Lecture notes for Feb 20, 2023
Shortest paths in general weighted digraphs and introduction of maximum flows

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1 Shortest paths in general weighted digraphs

The key requirement for Dijkstra's algorithm is that every edge has nonnegative weight. But we need negative weights for certain applications. For example, considering that we want to go from a vertex s to a vertex t in a digraph. When we traverse an edge e, we should pay $a(e)$ but we can gain $b(e)$. So the net profit for traversing e is $b(e) - a(e)$. Hence if we want to find a path from s to t that makes the most profit, then we can define the weight on e to be $b(e) - a(e)$, and we want to find a longest path from s to t. Equivalently, if we define $w(e) = a(e) - b(e)$ for each edge e, then we want to find a shortest path in (D, w) from s to t. Note that the weight can be negative in this case.

However, finding a shortest path in weighted graph with possibly negative weight is NP-hard, even for the special case for finding longest path in an unweighted graph. Given a graph G, a *Hamiltonian path* in G is a path that uses all vertices of G.

Theorem 1 *The problem of determining whether the input graph has a Hamiltonian path is NP-complete.*

As a corollary, if we can find the longest path in an unweighted graph G, then we can simply see whether its length equals $|V(G)| - 1$ to decide whether G has a Hamiltonian path or not. So finding longest path in an unweighted graph is NP-hard, and so does finding shortest path in weighted graph with possibly negative weight.

But the problem for finding shortest path in weighted graph is tractable if we assume that there is no cycle with negative weight.

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Bellman-Ford algorithm

Input: A weighted digraph (D, w) and a vertex $r \in V(D)$, where w is a function such that there exists no directed cycle C with $\sum_{e \in E(C)} w(e) < 0$.

Output: $d_{(D,w)}(r, v)$ for each $v \in V(D)$, and a shortest path in (D, w) from r to each $v \in V(D)$ with $d_{(D,w)}(r, v) \neq \infty$.

Procedure:

Step 1: Set $f_0(r) = 0$ and $f_0(v) = \infty$ for every $v \in V(D) - \{r\}$. Set $p(r) = r$.

Step 2: For $i = 1, 2, ..., |V(D)| - 1$, do the following:

- For every $v \in V(D)$, define $f_i(v) = \min\{f_{i-1}(v), f_{i-1}(u)+w((u, v)) : (u, v) \in E(D)\}$, and if $f_i(v) < f_{i-1}(v)$, then set $p(v)$ to be a vertex u with $(u, v) \in E(D)$ and $f_i(v) = f_{i-1}(u) + w((u, v))$.

Step 3: For every $v \in V(D)$, output $d_{(D,w)}(r, v) = f_{|V(D)|-1}(v)$, and if $f_{|V(D)|-1}(v) < \infty$, then output the reverse of the walk v-$p(v)$-$p(p(v))$-...-r.

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Lemma 2 *During the entire process, the following properties are preserved: For any i with $0 \leq i \leq |V(D)| - 1$ and $v \in V(D)$ with $f_i(v) \neq \infty$,*

1. *$p(v)$ is defined, $f_i(p(v)) \neq \infty$ and $(p(v), v) \in E(D)$,*
2. *the maximal walk v-$p(v)$-$p(p(v))$-... is well-defined and is the reverse of a directed path in D from r to v with length at most $f_i(v)$, and*
3. *$f_i(v) = \min_W w(W)$, where the minimum is over all directed walks from r to v with at most i edges.*

Proof. All properties clearly hold at the end of Step 1. Assume all properties hold after we did Step 2 $i-1$ times for some $i \in [|V(D)| - 1$. We shall prove that they also hold after we did Step 2 i times.

Fix $v \in V(D)$ with $f_i(v) \neq \infty$. If $p(v)$ is defined at the end of the $i-1$-th round of Step 2 and remains unchanged at the i-th round, then $p(v)$ is defined and $f_i(p(v)) \leq f_{i-1}(p(v)) < \infty$ by induction. If $p(v)$ was undefined at the end of the $i-1$-th round of Step 2 or $p(v)$ is changed in the i-th round, then $f_i(v) = f_{i-1}(u) + w((u,v))$ and $p(v)$ is changed to u for some $(u,v) \in E(D)$, so $p(v)$ is defined and $f_i(p(v)) = f_i(u) \leq f_{i-1}(u) < \infty$. So Property 1 holds.

