# Lecture notes for May 1, 2023
# Balanced separators

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When doing divide-and-conquer algorithms, we frequently divide an $n$-vertex graph into two parts $A$ and $B$ and then repeat the argument. Then the running time $T(n)$ satisfies the recurrence relation $T(n) = T(|A|) + T(|B|) + f(n)$ for some function $f$. And usually the running time is better when we can ensure that both $|A|$ and $|B|$ are significantly smaller than $n$. In addition, sometimes the term $f(n)$ in the running time can be replaced by $f(|A \cap B|)$ if $(A, B)$ is a separation (i.e. no edge is between $A - B$ and $B - A$).

Let $G$ be a graph. Let $\epsilon$ be a real number with $0 < \epsilon < 1$. An *$\epsilon$-balanced separator* of $G$ is a separation $(A, B)$ of $G$ such that $|A - B| \leq \epsilon|V(G)|$ and $|B - A| \leq \epsilon|V(G)|$. Note that it implies that $\max\{|A|, |B|\} \leq \epsilon|V(G)| + |A \cap B|$, so we also want $|A \cap B|$ is small. This implies that we should consider the case $\epsilon \geq 1/2$ only. Moreover, as we would like to repeatedly apply the divide-and-conquer argument, we also want $G[A]$ and $G[B]$ has balanced separators. For a function $f$, we say that $G$ *admits $\epsilon$-balanced separators of size $f$* if for every induced subgraph $H$ of $G$, there exists an $\epsilon$-balanced separator of $H$ of order at most $f(|V(H)|)$.

For a function $f$, we say that $G$ *admits balanced separators of size $f$* if $G$ admits $\frac{2}{3}$-balanced separators of size $f$. Note that we choose $\frac{2}{3}$ here is for convenience. Later (Proposition 8) we will show that actually choosing any constant between 1/2 and 1 is more or less equivalent.

## 1 Balanced separators and tree-width

**Theorem 1** *Let $w$ be a positive integer. If $G$ is a graph with tree-width at most $w$, then $G$ has a balanced separator of order at most $w + 1$.*

**Proof.** Let $(T, \mathcal{X})$ be a tree-decomposition of $G$ with width $w$. If there exists an edge $tt' \in E(T)$ with $X_t = X_{t'}$, then we can contract $tt'$ into a vertex and keep the same bag to obtain a tree-decomposition with the same width. So we may assume that for every edge $tt' \in E(T)$, $X_t \neq X_{t'}$. Hence every edge of $T$ gives a separation of $G$ of order at most $w$.

If there exists an edge $e$ of $T$ such that the separation given by $e$ is a balanced separator, then we are done. So we may assume that for every edge $xy$ of $T$, exactly one side of the separation $(A_{xy}, B_{xy})$ given by $xy$, say given by the side containing $X_y$, contains at least $\frac{2}{3}|V(G)| + |A_{xy} \cap B_{xy}|$ vertices, and we can direct $xy$ from $x$ to $y$. Hence $T$ becomes a directed graph. Since $\sum_{t \in V(T)} \deg_T^+(t) = |E(T)| = |V(T)| - 1$, there exists a vertex $t_0$ of $T$ with $\deg_T^+(t_0) = 0$. So for every component $C$ of $T - t_0$, $|\bigcup_{t \in V(C)} X_t - X_{t_0}| < \frac{2}{3}|V(G)|$.

If there exists a component $C$ of $T - t_0$ such that $|\bigcup_{t \in V(C)} X_t - X_{t_0}| \geq \frac{1}{3}|V(G)|$, then $|V(G) - (X_{t_0} \cup \bigcup_{t \in V(C)} X_t)| \leq |V(G)| - \frac{1}{3}|V(G)| \leq \frac{2}{3}|V(G)|$, so $(\bigcup_{t \in V(C)} X_t, V(G) - (X_{t_0} \cup \bigcup_{t \in V(C)} X_t))$ is a balanced separator of order $|X_{t_0}| \leq w + 1$. Hence we may assume that for every component $C$ of $T - t_0$, $|\bigcup_{t \in V(C)} X_t - X_{t_0}| < \frac{1}{3}|V(G)|$.

