647: Gravitational Physics

Problem Sheet 7

(**1a**) Consider the metric

$$ds^2 = 2dudv + dx^2 + dy^2 + H\,du^2\,,$$

where $H = H(u, x, y)$ is an arbitrary function of u, x and y. Show that it solves the vacuum Einstein equations (i.e. it is Ricci-flat, $R_{\mu\nu} = 0$) if

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = 0\,.$$

(The metric describes a gravitational plane wave.)

(**1b**) Calcuate the non-vanishing components $R_{\mu\nu\rho\sigma}$ of the Riemann tensor, and hence calculate $|\text{Riem}|^2 \equiv R^{\mu\nu\rho\sigma}\,R_{\mu\nu\rho\sigma}$ for these metrics.

Note: The Christoffel connection can be calculated quite easily using the geodesic equation method, but because the metric has non-diagonal components you will need to take certain combinations of the Euler-Lagrange equations in order to be able to read off the connection coefficients. It is convenient to use a labelling $x^\mu = (u, v, x, y)$ for $\mu = 0, 1, 2, 3$, and to split the μ index as $\mu = (0, 1, i)$ with $i = 2, 3$, so that the x and y directions can be handled as a package, preserving the manifest Euclidean covariance of the (x^2, x^3) plane. That is, you never need to write separate expressions for connection or curvature components in the "2" and "3" directions; they can always be written together more compactly and elegantly as i components, etc.

The "Weinberg formula" in eqn (4.75) of the lecture notes is probably the most convenient way to calculate the Riemann tensor in this problem. Note that there are rather few non-vanishing components of $R_{\mu\nu\rho\sigma}$. This, together with the simple nature of the inverse metric $g^{\mu\nu}$, results in an extremely simple (!) answer for $|\text{Riem}|^2$.

(**Caution: Make sure you calculate $g^{\mu\nu}$ correctly!**)

(**2**) Suppose that a metric $\tilde{g}_{\mu\nu}$ has Christoffel connection $\widetilde{\Gamma}^\mu{}_{\nu\rho}$ that is given by

$$\widetilde{\Gamma}^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\nu\rho} + W^\mu{}_{\nu\rho}\,,$$

where $\Gamma^\mu{}_{\nu\rho}$ is the Christoffel connection for a metric $g_{\mu\nu}$ and $W^\mu{}_{\nu\rho}$ is a $(1, 2)$ tensor with $W^\mu{}_{\nu\rho} = W^\mu{}_{\rho\nu}$. Calculate the Riemann tensor $\widetilde{R}^\mu{}_{\nu\rho\sigma}$ for the connection $\widetilde{\Gamma}^\mu{}_{\nu\rho}$ and show that it can be written in terms of the Riemann tensor $R^\mu{}_{\nu\rho\sigma}$ for the connection $\Gamma^\mu{}_{\nu\rho}$ as

$$\widetilde{R}^\mu{}_{\nu\rho\sigma} = R^\mu{}_{\nu\rho\sigma} + \nabla_\rho W^\mu{}_{\sigma\nu} - \nabla_\sigma W^\mu{}_{\rho\nu} + W^\mu{}_{\rho\lambda}\,W^\lambda{}_{\sigma\nu} - W^\mu{}_{\sigma\lambda}\,W^\lambda{}_{\rho\nu}\,,$$

where ∇_μ is the covariant derivative defined using the connection $\Gamma^\mu{}_{\nu\rho}$.

Turn over for comment, and for questions 3 and 4...

Note: In other words, one has to show explicitly that all the untilded connection terms that arise in the calculation either assemble into forming the untilded Riemann tensor or else into "covariantising" the partial derivatives acting on W, thereby turning them into covariant derivatives.

(**3**) An n-dimensional spacetime metric $\tilde{g}_{\mu\nu}$ is related to $g_{\mu\nu}$ by means of the conformal scaling

$$\tilde{g}_{\mu\nu} = e^{2\sigma}\, g_{\mu\nu}\,,$$

where σ is an arbitrary function of the spacetime coordinates. Show that the Christoffel connection for the metric $\tilde{g}_{\mu\nu}$ is given by

$$\widetilde{\Gamma}^{\mu}{}_{\nu\rho} = \Gamma^{\mu}{}_{\nu\rho} + W^{\mu}{}_{\nu\rho}$$

where $\Gamma^{\mu}{}_{\nu\rho}$ is the Christoffel connection of the metric $g_{\mu\nu}$ and where

$$W^{\mu}{}_{\nu\rho} = \delta^{\mu}_{\nu}\,\partial_\rho\sigma + \delta^{\mu}_{\rho}\,\partial_\nu\sigma - g_{\nu\rho}\, g^{\mu\lambda}\,\partial_\lambda\sigma\,,$$

Notice that $W^{\mu}{}_{\nu\rho}$ here is indeed a tensor.

(**4a**) Show that the Ricci tensor $\widetilde{R}_{\mu\nu}$ of the metric $\tilde{g}_{\mu\nu}$ in Qu. (3) is given in terms of the Ricci tensor $R_{\mu\nu}$ of the metric $g_{\mu\nu}$ by

$$\widetilde{R}_{\mu\nu} = R_{\mu\nu} - (n-2)\nabla_\mu\nabla_\nu\sigma - g_{\mu\nu}\,\Box\sigma + (n-2)\Big[\partial_\mu\sigma\,\partial_\nu\sigma - g_{\mu\nu}\,(\partial\sigma)^2\Big]\,,$$

where $\Box = \nabla^\mu\nabla_\mu$ is the covariant D'Alembertian (calculated in the untilded metric), and $(\partial\sigma)^2$ means $g^{\mu\nu}\,\partial_\mu\sigma\,\partial_\nu\sigma$.

(**4b**) Calculate also the Ricci scalar $\widetilde{R}$ of $\tilde{g}_{\mu\nu}$ in terms of the Ricci scalar R of $g_{\mu\nu}$.

Note: You can make use of the result obtained in Qu. (2) in order to do this question.

Due in class on Thursday 30th October