Lecture notes for Apr 26, 2023
Bounded expansion and meta-theorems

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1 Bounded expansion

We will try to relax Courcelle's theorem by considering more general graphs. Here we focus on "sparse" graphs.

Naturally, if the number of edges of a graph is small when comparing to its number of vertices, then it can be considered "sparse". The *average degree* of a graph G is defined to be $2|E(G)|/|V(G)|$. So graphs with small average degree are sparse.

But considering the average degree of G only does not really give enough information. For example, if G is obtained from a very dense graph H by adding k isolated vertices, then the average degree of G equals $|E(H)|/(|V(H)|+k)$, which can be very small if k is large. That is, even if G has small average degree, to solve a problem on G, we essentially have to solve the problem on H, but H can be arbitrary. So only knowing G has small average degree is not helpful.

According to the above example, we might want to require all induced subgraphs of G has small average degree. For an integer k, we say that a graph G is *k-degenerate* if every induced subgraph H of G contains a vertex with degree in H at most k. Note that 0-degenerate graphs are exactly edgeless graphs; 1-degenerate graphs are exactly forests; every induced subgraph of a k-degenerate graph is also k-degenerate. And if G is k-degenerate, then we can order the vertices of G as $v_1, v_2, ...$ such that for every i, v_i is adjacent in G to at most k vertices in $\{v_j : j > i\}$, so every induced subgraph H of G has at most $k|V(H)|$ edges and hence has average degree at most $2k$.

Having bounded degeneracy describes the sparsity of a graph, and this sparsity is "robust" under deleting vertices and edges. But in order to prove a variant of Courcelle's theorem, we need a more robust sparsity. More precisely, we want the sparity is robust under contracting edges as well. That is, we want to require that every minor of G has small average degree. There are many examples for such a sparsity, as we will see in this section.

All graphs are assumed to be simple in this section.

1.1 Minor-closed families

A set $\mathcal{F}$ of graphs is a *minor-closed family* if $H \in \mathcal{F}$ for every graph $G \in \mathcal{F}$ and for every graph H that is a minor of G.

For example, the set of planar graphs is a minor-closed family since no matter how we delete vertices or edges or contracting edges, the resulting graph remains planar.

Theorem 1 (Euler's formula) *Every simple planar graph G has at most $3|V(G)| - 6$ edges.*

Euler's formula implies that every planar graph has average degree at most 6. And every minor of a planar graph is planar. So every minor of a planar graph has average degree at most 6. Hence the set of simple planar graphs is a minor-closed family and enjoy the robust sparsity mentioned above.

Proposition 2 *Let w be a nonnegative integer. If G has tree-width at most w, then every minor of G has tree-width at most w. That is, the class of graphs with tree-width at most w is a minor-closed family.*

Proof. Let $(T, \mathcal{X})$ be a tree-decomposition of G with width at most w.

Note that $(T, \mathcal{X})$ is also a tree-decomposition of $G - e$ with width at most w, for every edge e of G. For every vertex v, if we remove v from every bag, then $(T, \mathcal{X})$ becomes a tree-decomposition of $G - v$ with width at most w. For every edge $e = uv$, if z is the vertex in G/uv obtained from contracting the edge uv, then replacing the appearance of u and v by z makes $(T, \mathcal{X})$ a tree-decomposition of G/uv with width at most w.

So every minor of G has tree-width at most w. ■

It is easy to show that every simple graph with tree-width at most w is w-degenerate by considering a vertex in the bag of the leaf in the tree in the tree-decomposition. Hence Proposition 2 implies that the class of simple graphs with tree-width at most w is a minor-closed family and enjoy the robust sparsity mentioned above.

Note that every cycle on at least three vertices contains K_3 as a minor. And it is easy to show that K_3 has tree-width at least two. So Proposition 2 implies that every cycle on at least three vertices has tree-width at least 2, and more generally, every non-forest has tree-width at least 2. Recall that we show every cycle has tree-width at most 2. It implies that every cycle on at least three vertices has tree-width exactly 2. And recall that we showed that every forest has tree-width at most 1. So graphs with tree-width 1 are exactly forests with at least one edge.

We have seen some examples for graph classes that are minor-closed and enjoy the robust sparsity mentioned above. It is not a coincidence.

Proposition 3 *For every positive integer t, every simple graph with average degree at least 2^t contains K_t as a minor.*

Proof. We prove this proposition by induction on t. It obviously holds when $t = 1$. We we may assume $t \geq 1$ and the proposition holds when t is smaller.

Let G be a graph with average degree 2^t. Let H be a minor of G such that H has average degree at least 2^t, and subject to this, $|V(H)| + |E(H)|$ is minimum. Note that H exists since G is a candidate. To show G contains K_t as a minor, it suffices to show that H contains K_t as a minor.

