611: Electromagnetic Theory

Homework 3

(**1a**) Complete the derivation of the discussion in section 2.4 of the lecture notes, by deriving the Lorentz transformation that gives $\vec{B}'$ in terms of $\vec{E}$ and $\vec{B}$, for an arbitrary Lorentz boost with velocity $\vec{v}$ (i.e. eqn (2.61) in the notes).

(**1b**) Specialise your result in part (1a) to the case where the boost is along the x direction, and similarly specialise eqn (2.60) in the lecture notes to a boost along x. Hence verify the expressions given in eqn (2.64) of the notes for the components of $\vec{B}$ and $\vec{E}$ after boosting along x.

(**2a**) Suppose that constant, uniform $\vec{E}$ and $\vec{B}$ fields are orthogonal, $\vec{E}\cdot\vec{B}=0$, with axes oriented so that $\vec{E}=(E_x,0,0)$ and $\vec{B}=(0,B_y,0)$ in the frame S.

Consider a Lorentz boost with a velocity orthogonal to both $\vec{E}$ and $\vec{B}$, *i.e.* $\vec{v}=(0,0,v)$. First, write down the general expressions, analogous to eqns (2.64), for the transformations of general electric and magnetic fields $\vec{E}$ and $\vec{B}$ under a such boost with $\vec{v}=(0,0,v)$. (*i.e.* obtain the analogues of eqns (2.64) in the notes, but now for a boost along z.)

Now appplying these boosts along z to the specific $\vec{E}$ and $\vec{B}$ fields specified in this problem, show that if the value of v is chosen appropriately, it can be arranged that in the frame S' the magnetic field vanishes, $\vec{B}\,'=0$.

(**2b**) Bearing in mind that any boost velocity must be less than the speed of light, find the condition involving E_x and B_y that must be satisfied in order for the boost in part (2a) to be possible.

(**2c**) Repeat the previous steps for the case where one tries instead to make the electric field vanish in the frame S'. Give the condition involving E_x and B_y for which this is possible.

(**2d**) Under what circumstances (expressed again as a condition involving E_x and B_y) is it impossible to find a boost of the form discussed in this question that allows one or other of $\vec{E}'$ or $\vec{B}'$ to be set to zero?

(**3**) Show by direct computation using the expressions for $\vec{E}'$ and $\vec{B}'$ in terms of $\vec{E}$ and $\vec{B}$ that are given in eqns (2.60) and (2.61) of the lecture notes, that $(\vec{B}^2-\vec{E}^2)$ is invariant under Lorentz boosts. Again using (2.60) and (2.61), show that $\vec{E}\cdot\vec{B}$ is also invariant under Lorentz boosts.

Turn over for problem 4

This next question shows another way of describing the Lorentz transformations:

(**4a**) Define the 2×2 complex matrix

$$X = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} . \tag{1}$$

Show that X is Hermitean, *i.e.* $X^\dagger = X$. Now consider the complex matrix A defined by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \qquad \text{where} \qquad ad - bc = 1 \, . \tag{2}$$

Show that it satisfies $\det A = 1$. Such matrices A are said to form the group $SL(2, \mathbb{C})$; i.e. complex 2×2 matrices with unit determinant. Because of the condition $ad - bc = 1$, the four complex numbers a, b, c d are subject to one complex equation and hence $SL(2, \mathbb{C})$ is parameterised by $4 - 1 = 3$ independent complex numbers, or, equivalently, 6 real numbers (which is the same as the number of parameters in the Lorentz transformations).

(**4b**) Show that $\det X = t^2 - x^2 - y^2 - z^2 = -\eta_{\mu\nu}\, x^\mu\, x^\nu$. Show also that if we define

$$X' = A\, X\, A^\dagger \, , \qquad \text{then} \qquad \det X' = \det X \, , \quad \text{and} \quad X'^\dagger = X' \, . \tag{3}$$

This action of $SL(2, \mathbb{C})$ on X in fact describes the Lorentz transformations on (t, x, y, z). (Note: Lorentz transformations, *by definition*, are the linear transformation of the coordinates x^μ that leave $\eta_{\mu\nu}\, x^\mu x^\nu = -t^2 + x^2 + y^2 + z^2$ invariant.) Next are some examples:

(**4c**) Consider the matrix

$$A = \begin{pmatrix} \cosh \frac{1}{2}\delta & -\sinh \frac{1}{2}\delta \\ -\sinh \frac{1}{2}\delta & \cosh \frac{1}{2}\delta \end{pmatrix} . \tag{4}$$

Show that it is an $SL(2, \mathbb{C})$ matrix, and show using (3) that it describes a Lorentz boost along x, with rapidity δ. (That is, the boost velocity is $v = \tanh \delta$.)

(**4d**) Consider the matrix

$$A = \begin{pmatrix} e^{\frac{i}{2}\theta} & 0 \\ 0 & e^{-\frac{i}{2}\theta} \end{pmatrix} . \tag{5}$$

Show that it is an $SL(2, \mathbb{C})$ matrix, and show using (3) that it describes a spatial rotation through angle θ in the (x, y) plane.

(**4e**) Find an $SL(2, \mathbb{C})$ matrix A that describes a Lorentz boost with rapidity δ along the z axis.

Turn over for some remarks about Qu. (4)...

Remarks:

The group $SL(2,\mathbb{C})$ actually describes a *double covering* of the Lorentz group $SO(1,3)$. This can be seen from the fact that a given $SL(2,\mathbb{C})$ matrix A and the matrix $\widetilde{A} := -A$ (which is also an $SL(2,\mathbb{C})$ matrix) both describe the *same* Lorentz transformation (see eqn (3)).

Consider, for example, the spatial rotation in the (x,y) plane described by the $SL(2,\mathbb{C})$ matrix (5) in Qu. (4d). If we increase the rotation angle θ from 0 up to 2π, this describes a complete rotation through 360 degrees, and returns the x and y coordinates to their original values. However, after sending θ from 0 to 2π, the matrix A in eqn (5) ends up becoming

$$A \longrightarrow -\mathbb{1}\,, \qquad \textit{i.e.} \quad A \longrightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6}$$

Thus although sending $\theta \longrightarrow \theta + 2\pi$ returns us to the same point in the group $SO(1,3)$, it does not return us to the same point in $SL(2,\mathbb{C})$. In fact it would be necessary to make *two* complete rotations in order to return to the original point in $SL(2,\mathbb{C})$.

This becomes important when one wishes to describe fermions in spacetime. Fermions are described by what are called *spinors*, which, unlike vectors and tensors, have the property that they change sign after performing a rotation through 360 degrees. The fact that they only return to their original value after a rotation through 720 degrees rather than 360 is a reflection of the fact that they have half-integer spin rather than integer spin.

Please be sure always to present all the key steps in all your answers, working logically from the given starting point to the required result. Don't just report that you did the calculation and got the stated answer!

Due Wednesday September 18th, in class.