The maximal walk v-$p(v)$-$p(p(v))$-... is well-defined by Property 1. Call this walk X. Suppose to the contrary that X is not a directed path. Since X traverses each edge backwards but is not a directed path, X contains a (directed) cycle C. So there exists $x \in V(C)$ such that $C = xp(x)p(p(x))...p^{(|C|)}(x)$ with $p^{(|C|)}(x) = x$, where $p^{(j)}(x) = p(p^{(j-1)})(x)$ for every $j \geq 1$ with $p^{(0)}(x) = x$. By the definition of p, for every $j \in [|C|-1]$, there exists $i_j \leq i-1$ such that $f_i(p^{(j)}(x)) < f_{i_j}(p^{(j)}(x))$ and $f_i(p^{(j)}(x)) = f_{i_j+1}(p^{(j)}(x)) = f_{i_j}(p^{(j+1)}(x)) + w((p^{(j)}(x), p^{(j+1)}(x))) \geq f_i(p^{(j+1)}(x)) + w((p^{(j)}(x), p^{(j+1)}(x)))$. By summing both sides of this inequality for all $j \in [|C|-1]$, we obtain $\sum_{j=0}^{|C|-1} f_i(p^{(j)}(x)) \geq \sum_{j=0}^{|C|-1}(f_i(p^{(j+1)}(x)) + w((p^{(j)}(x), p^{(j+1)}(x)))) = (\sum_{j=0}^{|C|-1}(f_i(p^{(j+1)}(x))) + w(C) = (\sum_{j=1}^{|C|} f_i(p^{(j)}(x))) + w(C) = (\sum_{j=0}^{|C|-1} f_i(p^{(j)}(x))) + w(C)$, where the last equality follows from $p^{(|C|)}(x) = x = p^{(0)}(x)$. So $w(C) \leq 0$. Since we assume that there is no cycle with negative weight, $w(C) = 0$. This implies that for every $j \in [|C|-1]$, $f_{i_j}(p^{(j+1)}(x)) = f_i(p^{(j+1)}(x))$. Since $i_j \geq i-1$ for each j, we know $p^{(j+1)}(x)$ at the i-th round of Step 2 is the same as $p^{(j+1)}(x)$ at the $i-1$-th round. But it is impossible by the induction hypothesis with taking $v = x$ at the $i-1$-th round.

So X is a directed path. Hence the end o of X other than v satisfies $p(o) = o$. So $o = r$. Hence X is a directed path from r to v. Denote X by $r_1 r_2 ... r_{|V(X)|}$, where $r_1 = r$. We prove that the subpath of X from r_1 to r_j, denoted by P_j, has length at most $f_i(r_j)$ for every $j \in [|V(X)|]$ by induction on j. It clearly holds when $j = 1$. When $j \geq 2$, let i_j be the integer such that $p(v)$ was defined or redefined at the i_j-th round and remains unchanged since then, so by the induction hypothesis, $w(P_j) = w(P_{j-1}) + w((r_{j-1}, r_j)) \leq f_i(r_{j-1}) + w((r_{j-1}, r_j)) \leq f_{i_j-1}(r_{j-1}) + w((r_{j-1}, r_j)) = f_{i_j}(p(r_j)) + w((p(r_j), r_j)) = f_i(r_j)$. Therefore, $w(X) = w(P_{|V(X)|}) \leq f_i(r_{|V(X)|}) = f_i(v)$. This proves Property 2.

Finally, we prove Property 3. Let $\ell = \min_W w(W)$, where the minimum is over all directed walks from r to v with at most i edges. Since $f_i(v) \neq \infty$, either $f_{i-1}(v) = f_i(v) \neq \infty$ or $f_i(v) = f_{i-1}(u) + w((u,v))$ for some $(u,v) \in E(D)$ with $f_{i-1}(u) \neq \infty$, so there exists a walk from r to v with at most i edges by the induction hypothesis. Hence $\ell \neq \infty$ and the length is attained by some directed walk Q from r to v with at most i edges. Since there is no cycle with negative weight, Q can be chosen to be a directed path from r to v with at most i edges. And since there is no cycle with negative weight, if $v = r$, then $\ell = 0 = f_i(v)$ and we are done. So we may assume Q has at least one edge. Note that for every $x \in V(Q) - \{v\}$, the subpath of Q from r to x has at most $i-1$ edges and has minimum length over all such walks. Let u be the neighbor of v in Q, so $\ell = w(Q) = w(Q - v) + w((u, v)) = f_{i-1}(u) + w((u, v)) \geq f_i(v)$ by the previous sentence and the induction hypothesis. So $f_i(v) \leq w(Q) = \ell$. Suppose to the contrary that $f_i(v) < \ell$. If $f_i(v) = f_{i-1}(v)$, then by the induction hypothesis, there exists a walk in D from r to v with at most $i - 1 < i$ edges with length $f_{i-1}(v) = f_i(v) < \ell$, contradicting to the definition of ℓ. So there exists $z \in V(D)$ with $(z, v) \in E(D)$ and $f_i(v) = f_{i-1}(z) + w((z, v))$. By the induction hypothesis, there exists a walk Q' from r to z with at most $i-1$ edges with length $f_{i-1}(z)$, so $Q' + (z, v)$ is a walk from r to v with at most i edges with length $f_{i-1}(z) + w((z, v)) = f_i(v) < \ell$, contradicting to the definition of ℓ. Hence Property 3 is preserved. ∎

