611: Electromagnetic Theory

Homework 6

(**1**) Consider an isolated system of electromagnetic fields, with (conserved) energy-momentum tensor $T_{\mu\nu}$. The conserved 4-momentum P^μ and conserved angular momentum $M^{\mu\nu}$ are defined as in eqns (4.81) and (4.92) in the lecture notes. Show that the conservation law for the M^{0i} components implies that

$$\frac{d\vec{R}}{dt} = \frac{\vec{P}}{\mathcal{E}}\,, \tag{1}$$

where $\vec{R}$ is the centre of mass of the electromagnetic field, defined by $\vec{R}\int W\, d^3x = \int \vec{r}\, W\, d^3x$, where W is the energy density, $\mathcal{E} = \int W\, d^3x$ is the total energy and $\vec{P}$ is the total relativistic 3-momentum.

(**2**) In a source-free region, the Maxwell field equations are $\partial_\mu F^{\mu\nu} = 0$. Writing $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, write down the wave equation that A_μ satisfies when one imposes the Lorenz gauge condition $\partial_\mu A^\mu = 0$.

(**2a**) Look for a solution of the form $A_\mu = a_\mu \sin(k\cdot x)$, where a_μ and k_μ are *constant* 4-vectors, and the notation $k\cdot x$ means $k_\mu\, x^\mu$. Derive the *two* equations that are implied by (**1**) the Lorenz gauge condition and (**2**) the wave equation for A_μ. (One of these equations will involve only k_μ, and the other equation will involve both k_μ and a_μ.)

(**2b**) Calculate the energy-momentum tensor $T_{\mu\nu} = \frac{1}{4\pi}\,(F_{\mu\rho}\, F_\nu{}^\rho - \frac{1}{4} F^{\rho\sigma} F_{\rho\sigma}\, \eta_{\mu\nu})$. [**Note**: Make use of the two conditions on a_μ and k_μ that you derived in part (2a). The final result is very simple!]

(**2c**) Show explicitly that your result for $T_{\mu\nu}$ satisfies $\partial^\mu T_{\mu\nu} = 0$.

(**2d**) Show that the electric field and the magnetic field make equal contributions to the energy density $W = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2)$ for this field.

Turn over for problems 3 and 4

(**3a**) Suppose that k^μ is a non-spacelike vector (so $k^\mu\, k_\mu \le 0$), and that $B_{\mu\nu}$ is an arbitrary antisymmetric tensor. Define $V^\mu \equiv B^\mu{}_\nu\, k^\nu$. Show that V^μ is a non-timelike vector, i.e. that $V^\mu\, V_\mu \ge 0$.

[**Hint**: An elegant way to prove this is by considering $(V^\mu - \lambda\, k^\mu)(V_\mu - \lambda\, k_\mu)$, where λ is an arbitrary quantity that you choose appropriately in order to obtain the desired result. (This problem is rather similar to Qu. (3) of Homework 2.)]

(**3b**) Make use of your result from Qu. (3a) in order to prove that

$$T_{\mu\nu}\, k^\mu\, k^\nu \ge 0\,, \tag{2}$$

where $T_{\mu\nu}$ is the energy-momentum tensor for an arbitrary electromagnetic field. (Recall from Qu. (3c) of HW 5 that we can write $T_{\mu\nu} = \frac{1}{8\pi}(F_{\mu\rho}\, F_\nu{}^\rho + {}^*F_{\mu\rho}\, {}^*F_\nu{}^\rho)$.)

Remark: An energy-momentum tensor that obeys the inequality in eqn (2) is said to obey the *Weak Energy Condition*. It is a property of any physically-reasonable matter system, and it plays an important role in the study of the evolution of gravitating systems in the general theory of relativity. Note also that an observer with 4-velocity U^μ relative in the inertial frame S will measure an energy density $T_{\mu\nu}\, U^\mu U^\nu$, and that physical observers must always have non-spacelike 4-velocity, since they cannot travel faster than light. Thus the result in eqn (2) shows that electromagnetic field always has positive energy, as seen by any observer.

(**4a**) Apply the method discussed in the lectures to calculate the energy-momentum tensor

$$\widetilde{T}_\rho{}^\nu = -\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\sigma)}\, \partial_\rho A_\sigma + \delta_\rho^\nu\, \mathcal{L} \tag{3}$$

for the source-free Proca theory, whose Lagrangian density is

$$\mathcal{L} = -\frac{1}{16\pi}\, F^{\mu\nu}\, F_{\mu\nu} - \frac{m^2}{8\pi}\, A^\sigma\, A_\sigma\,. \tag{4}$$

(The "unimproved" energy-momentum tensor in eqn (3) is denoted here with a tilde to signify that it is not symmetric when its first index is raised: $\widetilde{T}^{\mu\nu} \ne \widetilde{T}^{\nu\mu}$.)

(**4b**) Now construct the improved energy-momentum tensor $T^{\mu\nu}$ that *is* symmetric, by adding an appropriate term $\partial_\sigma \psi^{\mu\nu\sigma}$ to $\widetilde{T}^{\mu\nu}$, where $\psi^{\mu\nu\sigma} = -\psi^{\mu\sigma\nu}$. (Note: You can be guided by what was done in the Maxwell case in the lecture notes. But make sure you don't overlook any consequences of adding the $\partial_\sigma \psi^{\mu\nu\sigma}$ term! In particular, pay attention to the steps described in eqn (4.113) in the lecture notes.)

(**4c**) Check that the $T^{\mu\nu}$ you obtained in Qu. (4b) really does satisfy the conservation equation $\partial_\nu\, T^{\mu\nu} = 0$, upon using the Proca equation of motion (see Qu. (4a) of Homework 5). Note that any mistake made in obtaining $T^{\mu\nu}$ in Qu. (4b) is likely to be discovered here, when you check that $\partial_\nu T^{\mu\nu} = 0$.

Due Wednesday October 16th, in class