Lecture notes for Mar 27, 2023
Eulerian circuits and the Chinese Postman Problem

Chun-Hung Liu

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1 Eulerian circuits

Let G be a graph. A *trail* in G is a walk in G that does not have repeated edges. An *Eulerian trail* of G is a trail in G that uses all edges of G. An *Eulerian circuit* is a closed Eulerian trail.

Lemma 1 *Let G be a graph whose every vertex has even degree. Let $v \in V(G)$. Then every maximal trail in G starting at v is closed and contains all edges of G incident with v, and for every vertex x of G, W contains an even number of edges incident with x.*

Proof. Let W be a maximal trail in G starting at v. Let u be the end of W. If $u \neq v$, then W contains an odd number of edges incident with u, so some edge of u is not in W, and hence we can extend W by adding this edge, a contradiction. So $u = v$ and W is closed. And if W does not contain all edges of G incident with $v = u$, then we can extend W by adding this edge, a contradiction. It is clear that for every vertex x of G, W contains an even number of edges incident with x. ■

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Finding Eulerian circuits
Input: A connected graph G whose every vertex has even degree.
Output: A Eulerian circuit W.
Procedure:

Step 1: Pick a vertex v of G. Set W be the walk with single vertex v and with no edge. Put a token at v.

Step 2: Greedily find a maximal trail W_0 starting at v. Delete all edges in W_0 from G. Replace W by the walk by inserting W_0 (without the first entry) into W between the entry having the token and the entry right next to it.

Step 3: Repeatedly move the token to the next entry of W until the token is at a vertex u with non-zero degree in the current G. If such a vertex u can be found, then redefine v to be u, and do Step 2. Otherwise, output W and stop.

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Lemma 2 *The walk W output from the above algorithm is an Eulerian circuit.*

Proof. Clearly the walk W is a trail since once we include an edge into W, we delete this edge from G, so it cannot be added into W again in the future.

And by Lemma 1, every W_0 found in the process is a closed walk. Since W is a closed walk at the beginning, W is a closed walk during the entire process.

And once a vertex v of G is the entry that holds the token in W, if v has degree nonzero at that time, then Step 2 is executed at that moment, and after that, W include all edges of G incident with v by Lemma 1; if v has degree 0 at that time, then W already includes all edges of G incident with v. In particular, during the entire process, every vertex x that is an entry of W in front of the entry having the token, all edges of (the original) G incident with x are in W.

So W is an Eulerian circuit as long as W contains all vertices of G. When the algorithm stops, W contains all edges incident with a vertex in W, so W contains all edges of a connected component of G. Since G is connected, W contains all vertices of G. So W is an Eulerian circuit. ■

Theorem 3 *Given a connected graph G whose every vertex has even degree, one can find an Eulerian circuit of G in linear time.*

Proof. The correctness follows from Lemma 2. The time complexity is obvious. ■

Corollary 4 (Euler) *A connected graph G has an Eulerian circuit if and only if every vertex of G has even degree.*

Proof. ($\Rightarrow$) Walking along an Eulerian circuit W, whenever we must go into an internal vertex v, we may leave this vertex, so v has even degree. As we can shift W by using the second vertex of W as the first vertex, each vertex of G is an internal vertex of some Eulerian circuit and hence has even degree.

($\Leftarrow$) It immediately follows from Theorem 3. ■

A graph is *Eulerian* if all vertices have even degree.

2 Chinese Postman Problem

Let G be a graph. A *Chinese postman tour* is a closed walk in G that contains every edge of G at least once. The *Chinese Postman Problem* is to find a Chinese postman tour with minimum number of edges.

We can consider a more general version for weighted graphs: given a weighted graph (G, w), find a Chinese postman tour W with $\sum_{e \in E(W)} w(e)$ minimum. (Note that $E(W)$ is the multiset of edges in W such that for every $e \in E(G)$, the number of times that e appears in W equals the number of times that e appears in $E(W)$.)

Note that if there exists an edge with negative weight, then we can use this edge arbitrarily many times to obtain a tour with arbitrarily small weight. So we should assume the weight is nonnegative.

Lemma 5 *Let G be a connected graph. Let w be a nonnegative function on $E(G)$. Then there exists a minimum weighted Chinese postman tour W for (G, w) such that W uses every edge of G at most twice.*

Proof. Let W be a Chinese postman tour with minimum weight, and subject to this, $|E(W)|$ is minimum.

Suppose to the contrary that there exists an edge e of G used at least three times in W. Note that every loop is used exactly once in W. So e is not a loop. Let u, v be the ends of e.

We first assume that there exists a subwalk of W of the form $uevW'uev$, where W' is a subwalk of W from v to u without using e. Let W'' be the reverse of W'. Note that W'' is from u to v with $E(W'') = E(W')$. Then replacing $evW'ue$ by W'' results in another closed walk that uses every edge of $G-e$ the same number of times as in W, and uses e two times less than in W. Since e is used at least three times in W, the resulting walk is a Chinese postman walk better than W, a contradiction.

