Lecture notes for Apr 10, 2023
Disjoint Path Problems

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April 10, 2023

Note that the maximum-flow-minimum-cut theorem can be used to derive Menger's theorem.

Let G be a graph. A *separation* of G is an ordered pair (A, B) of subsets of $V(G)$ such that there exists no edge of G between $A - B$ and $B - A$. The *order* of a separation (A, B) is $|A \cap B|$.

Theorem 1 (Menger's theorem) *Let k be a nonnegative integer. Let G be a graph. Let X and Y be subsets of $V(G)$. Then there exist k disjoint paths from X to Y if and only if there exists no separation (A, B) of G with order less than k.*

The proof of Menger's theorem can be found in any standard graph theory course, so we do not repeat it here.

Menger's theorem characterizes when there exist k disjoint paths between X and Y. But it does not say what the ends of those k paths are. For example, the case $k = 2$ says that there are two disjoint paths P_1 and P_2 from $X = \{x_1, x_2\}$ to $Y = \{y_1, y_2\}$ such that $x_1 \in V(P_1)$, but we do not know which vertex in Y is in $V(P_1)$. This motivates the following Disjoint Path Problem.

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Disjoint Path Problem

Input: A positive integer k, a simple graph G, and $2k$ vertices $x_1, x_2, ..., x_k, y_1, y_2, ..., y_k$.

Output: Determine whether there exist k disjoint paths $P_1, P_2, ..., P_k$ in G such that for every $i \in [k]$, the ends of P_i are x_i and y_i.

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The Disjoint Path Problem is significantly more difficult than the "unpaired" version. (The "unpaired version" is solved by Menger's theorem and can be solved in polynomial time by adapting the maximum flow algorithm.)

Theorem 2 (Even, Itai, Shamir) *The Disjoint Path Problem is NP-complete.*

We will not prove the above theorem.

Let us consider the case when k is fixed.

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k-Disjoint Path Problem
Input: A simple graph G and $2k$ vertices $x_1, x_2, ..., x_k, y_1, y_2, ..., y_k$.
Output: Determine whether there exist k disjoint paths $P_1, P_2, ..., P_k$ in G such that for every $i \in [k]$, the ends of P_i are x_i and y_i.

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The 1-Disjoint Path Problem is easy: simply testing whether there is a component containing both x_1 and y_1, which can be done in linear time. The cases $k \geq 2$ are non-trivial. A seminal result of Robertson and Seymour states that it is polynomial time solvable for every k.

Theorem 3 (Robertson, Seymour) *There exists a function f such that for every positive integer k, the k-Disjoint Path Problem can be solved in time $f(k)n^3$.*

The time complexity was further improved to $f(k)n^2$ by Kawarabayashi, Kobayashi and Reed. The proof for general k is too complicated to be stated here.

In contrast, we will show that the 2-Disjoint Path Problem can be solved in polynomial time, which is significantly simpler than the general case. More precisely, we will characterize the negative instances of the 2-Disjoint Path Problem and then show how to recognize them in polynomial time.

1 Obstructions

Let G be a simple graph. Let x_1, x_2, y_1, y_2 be distinct vertices. Then we say that (x_1, x_2, y_1, y_2) is *feasible* in G if there exist two disjoint paths P_1 and P_2 in G such that for every $i \in [2]$, the ends of P_i are x_i and y_i; otherwise, we say that (x_1, x_2, y_1, y_2) is *infeasible* in G.

Proposition 4 *Let G be a planar graph such that there exists a cycle C bounding the outer face. Let x_1, x_2, y_1, y_2 be distinct vertices in C. If C passes through x_1, x_2, y_1, y_2 in the order listed, then (x_1, x_2, y_1, y_2) is infeasible.*

Proof. Suppose to the contrary that there exist two disjoint paths P_1 and P_2 in G such that for every $i \in [2]$, the ends of P_i are x_i and y_i. Let G' be the graph obtained from G by adding a new vertex v^* adjacent to the four vertices x_1, x_2, y_1, y_2. Note that G' is planar because we can draw v^* and the four new edges in the outer face of G. But $C \cup P_1 \cup P_2$ together with v^* gives a K_5-minor in G', a contradiction. ∎

Proposition 4 shows an obstruction for feasibility coming from the planarity. But there is another obstruction.

