Lecture notes for Feb 24, 2022
Flows, Grötzsch's theorem, and critical graphs

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February 24, 2022

1 Nowhere zero flows

1.1 6-flow theorem

We proved the following 2 lemmas last time.

Proposition 1 *Let $k \geq 3$ be a positive integer. Then the following statements are equivalent.*

1. *Every 2-edge-connected multigraph has a nowhere zero k-flow.*
2. *Every 3-edge-connected cubic graph has a nowhere zero k-flow.*

Lemma 2 *Let G be a 3-connected graph. Then $V(G)$ can be partitioned into parts $V_1, V_2, ..., V_m$ for some integer m such that*

1. *for each $i \in [m]$, either $|V_i| = 1$, or $G[V_i]$ contains a Hamiltonian cycle,*
2. *$|V_1| = 1$, and*
3. *for each $2 \leq i \leq m$, G has at least 2 edges between V_i and $V_1 \cup V_2 \cup ... \cup V_{i-1}$.*

Now we are ready to prove the existence of nowhere zero 6-flows.

Theorem 3 (Seymour) *Every 2-edge-connected graph has a nowhere zero 6-flow.*

Proof. By Proposition 1, it suffices to show that every 3-edge-connected cubic graph has a nowhere zero 6-flow. Let G be a 3-edge-connected cubic graph. So G is 3-connected. By Lemma 2, there exists a partition $(V_1, V_2, ..., V_m)$ of $V(G)$ satisfying the conclusion of Lemma 2. For every $2 \leq i \leq m$, choose 2 edges $e_{i,1}$ and $e_{i,2}$ of G between V_i and $\bigcup_{j=1}^{i-1} V_i$. For every $i \in [m]$ with $|V_i| \geq 2$, let C_i be a Hamiltonian cycle of $G[V_i]$. For every $i \in [m]$ with $|V_i| = 1$, we define C_i to be the empty graph.

Let G' be the spanning subgraph of G with $E(G') = \bigcup_{j=1}^{m} E(C_j) \cup \bigcup_{j=2}^{m}\{e_{j,1}, e_{j,2}\}$. For every $i \in [m]$, let $G_i = G'[\bigcup_{j=1}^{i} V_j]$. A simple induction on i shows that for each $2 \leq i \leq m$, G_i is connected and there exists a cycle Q_i in G_i containing $e_{i,1}$ and $e_{i,2}$.

Let D be an orientation of G.

Since G_m is connected, there exists a spanning tree T_m of G_m. Since $V(G_m) = V(G') = V(G)$, T_m is also a spanning tree of G. So there exists a $\mathbb{Z}_3$-flow f_m of D such that $\{e \in E(G) : f_m(e) = 0\} \subseteq E(T_m) \subseteq E(G_m) = E(G') = \bigcup_{j=1}^{m} E(C_j) \cup \bigcup_{j=2}^{m}\{e_{j,1}, e_{j,2}\}$. Hence f_m is a $\mathbb{Z}_3$-flow of D such that $\{e \in E(G) : f_m(e) = 0\} \subseteq \bigcup_{j=1}^{m} E(C_j) \cup \bigcup_{j=2}^{m}\{e_{j,1}, e_{j,2}\}$.

Let i be the minimum positive integer such that there exists a $\mathbb{Z}_3$-flow f_i of D such that $\{e \in E(G) : f_i(e) = 0\} \subseteq \bigcup_{j=1}^{m} E(C_j) \cup \bigcup_{j=2}^{i}\{e_{j,1}, e_{j,2}\}$. (We know $i \leq m$.) We shall prove that $i = 1$.

Suppose $i \geq 2$. Let $1_i : E(D) \to \mathbb{Z}_3$ be the function such that $1_i(e) = 1$ if $e \in E(Q_i)$, and $1_i(e) = 0$ otherwise. Since $|\{e_{i,1}, e_{i,2}\}| = 2$, one of $f_i, f_i + 1_i, f_i + 2 \cdot 1_i$, say g, is a $\mathbb{Z}_3$-flow of G such that $g(e_{i,1}) \neq 0 \neq g(e_{i,2})$. Note that $g(e) - f_i(e) \neq 0$ only when $e \in E(Q_i) \subseteq E(G_i) \subseteq \bigcup_{j=1}^{m} E(C_j) \cup \bigcup_{j=2}^{i}\{e_{j,1}, e_{j,2}\}$. Hence g is a $\mathbb{Z}_3$-flow of D such that $\{e \in E(G) : g(e) = 0\} \subseteq \bigcup_{j=1}^{m} E(C_j) \cup \bigcup_{j=2}^{i-1}\{e_{j,1}, e_{j,2}\}$. This contradicts the minimality of i.

Therefore, $i = 1$. In other words, f_1 is a $\mathbb{Z}_3$-flow of D such that $\{e \in E(G) : f_1(e) = 0\} \subseteq \bigcup_{j=1}^{m} E(C_j)$.

Since $\bigcup_{j=1}^{m} C_j$ is a 2-regular graph, there exists a nowhere zero $\mathbb{Z}_2$-flow f_0 of $\bigcup_{j=1}^{m} C_j$. Then (f_0, f_1) is a nowhere zero $\mathbb{Z}_2 \times \mathbb{Z}_3$-flow of D. So G has a nowhere zero 6-flow. ∎

1.2 3-flows

Can we get nowhere zero 3-flows? Petersen graph has no nowhere zero 4-flows, so it does not have nowhere zero 3-flows. Can we increase the edge-connectivity to ensure the existence of nowhere zero 3-flows?

