611: Electromagnetic Theory

Homework 4

(**1**) The relativistic Hamiltonian for a particle of mass m and electric charge e in the presence of an electromagnetic field described by potentials ϕ and $\vec{A}$ is

$$H = \sqrt{(\pi_j - eA_j)(\pi_j - eA_j) + m^2} + e\phi\,. \qquad (1)$$

Use Hamilton's equations $\dfrac{\partial H}{\partial \pi_i} = \dot{x}^i$ and $\dfrac{\partial H}{\partial x^i} = -\dot{\pi}_i$ to derive the equation of motion of the particle, and show that it is just the usual Lorentz force equation.

Note: The point of this question is to show that one can derive the equations of motion (*i.e.* Lorentz force equation) *purely* within the framework of the Hamiltonian description. In particular, you should *not* assume the relation between π_i and p_i that we previously found when using the Lagrangian description. Rather, the relation between π_i and p_i can itself be *derived* from the Hamiltonian H defined above, and the Hamilton equations. (You *are* allowed to use the definition $p_i = m\gamma\,\dot{x}^i$; this is purely a definition of the relativistic 3-momentum.)

(**2a**) This is a slightly more general variant of a problem in homework 3. The transformations of $\vec{E}$ and $\vec{B}$ under a general Lorentz boost with velocity $\vec{v}$ are

$$\vec{E}' = \gamma(\vec{E} + \vec{v}\times\vec{B}) - \frac{\gamma-1}{v^2}\,(\vec{v}\cdot\vec{E})\,\vec{v}\,, \qquad \vec{B}' = \gamma(\vec{B} - \vec{v}\times\vec{E}) - \frac{\gamma-1}{v^2}\,(\vec{v}\cdot\vec{B})\,\vec{v}\,.$$

Show that if $\vec{E}$ and $\vec{B}$ are constant fields (not necessarily orthogonal), then by choosing $\vec{v} = \lambda\,\vec{E}\times\vec{B}$ for a suitable constant λ (which you should determine), a Lorentz frame can in general be found in which the transformed fields $\vec{E}'$ and $\vec{B}'$ are *parallel*, i.e $\vec{E}'\times\vec{B}' = 0$.

(**2b**) Show that if $\vec{E}\cdot\vec{B} = 0$, then the required boost velocity will satisfy $|\vec{v}\,| = |\vec{E}|/|\vec{B}|$ or $|\vec{v}\,| = |\vec{B}|/|\vec{E}|$.

(**2c**) Why is it impossible to perform such a boost in the case that $\vec{E}\cdot\vec{B} = 0$ and $|\vec{E}| = |\vec{B}|$?

(**3a**) Show that the Lorentz force equation $m\dfrac{dU^\mu}{d\tau} = eF^\mu{}_\nu\,U^\nu$ implies that $U^\mu\,U_\mu =$ constant. [**Hint**: Multiply the equation by the 4-velocity U_μ.]

(**3b**) Writing the Lorentz force equation as $m\dfrac{d^2x^\mu}{d\tau^2} = eF^\mu{}_\nu\,\dfrac{dx^\nu}{d\tau}$, and taking the special case where $\vec{E} = (E,0,0)$ and $\vec{B} = 0$, where E is a constant, solve directly for the components of x^μ as functions of proper time τ. Show that by choosing constants of

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integration appropriately, the results reproduce the solution for the particle motion obtained in section (3.2.1) of the lectures. (Note that, as seen in Qu. (3a), although the Lorentz force equation implies that $\eta_{\mu\nu}\,\frac{dx^\mu}{d\tau}\,\frac{dx^\nu}{d\tau}$ = constant, it does not, of itself, fix the normalisation of $U^\mu = \frac{dx^\mu}{d\tau}$. (It couldn't, since U^μ appears linearly on both sides of the Lorentz force equation.) However, we know also that in fact $\eta_{\mu\nu}\,\frac{dx^\mu}{d\tau}\,\frac{dx^\nu}{d\tau} = -1$ for a massive particle.)

(4) Define the matrix $\mathcal{F} \equiv \eta^{-1}\,F$, where F is the Maxwell field strength matrix defined in the first equation in eqn (2.176) in the lecture notes. (So $\mathcal{F}$ is the matrix with components $F^\mu{}_\nu$.) Define U as the column vector with components U^μ. (So $U^T = (U^0, U^1, U^2, U^3)$, where the superscript T denotes the transpose.) Note that the Lorentz force equation for a particle of mass m and charge e can then be written in the matrix form

$$\frac{dU}{d\tau} = \frac{e}{m}\,\mathcal{F}\,U\,. \tag{2}$$

Note that in parts (4a), (4b) and (4c) of this problem you should just work with the abstract matrix $\mathcal{F}$; **do not complicate life by expressing it in terms of $\vec{E}$ and $\vec{B}$!**

(4a) Assuming that the components of $\mathcal{F}$ are *constant*, show that (2) can be integrated to give

$$U = \exp\Big(\frac{e}{m}\,\mathcal{F}\,\tau\Big)\,\overline{U}\,, \tag{3}$$

where $\overline{U}$ is a constant column vector (these are the "constants of integration"; they are equal to U at $\tau = 0$). Be sure to justify all matrix manipulations.

Note that here, the matrix $\frac{e}{m}\,\mathcal{F}\,\tau$ is being exponentiated; the exponential of a matrix M can be defined by its Taylor expansion:

$$\exp M = \sum_{n\ge 0} \frac{1}{n!}\,M^n = 1 + M + \tfrac{1}{2!}M^2 + \cdots\,.$$

(4b) We know that the 4-velocity should satisfy $U^T\eta\,U = -1$. Show that the solution (3) for U will satisfy this condition, provided that the integration constants $\overline{U}$ are chosen so that $\overline{U}^T\eta\,\bar{U} = -1$. (Showing this involves some matrix manipulations that depend upon the fact that the matrix F (not $\mathcal{F}$!) is antisymmetric.)

(4c) Writing X as the column vector with components x^μ (so $X^T = (x^0, x^1, x^2, x^3)$), and assuming that $\mathcal{F}$ is invertible, integrate (3) once more to obtain the solution for X.

(4d) Calculate $\det\mathcal{F}$ in terms of $\vec{E}$ and $\vec{B}$, and hence find the condition, in terms of $\vec{E}$ and $\vec{B}$, for $\mathcal{F}$ to be invertible.

Due Wednesday September 25th, in class.