would have been the popular word of choice to describe something like the delta-function bevaviour of the electric charge in parts (3a) and (3b). Mysteriously, the term "spike" was appropriated by Covid commentators to mean something more like a θ-function; a sudden increase but without any subsequent rapid fall again. Hence the use of the term "blip" here, to convey the idea of what used to be called a "spike."*]

(**4a**) The result for the retarded potential in 6 spacetime dimensions is

$$\phi(\vec{r},t) = \int \Big[\frac{\rho(\vec{r}\,',t-|\vec{r}-\vec{r}\,'|)}{|\vec{r}-\vec{r}\,'|^3} + \frac{\partial_t\rho(\vec{r}\,',t-|\vec{r}-\vec{r}\,'|)}{|\vec{r}-\vec{r}\,'|^2}\Big]\,d^5\vec{r}\,'\,, \qquad (6)$$

where $\partial_t \equiv \dfrac{\partial}{\partial t}$. Verify this by showing that this expression for $\phi(\vec{r},t)$ satisfies $\Box\phi(\vec{r},t) = -8\pi^2\,\rho(\vec{r},t)$. You may use without proof the fact that in 5-dimensional Euclidean space

$$\nabla^2\,\frac{1}{|\vec{r}-\vec{r}\,'|^3} = -8\pi^2\,\delta^5(\vec{r}-\vec{r}\,')\,. \qquad (7)$$

(Actually the proof is very simple, and proceeds by direct analogy with one that we have seen before in 3 spatial directions.) (Calculations in $6 = 5 + 1$ spacetime dimensions proceed almost identically to those in $4 = 3 + 1$ spacetime dimensions, except that what were formerly spatial 3-vectors are now spatial 5-vectors. The latin index range is 1 to 5 instead of 1 to 3. One still has $\partial_i r = \dfrac{x_i}{r}$, etc. $\Box = \nabla^2 - \partial_t^2$ and $\nabla^2 = \partial_i \partial_i$, with i now summed over 1 to 5. Of course now, one has $\partial_i x_i = 5$ rather than equalling 3, etc.)

(**4b**) Show that the retarded potential (6) can be written as

$$\phi(\vec{r}, t) = \int \int \rho(\vec{r}\,', t'') \Big[\frac{\delta(t'' - t + |\vec{r} - \vec{r}\,'|)}{|\vec{r} - \vec{r}\,'|^3} - \frac{\delta'(t'' - t + |\vec{r} - \vec{r}\,'|)}{|\vec{r} - \vec{r}\,'|^2} \Big] \, dt'' \, d^5\vec{r}\,' \,. \qquad (8)$$

Note that the prime on the delta function in the second term means the derivative of the delta function with respect to its argument.

In the final homework, this result will be used in order to calculate the Liénard-Wiechert potentials for an accelerating point charge in six spacetime dimensions.

Due Wednesday November 20th, in class