If $|V(G) - X_{t_0}| < \frac{1}{3}|V(G)|$, then the separation $(X_{t_0}, V(G))$ is a separation of $G$ of order $|X_{t_0}| \leq w + 1$ and satisfying $|X_{t_0} - V(G)| = 0 \leq \frac{2}{3}|V(G)|$ and $|V(G) - X_{t_0}| \leq \frac{1}{3}|V(G)|$, so $(X_{t_0}, V(G))$ is a balanced separator of order at most $w + 1$. So we may assume $|V(G) - X_{t_0}| \geq \frac{1}{3}|V(G)|$.

Let $C_1, C_2, ..., C_k$ be the components of $T - t_0$. Since $|V(G) - X_{t_0}| \geq \frac{1}{3}|V(G)|$, there exists a smallest integer $q$ such that $\sum_{i=1}^{q} |\bigcup_{t \in V(C_i)} X_t - X_{t_0}| \geq \frac{1}{3}|V(G)|$. Note that $q \geq 2$. So $\sum_{i=1}^{q} |\bigcup_{t \in V(C_i)} X_t - X_{t_0}| = \sum_{i=1}^{q-1} |\bigcup_{t \in V(C_i)} X_t - X_{t_0}| + |\bigcup_{t \in V(C_q)} X_t - X_{t_0}| \leq \frac{1}{3}|V(G)| + \frac{1}{3}|V(G)| = \frac{2}{3}|V(G)|$. Moreover, $|\bigcup_{i=q+1}^{k} \bigcup_{t \in V(C_i)} X_t - X_{t_0}| = |V(G) - X_{t_0}| - \sum_{i=1}^{q} |\bigcup_{t \in V(C_i)} X_t - X_{t_0}| \leq |V(G)| - \frac{1}{3}|V(G)| = \frac{2}{3}|V(G)|$. Hence $(\bigcup_{i=1}^{q} \bigcup_{t \in V(C_i)} X_t, \bigcup_{i=q+1}^{k} \bigcup_{t \in V(C_i)} X_t)$ is a balanced separator of order $|X_{t_0}| \leq w + 1$. ∎

Using Theorem 1, we can show that some graph has large tree-width.

**Proposition 2** *Let $k$ be a positive integer with $k \geq 4$. Let $G$ be the $k \times k$-grid. Then $G$ has tree-width at least $\lfloor k/4 \rfloor$.*

**Proof.** Suppose that $G$ has tree-width at most $\lfloor k/4 \rfloor - 1$. By Theorem 1, there exists a balanced separation $(A, B)$ of $G$ of order at most $\lfloor k/4 \rfloor$.

Since $|A \cap B| \leq k/4$, at most $k/4$ rows intersect both $A$ and $B$. So there are at least $3k/4$ rows each contained in $A$ or contained in $B$. If there exist a row contained in $A$ and another row contained in $B$, then there are $k$ disjoint paths from $A$ to $B$, contradiction. So we may assume that there are at least $3k/4$ rows contained in $B$. Hence $|B| \geq 3k^2/4$. So $|B-A| = |B| - |A \cap B| \geq 3k^2/4 - k/4 > 2k^2/3 = \frac{2}{3}|V(G)|$ since $k \geq 4$, a contradiction. ■

Note that it is not very hard to show that the bound $k/4$ can be improved to $k$, but it requires other notions and we will not do it here. The important message from the above proposition is that if a graph has small tree-width, then it cannot contain a large grid as a minor. The converse statement is also true, but we will not prove it here.

**Theorem 3 (Grid Minor Theorem (Robertson, Seymour))** *There exists a function $f$ such that for every positive integer $w$, if a graph $G$ has tree-width at least $w$, then it contains a $f(w) \times f(w)$-grid as a minor.*

Theorem 3 tells us exactly when a graph has large tree-width.

On the other hand, the converse of Theorem 1 holds approximately.

**Theorem 4 (Dvořák, Norin)** *Let $w$ be a positive integer. If $G$ admits balanced separators of order $w$, then the tree-width of $G$ is at most $15w$.*

# 2 Strongly sublinear balanced separators

Recall that we show that if $G$ has tree-width at most $w$, then we can find a maximum stable set in $G$ in time $O(w4^w|V(G)|)$. It implies that if $G$ has sublinear tree-width (in $|V(G)|$), then a maximum stable set can be found in subexponential time. So it is reasonable to consider what kind of graphs have sublinear tree-width. By Theorem 4, it is equivalent to consider what kind of graphs have balanced separators with sublinear size.

