PHYS 611: Electromagnetic Theory

Homework 2

(1a) Show that a (real) Cartesian vector V^i in 3-dimensional Euclidean space obeys the inequality $V^i V^i \geq 0$. Show also that if $V^i\, V^i = 0$, then it must be that $V^i = 0$.

(1b) Show that if a (real) 4-vector V^μ in 4-dimensional Minkowski spacetime satisfies $V^\mu V_\mu = 0$, then V^μ need not be zero.

(1c) A particle following the worldline $x^\mu = x^\mu(\tau)$ has 4-velocity $U^\mu = \dfrac{dx^\mu(\tau)}{d\tau}$ and 4-acceleration $a^\mu = \dfrac{dU^\mu}{d\tau}$, where τ is the proper time. Show that $U^\mu\, a_\mu = 0$.

(2a) We are told that $W^\mu \equiv T^{\mu\nu}\, V_\nu$ is a 4-vector, where V_ν is any *arbitrary* 4-vector. We are not told any properties of $T^{\mu\nu}$. *Prove* that in fact $T^{\mu\nu}$ must be a 4-tensor.

Note: The whole point of the question is to *prove* from the given information that $T^{\mu\nu}$ must be a 4-tensor; you must not assume it is a 4-tensor without proving it! The required proof will make use of the given facts that W^μ and V_ν transform as 4-vectors under Lorentz transformations, and, crucially, that V_ν can be chosen *arbitrarily.*

This example is a particular case of a more general *quotient theorem* for tensors.

(2b) Show that the tensor $S_{\mu\nu} \equiv P_{\mu\rho} P_\nu{}^\rho$ is always symmetric, regardless of any symmetry properties of the tensor $P_{\mu\nu}$.

(3) A 4-vector V^μ is called *timelike* if $V^\mu V_\mu < 0$; *spacelike* if $V^\mu V_\mu > 0$; and *null* (or *lightlike*) if $V^\mu V_\mu = 0$. Suppose that k^μ is a null vector. Show that if a *non-spacelike* vector V^μ is orthogonal to k^μ (i.e. $k^\mu V_\mu = 0$), then it must be that V^μ is just a multiple of k^μ. (Note that a "non-spacelike vector" means that it is either timelike or null.)

[**Hint:** A classic and elegant way to solve a problem like this is to consider a judiciously-chosen "perfect square," which gives something useful when expanded out, and which also has a useful property when written as a square. Here, you could consider $W \equiv (V^\mu - \lambda\, k^\mu)(V_\mu - \lambda\, k_\mu)$, where λ is an as-yet arbitrary quantity that can then be chosen in an optimal manner in order to establish the required result.]

Turn over for problem 4

(**4a**) Define the scalar quantity R by $R^2 = \eta_{\mu\nu}\, x^\mu x^\nu$. Show that $\partial_\mu R = \eta_{\mu\nu}\, \dfrac{x^\nu}{R}$.

(**4b**) Hence show that

$$\Box \frac{1}{R^2} = 0\,,$$

where $\Box$ is the D'Alembertian $\partial^\mu \partial_\mu$. (Assume that $R \neq 0$).

(**4c**) Derive the condition on the constant 4-vector k_μ such that the scalar ϕ defined by

$$\phi \equiv e^{\mathrm{i}\, k_\mu x^\mu}$$

solves the wave equation $\Box\phi = 0$.

Due Wednesday 11th September, in class.