611: Electromagnetic Theory

Homework 5

(**1a**) Show that the Bianchi identity $\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$ can be written as

$$\partial_\mu\, {}^*F^{\mu\nu} = 0\,,$$

where ${}^*F^{\mu\nu}$ is the *Hodge dual* of $F_{\mu\nu}$, defined by ${}^*F^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$.

(**1b**) Show that the vector $V^\mu \equiv {}^*F^{\mu\nu} A_\nu$ has the property that

$$\partial_\mu V^\mu = \tfrac{1}{4}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}\,.$$

(**2**) This problem is concerned with establishing some results for the $\epsilon_{\mu\nu\rho\sigma}$ tensor. It will begin by following the same steps for the slightly simpler case of ϵ_{ijk} in 3 dimensions, where it is easier to see what is going on. We shall need some notation for indicating an antisymmetrisation over a set of indices. For any tensor $W_{\mu_1\cdots\mu_n}$ we define

$$W_{[\mu_1\cdots\mu_n]} \equiv \frac{1}{n!}\,(W_{\mu_1\cdots\mu_n} + \text{even permutations} - \text{odd permutations})\,. \qquad (1)$$

Thus, for example,

$$W_{[\mu\nu]} = \frac{1}{2!}\,(W_{\mu\nu} - W_{\nu\mu})\,,\quad W_{[\mu\nu\rho]} = \frac{1}{3!}\,(W_{\mu\nu\rho} + W_{\nu\rho\mu} + W_{\rho\mu\nu} - W_{\nu\mu\rho} - W_{\mu\rho\nu} - W_{\rho\nu\mu})\,,$$

and so on. Analogous definitions apply in three dimensions. Define also

$$\delta^{\nu_1\cdots\nu_n}_{\mu_1\cdots\mu_n} \equiv \delta^{[\nu_1}_{\mu_1}\,\delta^{\nu_2}_{\mu_2}\,\delta^{\nu_3}_{\mu_3}\cdots\delta^{\nu_n]}_{\mu_n}\,. \qquad (2)$$

(Note that this is also automatically antisymmetric in the $\mu_1,\ldots,\mu_n$ indices.)

(**2a**) Show that $\epsilon^{ijk}\,\epsilon_{\ell mn} = 6\,\delta^{ijk}_{\ell mn}$. **Hint**: Observe that the left-hand side is antisymmetric in ijk and in ℓmn, and that the right-hand side is also antisymmetric in ijk and in ℓmn. From this, you can argue that it is only necessary to check the equation for one non-trivial assignment of the six index values in order to verify it completely. (Make sure you actually present an argument; don't just state it without proof!)

(**2b**) Take the identity in Qu. (2a) and set $n = k$. Show by this means that

$$\epsilon^{ijk}\,\epsilon_{\ell mk} = \delta^i_\ell\,\delta^j_m - \delta^i_m\,\delta^j_\ell = 2\delta^{ij}_{\ell m}\,. \qquad (3)$$

(**2c**) Using an analogous argument to that in Qu. (2a), show that

$$\epsilon^{\mu\nu\rho\sigma}\,\epsilon_{\alpha\beta\gamma\lambda} = -24\,\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\lambda}\,. \qquad (4)$$

(Pay attention to that minus sign!)

(**2d**) By contracting σ and λ, show that

$$\epsilon^{\mu\nu\rho\sigma}\,\epsilon_{\alpha\beta\gamma\sigma} = -\delta^\mu_\alpha\delta^\nu_\beta\delta^\rho_\gamma - \delta^\nu_\alpha\delta^\rho_\beta\delta^\mu_\gamma - \delta^\rho_\alpha\delta^\mu_\beta\delta^\nu_\gamma + \delta^\nu_\alpha\delta^\mu_\beta\delta^\rho_\gamma + \delta^\mu_\alpha\delta^\rho_\beta\delta^\nu_\gamma + \delta^\rho_\alpha\delta^\nu_\beta\delta^\mu_\gamma = -6\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\,. \qquad (5)$$

Turn over for problems 3 and 4

(3a) Using the result in part (2d), show that $\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\rho\sigma} = -2\delta^\mu_\alpha\delta^\nu_\beta + 2\delta^\mu_\beta\delta^\nu_\alpha = -4\delta^{\mu\nu}_{\alpha\beta}$.

(3b) Use the result in part (3a) to show that ${}^*({}^*F_{\mu\nu}) = -F_{\mu\nu}$. (i.e. that the Hodge dual of the Hodge dual of a 2-index antisymmetric tensor is equal to minus the original tensor.)

(3c) Define the tensor $T_{\mu\nu} = (4\pi)^{-1}\Big(F_{\mu\rho}\,F_\nu{}^\rho - \frac{1}{4}F^{\rho\sigma}\,F_{\rho\sigma}\,\eta_{\mu\nu}\Big)$, where $F_{\mu\nu}$ is the usual electromagnetic field tensor. (We shall soon be encountering $T_{\mu\nu}$ in the lectures; it is the energy-momentum tensor for the electromagnetic field.) Show that $T_{\mu\nu}$ can be written as

$$T_{\mu\nu} = \frac{1}{8\pi}(F_{\mu\rho}\,F_\nu{}^\rho + {}^*F_{\mu\rho}\,{}^*F_\nu{}^\rho)\,.$$

(**Hint**: Work backwards from the answer; *i.e.* start by considerering ${}^*F_{\mu\rho}\,{}^*F_\nu{}^\rho$, and then make use of the result in Qu. (2d).)

(4a) Derive the equation of motion (Euler-Lagrange equation) that follows from the Lagrangian density

$$\mathcal{L} = -\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} - \frac{m^2}{8\pi}A^\mu A_\mu + J^\mu A_\mu$$

where m is a constant, and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. This theory is a massive generalisation of Maxwell's theory, known as the Proca theory. Note that the Proca equation of motion is not gauge invariant.

(4b) Assuming that the 4-current J^μ is conserved (i.e. $\partial_\mu J^\mu = 0$), show that the Proca equation of motion implies that A^μ obeys $\partial_\mu A^\mu = 0$.

Due Wednesday October 2nd, in class.