Conjecture 4 (Tutte's 3-Flow Conjecture) *Every 4-edge-connected graph has a nowhere zero 3-flow.*

Tutte's 3-Flow Conjecture remains open. The following two theorems are the currently best results.

Theorem 5 (Kochol) *Tutte's 3-Flow Conjecture* $\Leftrightarrow$ *every 5-edge-connected graph has a nowhere zero 3-flow.*

Theorem 6 (Lovász, Thomassen, Wu, Zhang) *Every 6-edge-connected graph has a nowhere zero 3-flow.*

2 Grötzsch's theorem

Recall that Tutte's 3-flow conjecture states that every 4-edge-connected graph has a nowhere zero 4-flow. Restricting to planar graphs, the dual version states that every graph with girth at least 4 is properly 3-colorable, which is Grötzsch's theorem.

Theorem 7 (Grötzsch) *If* G *is a planar graph with no triangle, then* $\chi(G) \leq 3$.

There are many different proofs of Grötzsch's theorem in the literature. Here we present a proof that uses the density of critical graphs.

Intuitively, if a graph is sparse, then it is "easy" to be colored so that it has small chromatic number. So every graph with large chromatic number is expected to be dense, in the sense that the average degree is large. However, this intuition is not quite correct because we can attach a tree to a graph with large chromatic number to make the average degree small without decreasing the chromatic number. Hence, we should consider the density of minimal subgraphs with large chromatic number.

For a positive integer k, a graph is *k-critical* if it is not $(k-1)$-colorable but every its subgraph is $(k-1)$-colorable. Note that every k-critical graph is k-colorable. The following theorem gives a quantitative bound for the density of a 4-critical graph.

Theorem 8 (Kostochka, Yancey) *If* G *is a 4-critical graph, then* $|E(G)| \geq \frac{5}{3}|V(G)| - \frac{2}{3}$.

Grötzsch's theorem is a simple corollary of this theorem (we will prove both theorems next time).

3 Critical graphs

Before proving Theorem 8, we give some background about critical graphs.

A *clique cut-set* of a graph G is a clique S in G such that $G - S$ is disconnected. We have the following simple properties.

Proposition 9 *Let k be a positive integer. If G is a k-critical graph, then G has minimum degree at least $k-1$, and G has no clique cut-set. In particular, G is 2-connected.*

Note that the minimum degree property in the above proposition implies that for every k-critical graph G, $|E(G)| \geq \frac{k-1}{2}|V(G)|$. So we get a lower bound for the edge-density for k-critical graphs. This bound can be improved.

Let's first consider the best possible lower bound for the edge-density. To do so, we want to get the largest function f such that there are infinitely many k-critical graphs G with $|E(G)| \geq f(|V(G)|)$.

We know K_k is a k-critical graph. And there is a way to generate infinitely many k-critical graphs.

For graphs G_1 and G_2, the *Hajós' construction from G_1 and G_2* is the graph obtained from a disjoint union of G_1 and G_2 by

- first taking an edge u_1v_1 of G_1 and an edge u_2v_2 of G_2,
- then identifying u_1 and u_2 into a vertex u^*, and
- then deleting u_1v_1 and u_2v_2, and adding the edge v_1v_2.

Proposition 10 *Let $k \geq 3$ be a positive integer. If G_1, G_2 are k-critical graphs, then the Hajós-construction from G_1 and G_2 is also a k-critical graph.*

Proof. It is a simple exercise. ■

Using Hajós' construction, we obtain the following result about the edge-density of k-critical graphs.

Proposition 11 *Let $k \geq 3$ be a positive integer. Then there are infinitely many k-critical graphs G with $|V(G)| \equiv 1 \pmod{k-1}$ and*

$$|E(G)| = \frac{(k+1)(k-2)}{2(k-1)}|V(G)| - \frac{k(k-3)}{2(k-1)}.$$

Proof. Let G be the k-critical graph obtained by doing Hajós' construction t times. Then $|V(G)| = k+t(k-1)$ and $|E(G)| = \binom{k}{2}+t(\binom{k}{2}-1)$. Therefore, $|E(G)| = \binom{k}{2} + (\binom{k}{2} - 1) \cdot \frac{|V(G)|-k}{k-1}$. This equivalent to the statement of the proposition. ■

When $|V(G)| \not\equiv 1 \pmod{k-1}$, the bound can possibly be improved. Gallai conjectured that the bound in Proposition 11 is optimal when $|V(G)| \not\equiv 1 \pmod{k-1}$. Kostochka and Yancey proved this conjecture.

Theorem 12 (Kostochka, Yancey) *Let $k \geq 4$ be an integer. If G is a k-critical graph, then*

$$|E(G)| \geq \frac{(k+1)(k-2)}{2(k-1)}|V(G)| - \frac{k(k-3)}{2(k-1)}.$$

Note that Theorem 8 is exactly the case $k = 4$ of Theorem 12. We will only prove the $k = 4$ case in this course. Note that we did not include the $k = 3$ case in Theorem 12, but it also holds for $k = 3$, since it is easy to prove that the 3-critical graphs are exactly the odd cycles.