If there exists an edge uv such that u and v have at most $2^{t-1}-1$ common neighbors, then the average degree of H/uv is $2|E(H/uv)|/|V(H/uv)| \geq 2(|E(H)|-1-(2^{t-1}-1))/(|V(H)|-1) \geq 2(2^{t-1}|V(H)|-2^{t-1})/(|V(H)|-1) \geq 2^t$, contradicting the minimality of H. So for every edge of H, its ends have at least 2^{t-1} common neighbors.

Let v be a vertex of H. Let $H' = H[N_H(v)]$. For every $u \in V(H')$, $uv \in E(H)$, so u and v has at least 2^{t-1} common neighbors in H, so u has degree at least 2^{t-1} in H'. So $2|E(H')|/|V(H')| \geq 2^{t-1}$. By the induction hypothesis, H' contains K_{t-1} as a minor. Since $V(H') = N_H(v)$, H contains K_t as a minor. ■

Note the class of graphs is a minor-closed family. But this class is not interesting. We say that a set of graphs is a *proper minor-closed family* if it is minor-closed and does not contain all graphs.

Theorem 4 *For every proper minor-closed family $\mathcal{F}$, there exists an integer t such that every graph in $\mathcal{F}$ has average degree at most t.*

Proof. Since $\mathcal{F}$ is a proper minor-closed family, there exists a graph $H \notin \mathcal{F}$, so $K_{|V(H)|} \notin \mathcal{F}$. So every graph in $\mathcal{F}$ has average degree at most $2^{|V(H)|}$ by Proposition 3. ■

1.2 Shallow minors

We have seen that every minor-closed family has a sparsity that is robust with respect to deleting vertices and edges and contracting edges.

The class of graphs with bounded maximum degree does not have this property. For example, it is known that for every integer t, there exists a graph with maximum degree at most 3 containing K_t as a minor. So graphs with maximum degree 3 can be made arbitrarily dense by contracting edges. However, if G has maximum degree at most d, then contracting a subgraph with radius at most r can only create a vertex with degree at most $d + d^2 + d^3 + ... + d^r \leq d^{r+1}$. So if we want to obtain a minor H of G with average degree t, we have to contract subgraphs with radius $\Omega(\log_d t)$.

Let G be a graph. Let r be a nonnegative integer (or ∞). We say that H is an *r-shallow minor* of G if there exist a set $\{X_v : v \in V(H)\}$ of pairwise disjoint subsets of $V(G)$ such that $G[X_v]$ has radius at most r, and for every edge $xy \in E(H)$, there exists an edge of G between X_x and X_y. Note that 0-shallow minors of G are exactly subgraphs of G; ∞-shallow minors of G are exactly minors of G.

Let $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ be a function. We say that a set $\mathcal{F}$ of graphs has *expansion* f if every r-shallow minor of G has average degree at most $f(r)$ for every $G \in \mathcal{F}$ and $r \in \mathbb{N} \cup \{0\}$.

Examples:

- The set of planar graphs have expansion f, where f is the constant function 6.
- The set of graphs with tree-width at most w has expansion f, where f is the constant function $2w$.
- The set of graphs with maximum degree at most d has expansion f, where $f(x) = d^{x+1}$ for every nonnegative integer x.

We say that a set of graphs has *bounded expansion* if there exists a function f such that it has expansion f. Bounded expansion classes are very general.

Examples of graph classes with bounded expansion:

- Every proper minor-closed family.

- Every proper topological minor-closed family.

  (For example, for any fixed surface Σ and integer k, the set of graphs that can be drawn in Σ with at most k crossings.)

- For any positive integers d and k, the set of intersection graphs of a set S of closed balls in $\mathbb{R}^d$ such that every point in $\mathbb{R}^d$ is contained in the interior of at most k balls in S.

Here is an analog of Courcelle's theorem.

Theorem 5 (Dvořák, Kral', Thomas) *For every set $\mathcal{F}$ of graphs with bounded expansion, and for every graph property P that can be expressed in FO, testing whether a graph in $\mathcal{F}$ satisfies P can be done in linear time.*

In particular, for any fixed k, testing whether a planar graph has a stable set of size at least k or has a dominating set with size at most k can be done in linear time.

1.3 Hereditary properties

We will see another application of Theorem 5 in this subsection.

A graph property P is *hereditary* if it is closed under deleting vertices. That is, if G satisfies P and H is obtained from G by deleting vertices, then H satisfies P.

Examples of hereditary properties:

- "Being k-colorable", where k is a fixed integer.

