# 647: Gravitational Physics

Problem Sheet 3

(**1a**) Suppose $A_\mu$ is a general-coordinate co-vector. Show that $W_{\mu\nu} \equiv \partial_\mu A_\nu$ is *not* a general-coordinate tensor. (i.e. show explicitly that it does not transform as a tensor under general coordinate transformations.)

(**1b**) On the other hand, show explictly that $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ *does* transform as a general-coordinate tensor (*i.e.* show explicitly that it transforms as in eqn (4.20) of the notes, for a $(p,q) = (0,2)$ tensor.)

(**1c**) Suppose $G_{\mu\nu}$ is an **antisymmetric** $(0,2)$ tensor. Show explicitly that the $W_{\mu\nu\rho}$, defined by

$$W_{\mu\nu\rho} \equiv \partial_\mu G_{\nu\rho} + \partial_\nu G_{\rho\mu} + \partial_\rho G_{\mu\nu}\,,$$

is a tensor. That is, show explicitly that $W_{\mu\nu\rho}$ transforms as in eqn (4.20) in the notes for a general-coordinate $(0,3)$ tensor.

(**2a**) Consider the transformation from a Minkowski coordinate system $(\tilde t, \tilde x, \tilde y, \tilde z)$ to a non-inertial coordinate system $(t, x, y, z)$ where

$$\tilde t = t\,, \qquad \tilde x = x + \tfrac{1}{2} a t^2\,, \qquad \tilde y = y\,, \qquad \tilde z = z\,,$$

and where $a$ is a constant. Calculate the components of the Christoffel connection $\Gamma^\mu{}_{\nu\rho}$, using eqn (3.11) in the notes.

(**2b**) Show that the geodesic equation (eqn (3.10) in the notes) reduces in the Newtonian low-velociy limit to $d^2x^i/dt^2 + \Gamma^i{}_{00} = 0$. Look at this limit for the connection you found in part (2a), and show that it reproduces the expected Newton second law for a particle in a uniform gravitational field.

(In the low-velocity Newtonian limit the 4-velocity $dx^\mu/d\tau = U^\mu = (\gamma, \gamma\vec u\,)$ reduces to $U^\mu \approx (1, \vec u)$ with $|\vec u\,| << 1$, and $d\tau \approx dt$. (Here $\vec u$ is the 3-velocity; i.e. $u^i = dx^i/dt$.))

(**3a**) Consider the transformation from a Minkowski coordinate system $(\tilde t, \tilde x, \tilde y, \tilde z)$ to a rotating non-inertial coordinate system $(t, x, y, z)$ where

$$\tilde t = t\,, \qquad \tilde x = x\, \cos\omega t - y\, \sin\omega t\,, \qquad \tilde y = x\, \sin\omega t + y\, \cos\omega t\,, \qquad \tilde z = z\,,$$

where $\omega$ is a constant. Calculate the metric expressed in terms of the untilded coordinates. (The expression is quite simple.)

**Turn over for questions (3b) and 4...**

**(3b)** Calculate the Christoffel connection for the metric you obtained in part (3a). Show that the only non-vanishing components are

$$\Gamma^1{}_{00} = -\omega^2\, x\,, \qquad \Gamma^1{}_{02} = \Gamma^1{}_{20} = -\omega\,, \qquad \Gamma^2{}_{00} = -\omega^2\, y\,, \qquad \Gamma^2{}_{01} = \Gamma^2{}_{10} = \omega\,. \quad (1)$$

Show that in the Newtonian low-velocity limit, the geodesic equation produces the expected equations for a particle that experiences centrifugal and Coriolis forces.

**(4a)** Calculate the metric in Euclidean space, $ds^2 = dx^2 + dy^2 + dz^2$, when expressed in terms of the spherical polar coordinates $(r, \theta, \varphi)$, where

$$x = r\, \sin\theta\, \cos\varphi\,, \qquad y = r\, \sin\theta\, \sin\varphi\,, \qquad z = r\, \cos\theta\,. \qquad (2)$$

The metric has the form $ds^2 = dr^2 + r^2\, d\Omega^2$, where $d\Omega^2$ is the metric on a unit-radius sphere. Read off $d\Omega^2$ in terms of the $\theta$ and $\varphi$ coordinates.

**(4b)** The *Mercator projection* maps points on the sphere (for example, the earth) onto a rectangular Cartesian plane with coordinates $(u, v)$ defined by

$$u = \varphi\,, \qquad v = \log\cot\frac{\theta}{2}\,. \qquad (3)$$

Calculate the metric $ds_2^2 = du^2 + dv^2$ on the Mercator plane in terms of $\theta$ and $\varphi$, and show that it is conformal to the metric $d\Omega^2$ on the unit sphere, i.e. $ds_2^2 = \Psi^2\, d\Omega^2$, with a conformal factor $\Psi^2$ that you should find explicitly.

**Remark**: *This conformal property shows that the Mercator projection has the useful feature that the angle between two intersecting paths on the surface of the earth will be the same as the angle between the corresponding paths plotted on the Mercator map.*

**(4c)** Show that the Mercator projection has the property that straight lines in the $(u, v)$ plane correspond to trajectories on the unit sphere (or the earth) that maintain a constant compass bearing (i.e. relative to true north).

**Note**: "Maintaining a constant compass bearing" means that for each point along the trajectory, the angle between the tangent to the trajectory and the line of longitude that passes through that point will be the same all along the trajectory.

**Remark**: *The property in (4c) is very useful for a world navigator who wants to figure out how to steer a course that will arrive at a specific location.*[1]

**Due in class on Thursday 25th September**

[1] Although maps using the Mercator projection have the nice features established in (4b) and (4c) above, they have the disadvantage that regions nearer the poles appear disproportionately larger in area relative to those nearer the equator, culminating in becoming completely useless at the north and south poles. Equal-area map projections do exist, but they have other disadvantages. For example, as was proved by Gauss, they are necessarily non-conformal, so angles and shapes are not preserved. There is no such thing as a "perfect" map projection from the sphere to the plane that embodies all the features one might want.