Let (A, B) be a separation of G of order at most three. The *3-reduction of* (A, B) is the graph obtained from $G[A]$ by adding edges such that A becomes a clique.

Proposition 5 *Let G be a graph. Let x_1, x_2, y_1, y_2 be distinct vertices of G. Let (A, B) be a separation of G of order at most three such that $\{x_1, x_2, y_1, y_2\} \subseteq A$. Assume that either $|A \cap B| \leq 1$, or there exists no separation (A', B') of order at most two such that $A' \supseteq A$ and $B' - A' \neq \emptyset$. Then (x_1, x_2, y_1, y_2) is feasible in G if and only if (x_1, x_2, y_1, y_2) is feasible in the 3-reduction of (A, B).*

Proof. $(\Rightarrow)$ Since $|A \cap B| \leq 3$, it is impossible to have two disjoint paths P_1, P_2 with both ends in $\{x_1, x_2, y_1, y_2\} \subseteq A$ intersecting $B - A$. So if (x_1, x_2, y_1, y_2) is feasible in G, then (x_1, x_2, y_1, y_2) is feasible in the 3-reduction of (A, B).

$(\Leftarrow)$ Now we show the converse direction. When $|A \cap B| \leq 1$, no new edge is added when constructing the 3-reduction of (A, B), so if (x_1, x_2, y_1, y_2) is feasible in the 3-reduction of (A, B), then (x_1, x_2, y_1, y_2) is feasible in G. So we may assume $|A \cap B| \geq 2$. Hence there exists no separation (A', B') of order at most two such that $A' \supseteq A$ and $B' - A' \neq \emptyset$ by assumption. So for any two vertices $u, v \in A \cap B$, there exists a path in $G[B]$ from u to v internally disjoint from $A \cap B$. Moreover, if P is a path in the 3-reduction of (A, B) containing at least two vertices in $A \cap B$, then we can replace a subpath Q of P with both ends in $A \cap B$ by a path Q_B in $G[B]$ connecting the same ends as Q to obtain a path P' in G that has the same ends as P, where $V(Q_B) \cap A \subseteq A \cap B$. Since $|A \cap B| \leq 3$, no two disjoint paths each containing two vertices in $A \cap B$ exist, so if (x_1, x_2, y_1, y_2) is feasible in the 3-reduction of (A, B), then (x_1, x_2, y_1, y_2) is feasible in G. ∎

Proposition 5 gives a way to construction non-planar infeasible graphs: Take a plane graph G such that x_1, x_2, y_1, y_2 are passed through by the outer cycle in the order listed. Assume that G has a face f that is a triangle. Attach a highly non-planar graph on the 3 vertices of f to get a new graph G'. Then we obtain a separation (A, B) of G', where $A = V(G)$ and B consists of the 3 vertices of f and the new vertices. So G is a 3-reduction of G'. Since (x_1, x_2, y_1, y_2) is infeasible in G, (x_1, x_2, y_1, y_2) is feasible in G' by Proposition 5, even though G' is non-planar.

Let G be a graph. Let x_1, x_2, y_1, y_2 be distinct vertices of G. A *3-reduction of* (x_1, x_2, y_1, y_2) *(for G)* is the 3-reduction of a separation (A, B) of G of order at most three with $\{x_1, x_2, y_1, y_2\} \subseteq A$ and $B - A \neq \emptyset$ such that either $|A \cap B| \leq 1$, or there exists no separation (A', B') of order at most two such that $A' \supseteq A$ and $B' - A' \neq \emptyset$. If no separation (A, B) mentioned above exists, then we say G is *3-irreducible with respect to* (x_1, x_2, y_1, y_2). A *full 3-reduction of* (x_1, x_2, y_1, y_2) *(for G)* is a 3-irreducible graph G' with respect to (x_1, x_2, y_1, y_2) such that there exists a sequence of graphs $G_1, G_2, ..., G_t$ such that $G_1 = G$, $G_t = G'$, and G_{i+1} is a 3-reduction of (x_1, x_2, y_1, y_2) for G_i for every $1 \leq i \leq t-1$.