So there exists a subwalk of W of the form $uevW_1veuW_2uev$, where W_1 is a subwalk of W from v to v without using e, and W_2 is a subwalk of W from u to u without using e. Then replacing evW_1veuW_2ue by W_2uevW_1 results in another closed walk that uses every edge of $G-e$ the same number of times as in W, and uses e two times less than in W. Since e is used at least three times in W, the resulting walk is a Chinese postman walk better than W, a contradiction. ∎

2.1 Reducing to T-joins

Let G be a graph and $T \subseteq V(G)$. A *T-join* of G is a subset J of $E(G)$ such that for every $v \in V(G)$, $|\delta(v) \cap J|$ is odd if and only if $v \in T$.

Lemma 6 *Let G be a connected graph. Let W be a Chinese postman tour of G that uses every edge of G at most twice. Let $T = \{v \in V(G) : \deg_G(v)$ is odd$\}$. Let G' be the graph with $V(G') = V(G)$ and $E(G') = E(W)$. Then G' is Eulerian and $E(G')$ is a disjoint union of $E(G)$ and a T-join of G.*

Proof. Clearly, W is an Eulerian circuit of G', so G' is Eulerian.

Let $H = G' - E(G)$. For every $v \in V(G) = V(G')$, $\deg_{G'}(v) = \deg_G(v) + \deg_H(v)$, and $\deg_{G'}(v)$ is even, so $\deg_H(v) \equiv \deg_G(v) \pmod 2$. Since every edge of G is used in W at most twice, it is used in G' at most twice, so it is used in H at most once. Hence every edge in H can be viewed as a copy of an edge in G, and no edge of G is copied twice. So $E(H)$ is a copy of a T-join of G. ∎

Lemma 7 *Let G be a connected graph. Let w be a nonnegative function on $E(G)$. Let $T = \{v \in V(G) : \deg_G(v)$ is odd$\}$. Then there exists a minimum weighted Chinese postman tour that is a disjoint union of $E(G)$ and a T-join of G.*

Proof. It immediately follows from Lemmas 5 and 6. ∎

Lemma 8 *Let G be a connected graph. Let $T = \{v \in V(G) : \deg_G(v)$ is odd$\}$. Let J be a T-join. Let G' be the graph obtained from G by duplicating each edge in J once. Then G' is Eulerian, and every Eulerian circuit of G' is a Chinese postman tour of G using every edge in J exactly twice and every edge in $E(G) - J$ exactly once.*

Proof. For every $v \in V(G) = V(G')$, $\deg_{G'}(v) = \deg_G(v) + |\delta(v) \cap J| \equiv 0 \pmod 2$. So G' is Eulerian. And it is clear that every Eulerian circuit of G' is a Chinese postman tour of G using every edge in J exactly twice and every edge in $E(G) - J$ exactly once. ∎

Theorem 9 *Let G be a graph. Let $w : E(G) \to \mathbb{R}_{\geq 0}$. Let $T \subseteq V(G)$. Let J be a T-join of G with $\sum_{e \in J} w(e)$ minimum. Let G' be the graph obtained from G by duplicating each edge in J once. Then every Eulerian circuit of G' is a minimum weighted Chinese postman tour of G.*

Proof. Let W be an Eulerian circuit of G'. By Lemma 8, W is a Chinese postman tour with weight $w(E(G)) + w(J)$.

By Lemma 7, there exists a minimum weighted Chinese postman tour W^* that is a disjoint union of $E(G)$ and a T-join J' of G. Note that $w(E(G)) + w(J) = w(W) \geq w(W^*) = w(E(G)) + w(J')$. So $w(J) \geq w(J')$. But J is a minimum weighted T-join, so $w(J) = w(J')$. Hence $w(W) = w(W^*)$. Therefore, W is a minimum weighted Chinese postman tour. ∎

2.2 Finding a minimum T-join

By Theorem 9, to find a minimum weighted Chinese postman tour, it suffices to find a minimum weighted T-join, where $T = \{v \in V(G) : \deg_G(v) \text{ is odd}\}$.

In the next lecture, we will describe how to find a minimum weighted T-join, for any subset T of $V(G)$ for which a T-join exists. We first characterize the existence of a T-join.

Proposition 10 *Let G be a graph. Let $T \subseteq V(G)$. Then there exists a T-join of G if and only if for every component C of G, $|T \cap V(C)|$ is even.*

Proof. ($\Rightarrow$) Let J be a T-join. Let H_J be the graph with $V(H_J) = V(G)$ and $E(H_J) = J$. For every component C of G, $T \cap V(C)$ is the set of odd degree vertices in C, so its size must be even by the hand-shake lemma.

($\Leftarrow$) We prove it by induction on $|T|$. There is nothing to prove when $|T| = 0$. So we may assume $|T| \geq 1$ and the proposition holds when $|T|$ is smaller. Hence there exists a component C of G with $|T \cap V(C)| \geq 2$. Let x, y be distinct vertices in $T \cap V(C)$. Let P be a path in C connecting x and y. Note that for every component Q of G, $|(T - \{x, y\}) \cap V(Q)|$ is even. So by the induction hypothesis, there exists a $(T - \{x, y\})$-join J' of G. Then $J' \Delta E(P)$ is a T-join of G. $\blacksquare$