Lemma 3 *At the end of the Bellman-Fold algorithm, for every $v \in V(D)$, $d_{(D,w)}(r, v) = f_{|V(D)|-1}(v)$, and if $d_{(D,w)}(r, v) \neq \infty$, then the reverse of the maximal path v-$p(v)$-$p(p(v))$-... is a shortest path in D from r to v.*

Proof. Note that no path in D has more than $|V(D)| - 1$ edges. And the length of a shortest walk in D from r to v is attained by a path in D from r to v since there exists no cycle with negative weight. So Property 3 in Lemma 2, $d_{(D,w)}(r, v) = f_{|V(D)|-1}(v)$. And if $d_{(D,w)}(r, v) \neq \infty$, then $f_{|V(D)|-1}(v) \neq \emptyset$, so the maximal path v-$p(v)$-$p(p(v))$-... is well-defined, and its reverse is a directed path from r to v and has length at most (and hence equal to) $f_{|V(D)|-1}(v) = d_{(D,w)}(r, v)$ by Property 2 in Lemma 2. ∎

Theorem 4 *Bellman-Ford algorithm works correctly and runs in time $O(|V(D)|^2 + |V(D)||E(D)|)$.*

Proof. The correctness follows from Lemma 3. And each round of Step 2 takes time $O(\sum_{v \in V(D)}(1 + \deg_D^-(v)) = O(|V(D)| + |E(D)|)$. So the algorithm takes time $O(|V(D)|(|V(D)| + |E(D)|))$. ∎

2 Maximum flow and minimum cut

Definition:

- In a digraph D, for every $S \subseteq V(D)$,
  - we define $\delta^+(S)$ to be the set of edges of D whose tails are in S but whose heads are not in S, and
  - we define $\delta^-(S)$ to be the set of edges of D whose heads are in S but whose tails are not in S.
- For every vertex v,
  - we define $\delta^+(v)$ to be the set of edges of D whose tails are v, and
  - we define $\delta^-(v)$ to be the set of edges of D whose heads are v.

Note that when D is loopless, $\delta^+(v) = \delta^+(\{v\})$ and $\delta^-(v) = \delta^-(\{v\})$. When we work on network flows, the existence of loops does not affect our argument. So you can consider loopless digraph only if you think the notations are too complicated.

Definition:

- A *network* is a 4-tuple (D, s, t, c), where D is a digraph, s, t are distinct vertices of D and $c : E(D) \to \mathbb{R}_{\geq 0}$. (Here $\mathbb{R}_{\geq 0}$ denotes the set of all nonnegative real numbers.) The vertex s is called the *source* and t is called the *sink*.
- A *flow* in a network (D, s, t, c) is a function $f : E(D) \to \mathbb{R}_{\geq 0}$.
- A flow f is *feasible* if it satisfies
  - (Capacity condition:) $0 \leq f(e) \leq c(e)$ for every $e \in E(D)$, and
  - (Conservation condition:) $\sum_{e \in \delta^+(v)} f(e) = \sum_{e \in \delta^-(v)} f(e)$ for every $v \in V(D) - \{s, t\}$.
- The *value* of a flow f is $\sum_{e \in \delta^+(s)} f(e) - \sum_{e \in \delta^-(s)} f(e)$. That is, the value is the "net amount" flowing out from the source. We denote the value of f by $\text{val}(f)$.
- If f is a feasible flow with maximum value, then f is called a *maximum flow*.