Proposition 6 *Let G be a graph. Let x_1, x_2, y_1, y_2 be distinct vertices of G. Then the following statements are equivalent:*

1. *(x_1, x_2, y_1, y_2) is feasible in G.*

2. *(x_1, x_2, y_1, y_2) is feasible in any 3-reduction of (x_1, x_2, y_1, y_2).*

3. *(x_1, x_2, y_1, y_2) is feasible in any full 3-reduction of (x_1, x_2, y_1, y_2).*

Proof. It immediately follows from Proposition 5. ∎

2 A characterization

Now we can characterize the infeasible cases. It shows that the planar obstruction mentioned in Proposition 4 is essentially the unique obstruction up to 3-reductions.

Theorem 7 (Seymour; Thomassen) *Let G be a graph. Let x_1, x_2, y_1, y_2 be distinct vertices of G. Then (x_1, x_2, y_1, y_2) is infeasible in G if and only if for every full 3-reduction G' of (x_1, x_2, y_1, y_2) for G, the graph $G'+x_1x_2y_1y_2x_1$ can be drawn in the plane such that the 4-cycle $x_1x_2y_1y_2x_1$ bounds the outer face.*

We will prove Theorem 7 next time. We show how to use Theorem 7 to solve the 2-Disjoint Path Problem.

Corollary 8 *The 2-Disjoint Path Problem can be solved in polynomial time.*

Proof. Let G be a simple graph, and let $\{x_1, x_2, y_1, y_2\}$ be distinct vertices of G.

For every subset S of $V(G)$ with $|S| \leq 3$, we can test whether there exists a separation (A, B) with $\{x_1, x_2, y_1, y_2\} \subseteq A$, $B - A \neq \emptyset$ and $A \cap B = S$ by checking whether there exists a components of $G - S$ disjoint from $\{x_1, x_2, y_1, y_2\} - S$, which takes time $O(|V(G)| + |E(G)|) = O(|V(G)|^2)$. So we can test whether G is 3-irreducible by considering all subsets S of $V(G)$ with $|S| \leq 3$. Since there are at most $O(|V(G)|^3)$ subsets of $V(G)$ with size at most three, we know that in time $O(|V(G)|^5)$, we can test whether G is 3-irreducible, and if G is not 3-irreducible, then we can find a 3-reduction of (x_1, x_2, y_1, y_2) for G.

Hence in time $O(|V(G)|^6)$, we can obtain a full 3-reduction G' of (x_1, x_2, y_1, y_2) for G. And we can test whether $G'+x_1x_2y_1y_2x_1$ in the plane such that the 4-cycle $x_1x_2y_1y_2x_1$ bounds the outer face in polynomial time. (Note that it can be done by transforming a proof of Kuratowski's theorem into an algorithm. And in fact, it can be done in linear time.)

If $G' + x_1x_2y_1y_2x_1$ can be drawn in the plane such that the 4-cycle $x_1x_2y_1y_2x_1$ bounds the outer face, then we output "no"; otherwise we output "yes".

If (x_1, x_2, y_1, y_2) is infeasible in G, then $G' + x_1x_2y_1y_2x_1$ can be drawn in the plane such that the 4-cycle $x_1x_2y_1y_2x_1$ bounds the outer face by Theorem 7, so our output is correct. If (x_1, x_2, y_1, y_2) is feasible in G, then (x_1, x_2, y_1, y_2) is feasible in G' by Proposition 6, so $G'+x_1x_2y_1y_2x_1$ cannot be drawn in the plane such that the 4-cycle $x_1x_2y_1y_2x_1$ bounds the outer face by Proposition 4. Therefore, our algorithm correctly solves the 2-Disjoint Path Problem in polynomial time. ∎