611: Electromagnetic Theory

Homework 7

(**1**) Consider the expression for the electric field due to a charge e moving with uniform velocity $\vec{v}$, as derived in the lectures. Evaluate the Gaussian surface integral

$$Q = \frac{1}{4\pi} \int_S \vec{E} \cdot d\vec{S}$$

explicitly over a convenient spherical surface that encloses the moving charge at some particular instant. (It is convenient to choose a sphere that is centred on the point where the particle is located at the instant.) Discuss why you answer is physically reasonable.

Note: You must actually evaluate the surface integral explicitly here; you can use the expression for $\vec{E}$ in eqn (5.41) in the lecture notes. You should also present the evaluation of the integral explicitly (it is not a difficult integral); don't just report that "Mathematica says the answer is..."

(**2**) A particle of charge e moves along the x axis with (relativistic) constant velocity v, and so the electric field is given by eqn (5.41) in the lecture notes. A second particle, carrying charge q, is fixed at the point $y = a$ on the y axis (i.e. at the coordinate position $(0, a, 0)$). Calculate the total *impulse* $\vec{I} \equiv \int_{-\infty}^{\infty} \vec{F}\, dt$ that is experienced by the fixed particle over the period from $t = -\infty$ to $t = \infty$. ($\vec{F}$ is the 3-force exerted on the fixed particle by the moving particle.)

(**3a**) Consider the **non-relativistic** motion of a particle of mass m moving under the influence of a central force $\vec{F} = \kappa \vec{r}/r^3$. Show, using the equations of motion, that the Runge-Lenz vector

$$\vec{W} = \vec{p} \times \vec{L} + \frac{m\, \kappa\, \vec{r}}{r}$$

is conserved, i.e. that $d\vec{W}/dt = 0$, where $\vec{L} = \vec{r} \times \vec{p}$ is the angular momentum of the particle about the centre of the force at $r = 0$. (The calculation is most easily done in Cartesian coordinates.)

(**3b**) Suppose now that the orbit is an ellipse (as given by eqn (5.83) in the notes); that is, $\kappa < 0$ and $-m\kappa^2/(2\ell^2) < E < 0$. Show that the Runge-Lenz vector $\vec{W}$ is aligned with the major axis of the ellipse.

[**Note**: By making use of the result from Qu. (3a) that $\vec{W}$ is conserved, you can in fact establish the direction of $\vec{W}$ merely by knowing that that the orbit is an ellipse with the central charge at the focus, without needing the full details of the result in eqn (5.83). (As long as you give a reasoned argument for how you obtained the result.)

Turn over ...

It is fine instead to use eqn (5.83) if you prefer.]

(**4**) Consider the **relativistic** orbital motion of a particle of charge e in the Coulomb potential of a charged scattering centre of charge Q, in the case where $\ell^2 > \kappa^2$ and $\kappa := e\,Q > 0$ (repulsive force), so it is described by eqn (5.79) in the lecture notes. Show that the scattering angle ψ for a particle with impact parameter b, and whose 3-velocity at infinity is v, is given by

$$\psi = \pi - \frac{2\Phi}{\sqrt{1 - z^{-1}}}\,, \tag{1}$$

where we define

$$\Phi := \arctan\xi\,, \qquad \xi := v\,\sqrt{z-1}\,, \qquad z := \Big(\frac{b}{a}\Big)^2\,\frac{v^2}{1-v^2}\,, \qquad a := \frac{\kappa}{m}\,. \tag{2}$$

Note: The particle is sent in from infinity, passes close to the scattering Coulomb potential, and then passes out to infinity again. The *scattering angle* is defined as the change in angle between the ingoing and outgoing asymptotic paths (so it would be zero if there were no charge Q present). The *impact parameter* is defined as the perpendicular distance from the scattering centre to the "ingoing" straight-line asymptote. (*i.e.* it is the distance by which the ingoing particle would "miss" the scattering centre if the charge Q were zero. Note that this is **not** the same as the particle's distance of closest approach to the scattering centre when it follows its actual path!)

Hints: It is probably simplest to orient the axes so that the point of closest approach on the particle's trajectory occurs at $\varphi = 0$ (as in eqn (5.79) in the notes). In tackling this problem it is very helpful to **draw a diagram**, and, in particular, to make sure you correctly identify what is meant by the impact parameter. Note that there are two parameters (constants of integration) characterising the orbital path in eqn (5.79), namely ℓ (related to the angular momentum) and $\mathcal{E}$ (the energy). These need to be re-expressed in terms of the data specified in the problem, namely the impact parameter b and the velocity at infinity, v. It is useful to remember that $\mathcal{E}$ and ℓ are conserved quantities, so they can be evaluated at any convenient point on the path of the particle. The easiest way to find $\mathcal{E}$ and ℓ is to evaluate them when the particle is out at infinity. By this means you should be able to show that $\mathcal{E} = m\,(1-v^2)^{-1/2}$ and $\ell = \mathcal{E}\,b\,v$. (Derive these results, presenting the steps in the argument; don't just state them!)

Having found $\mathcal{E}$ and ℓ, the rest of the problem comes down to using eqn (5.79) in the notes to figure out the angles φ corresponding to when the particle is out at infinity, and then making a few manipulations to arrive at the formula (1) in the question.

Historical Remark: This calculation was first done by Charles Galton Darwin (the grandson of "the" Charles Darwin) in 1913.

Due Wednesday October 23rd, in class.