- "Having girth at least k", where k is a fixed integer.

- "Being claw-free".

- "Being chordal".

- "Being a line-graph".
- "Being a cograph", where a graph is a *cograph* if it is K_1 or can be obtained by taking disjoint union of two cographs or obtained by taking complement of a cograph.
- "Can be drawn in Σ such that every edge contains at most k crossings", where Σ is a fixed surface and k is a fixed integer.

Let P be a hereditary property. A graph G is a *hereditary-minimal obstruction* with respect to P if G does not satisfy P but every proper induced subgraph of G satisfies P. Let $\mathrm{Forb}(P)$ be the set of hereditary-minimal obstructions with respect to P.

Proposition 6 *Let P be a hereditary property. Then a graph G satisfies P if and only if G does not contain any graph H in* $\mathrm{Forb}(P)$ *as an induced subgraph.*

Proof. ($\Rightarrow$) If G satisfies P and contains some graph $H \in \mathrm{Forb}(P)$ as an induced subgraph, then H satisfies P (since P is hereditary), a contradiction.

($\Leftarrow$) Let G be a graph that does not satisfy P. Let H be an induced subgraph of G not satisfying P, and subject to this, $|V(H)|$ is minimal. Note that H exists since G is a candidate. If $H \notin \mathrm{Forb}(P)$, then some proper induced subgraph of H does not satisfy P by the definition of $\mathrm{Forb}(P)$, but it contradicts the minimality of $|V(H)|$. ■

Some hereditary property has finitely many hereditary-minimal obstructions. For example, if P is "being claw-free", then $\mathrm{Forb}(P) = \{K_{1,3}\}$ by definition; if P is "being a line graph", then $|\mathrm{Forb}(P)| = 9$ by a theorem of Beineke; if P is "being a cograph", then $\mathrm{Forb}(P) = \{P_4\}$. However, other hereditary properties mentioned above have infinitely many hereditary-minimal obstructions.

Corollary 7 *If P is a hereditary property such that there are only finitely many hereditary-minimal obstructions, then for every graph class $\mathcal{F}$ with bounded expansion, testing whether a graph in $\mathcal{F}$ satisfies P can be done in linear time.*

Proof. Recall that we showed that for every graph H, "containing H as an induced subgraph" can be expressed in FO, so "not containing H as an induced subgraph" can be expressed in FO. Since Forb(P) is finite, "not containing any graph in Fobr(P) as an induced subgraph" can be expressed in FO. By Proposition 6, "satisfying P" can be expressed in FO. So this corollary follows from Theorem 5. ■

2 Testing minor-closed property

We have seen Courcelle's theorem and the theorem of Dvořák, Kral' and Thomas about properties that can be expressed by logic expressions. They are examples of meta-theorems. We will see other meta-theorems in this section.

A property P is *minor-closed* if it is closed under taking minor. That is, it is closed under deleting vertices, deleting edges and contracting edges. Equivalently, if G satisfies P and H is a minor of G, then H satisfies P. Note that every minor-closed property is hereditary.

Examples of minor-closed properties:

- "Can be drawn in Σ without crossing", where Σ is a fixed surface.
- "Can be made planar by deleting at most k vertices", where k is a fixed integer.
- "Being a tree".
- "Having tree-width at most w", where w is a fixed integer.
- "Having a vertex-cover with size at most k", where k is a fixed integer.
- "Can be embedded in $\mathbb{R}^3$ such that every cycle forms a trivial knot".

For a minor-closed property P, a graph G is a *minor-minimal obstruction* with respect to P if G does not satisfy P but every proper minor of G satisfies P. Unlike hereditary properties, the set of minor-minimal obstructions is always finite.

Theorem 8 (Graph Minor Theorem (Robertson, Seymour)) *For every minor-closed property P, there are only finitely many minor-minimal obstructions with respect to P.*

By the Graph Minor Theorem, to test whether a graph G satisfies a minor-closed property P, it suffices to test whether G contains H as a minor or not, for finitely many graphs H. Recall that "containing H as a minor" is a MSO_1-property, so it can be tested in linear time for graphs with bounded tree-width. But it is known that it can always be done in polynomial time.

Theorem 9 (Robertson, Seymour) *There exists a function f such that for every graph H, testing whether an input graph G contains H as a minor can be done in $f(H)|V(G)|^3$ time.*

Corollary 10 *Every minor-closed property can be test in $O(n^3)$ time.*

Proof. It immediately follows from Theorems 8 and 9. ■

We remark that Kawarabayashi, Kobayashi and Reed improved Theorem 9 to time $f(H)|V(G)|^2$. So the above corollary can be improved to time $O(n^2)$.