611: Electromagnetic Theory

Homework 9

(**1a**) Construct the wavefunctions $\psi = B_z$ and corresponding eigenvalues Ω^2_{mn} for TE modes in a rectangular waveguide with sides at $x = 0$, $x = a$ and at $y = 0$, $y = b$.

(**1b**) Consider the special case of a rectangular waveguide when $b = a$. Show how to construct the wavefunctions $\psi = B_z$ for TE modes in the isosceles right triangle defined by the three vertices $(x, y) = (0, 0)$; $(x, y) = (a, 0)$; and $(x, y) = (a, a)$, in terms of linear combinations of the wavefunctions for the square waveguide. Show explicitly that the required boundary conditions are satisfied on all three sides.

(**1c**) Spell out the restrictions on the integers m and n in the expressions for the eigenvalues Ω^2_{mn} in the four cases: (1) TM modes in the square; (2) TE modes in the square; (3) TM modes in the isosceles right triangle; (4) TE modes in the isosceles right triangle. List the first three non-vanishing eigenvalues in each case, and give their degeneracies. (For the TM modes, you can use results in the notes.)

(**2a**) Let the vertices of an equilateral triangle be at $(x, y) = (0, 0)$; $(x, y) = (\frac{h}{\sqrt{3}}, h)$; and $(x, y) = (-\frac{h}{\sqrt{3}}, h)$. Show that the functions

$$\psi_{mn} = W\Big[\frac{(m-n)\pi x}{\sqrt{3}\,h}\Big]\,\sin\frac{\ell\pi y}{h} + W\Big[\frac{(n-\ell)\pi x}{\sqrt{3}\,h}\Big]\,\sin\frac{m\pi y}{h} + W\Big[\frac{(\ell-m)\pi x}{\sqrt{3}\,h}\Big]\,\sin\frac{n\pi y}{h}\,,$$

are eigenfunctions of the 2-dimensional Laplacian $\nabla^2_\perp$, where $\ell \equiv -m - n$ n in of the two cases $W(u) = \sin u$ and $W(u) = \cos u$. (*i.e.* Either all three W's are sine, or else all three are cosine.) Calculate the corresponding eigenvalues Ω^2_{mn} of $\nabla^2_\perp\,\psi_{mn} + \Omega^2_{mn}\,\psi_{mn} = 0$.

(**2b**) Show that ψ_{mn} in part (**2a**) satisfies the proper boundary conditions for TM modes on **all three** sides of the triangle, for both choices for W. (Note: The calculation does require a little non-trivial trigonometric manipulation. Show this explicitly.)

(**2c**) What is the lowest eigenvalue $\Omega^2_{\rm min}$ for TM modes in this waveguide? Is it degenerate?

(**3a**) Consider a hollow conducting sphere of radius a. Use spherical polar coordinates thoughout this problem. Consider TM electromagnetic waves with angular frequency ω inside the cavity, so the radial component of the magnetic field vanishes, $B_r = 0$. **For simplicity, restrict attention to fields that are independent of the azimuthal angle** φ. Show that the Maxwell equations imply that it is consistent to set also $B_\theta = 0$, and so the magnetic field is then given by the one non-vanishing component

$$B_\varphi = \psi(r, \theta)\, e^{-{\rm i}\,\omega\, t}\,. \qquad (1)$$

Qu. (3) and Qu. (4) continue on next page...

Use the Maxwell equations to derive the second-order partial differential equation satisfied by ψ. Obtain expressions also for the components E_r, E_θ and E_φ of the electric field, in terms of the function ψ. [You can refer to the expressions for $\vec\nabla\cdot\vec V$ and $\vec\nabla\times\vec V$ in spherical polar coordinates that were given in HW8.]

(**3b**) Separate variables in the standard way, by writing $\psi(r,\theta)=R(r)\,P(\theta)$, and obtain the second-order ordinary differential equations satisfied by $R(r)$ and $P(\theta)$. Show that the solutions for $R(r)$ are given by the spherical Bessel functions $j_\ell(\omega\, r)$ and $n_\ell(\omega\, r)$, and that the (regular) solutions for $P(\theta)$ are given by certain of the associated Legendre functions $P_\ell^m(\cos\theta)$. (The spherical Bessel functions are described below. The Associated Legendre equation and the standard Bessel equation, and their solutions, are discussed in chapters 4 and 5 of my PHYS 603 lectures, on my webpage.)

(**4a**) Continuing with the set-up in Qu. (3), impose the boundary condition at $r=a$ (tangential components of $\vec E$ vanish at $r=a$), together with regularity at $r=0$. Obtain the transcendental equations involving a, ω and j_ℓ that will restrict the allowed values of ω to a discrete, infinite set $\omega_{\ell\,,n}$, with $\ell=1,2,3,\ldots$ and $n=1,2,3,\ldots$. To see how this works, consider as an example the lowest mode, $\ell=1$, and obtain the corresponding transcendental equation that determines the allowed values for $\omega_{1\,,n}$. Draw graphs to show how the discrete infinity of these $\omega_{1\,,n}$ eigenvalues occurs. (The procedure is quite similar to the way one can find the energy levels of a quantum-mechanical particle in a square-well potential.)

(**4b**) Find an approximate numerical value for the lowest eigenfrequency, $\omega_{1\,,1}$, among the $\ell=1$ TM modes. (Don't just state an answer! Show briefly how you did it; graphically, Newton-Raphson, or ... .)

The first few spherical Bessel functions $j_\ell(x)=\sqrt{\frac{\pi}{2x}}\,J_{\ell+\frac12}(x)$ (regular at $x=0$) and $n_\ell(x)=\sqrt{\frac{\pi}{2x}}\,Y_{\ell+\frac12}(x)$ (singular at $x=0$) are:

$$
\begin{aligned}
j_0(x) &= \frac{\sin x}{x}\,,\quad j_1(x)=\frac{\sin x}{x^2}-\frac{\cos x}{x}\,,\quad j_2(x)=\Big(\frac{3}{x^2}-1\Big)\frac{\sin x}{x}-\frac{3\cos x}{x^2}\,,\\
n_0(x) &= -\frac{\cos x}{x}\,,\quad n_1(x)=-\frac{\cos x}{x^2}-\frac{\sin x}{x}\,,\quad n_2(x)=\Big(1-\frac{3}{x^2}\Big)\frac{\cos x}{x}-\frac{3\sin x}{x^2}\,.
\end{aligned}
$$

Note that we must reject the $n_\ell(x)$ solutions because they are singular at $x=0$.

Due Wednesday November 13th, in class.