The following is a famous example.

**Theorem 5 (Lipton, Tarjan)** *Every planar graph admits balanced separators of size $f$, where $f$ is the function $f(x) = 2\sqrt{2}\sqrt{x}$.*

It is strengthened to another famous theorem.

**Theorem 6 (Alon, Seymour, Thomas)** *For every proper minor-closed family $\mathcal{F}$, there exists a constant $c$ such that every graph in $\mathcal{F}$ admits balanced separators of size $f$, where $f$ is the function $f(x) = c\sqrt{x}$.*

We usually want a bit more than just being sublinear. That is, we want the order of the separators is "polynomially better than linear", like the above two examples. For a class $\mathcal{F}$ of graphs, we say that $\mathcal{F}$ *admits strongly sublinear balanced separators* if there exists a function $f(x) := cx^\beta$ for some constants $c > 0$ and $0 \leq \beta < 1$ such that every graph in $\mathcal{F}$ admits balanced separators with size $f$.

The following theorem characterizes the existence of strongly sublinear balanced separators, via graph expansion.

**Theorem 7 (Dvořák, Norin)** *Let $\mathcal{F}$ be a set of graphs. Then $\mathcal{F}$ admits strongly sublinear balanced separators if and only if $\mathcal{F}$ has expansion $g$ for some polynomial $g$.*

# 3 Choices of constants

Now we show that the choice of $\epsilon$ for $\epsilon$-separators does not matter.

**Proposition 8** *Let $f$ be a nondecreasing function. Let $\epsilon$ be a real number with $\frac{1}{2} < \epsilon < 1$. If $G$ admits $\epsilon$-balanced separators of size $f$, then for every $\frac{1}{2} < \delta < 1$, $G$ admits $\delta$-balanced separators of size $g$, where $g$ is the function such that $g(x) = \sum_{i=0}^{\lceil \log_\epsilon(\frac{\delta - \frac{1}{2}}{2}) \rceil - 1} 2^i f(\epsilon^i x)$ for every $x$.*

*In particular, if there exist constants $c > 0$ and $0 \leq \beta < 1$ such that $f(x) \leq cx^\beta$, then there exists a constant $c' > 0$ such that $g(x) \leq c'x^\beta$.*

**Proof.** Since $\frac{1}{2} < \delta < 1, 0 < \frac{\delta - \frac{1}{2}}{2} < \frac{1}{4}$. Since $\frac{1}{2} < \epsilon < 1, \lceil \log_\epsilon(\frac{\delta - \frac{1}{2}}{2}) \rceil \geq 2$. In particular, $g \geq f$. So there is nothing to prove if $\delta \geq \epsilon$.

Hence we may assume $\delta < \epsilon$. And it suffices to prove that there exists a separation $(A^*, B^*)$ of $G$ of order at most $f(|V(G)|)$ such that $|A^* - B^*| \leq \delta|V(G)|$ and $|B^* - A^*| \leq \delta|V(G)|$, as we can apply the same argument to any induced subgraph of $G$.

Let $(A, B)$ be an $\epsilon$-balanced separator of $G$ of order $f(|V(G)|)$. Since $G[A]$ is an induced subgraph of $G$, there exists an $\epsilon$-separator $(A_A, B_A)$ of $G[A]$ of order $f(|A|) \leq f(\epsilon|V(G)|)$. Similarly, $G[B]$ has an $\epsilon$-balanced separator

$(A_B, B_B)$ of $G[B]$ of order $f(|B|) \leq f(\epsilon|V(G)|)$. Let $Z_1 = (A \cap B) \cup (A_A \cap B_A) \cup (A_B \cap B_B)$. Then $|Z_1| \leq f(|V(G)|) + f(|A|) + f(|B|) \leq f(|V(G)|) + 2f(\epsilon|V(G)|)$.

We can repeat this process to obtain a set $Z_k \subseteq V(G)$ with $|Z_k| \leq \sum_{i=0}^{k} 2^i f(\epsilon^i |V(G)|)$ and $2^{k+1}$ induced subgraphs $H_1, H_2, ..., H_{2^{k+1}}$ of $G$ such that each $H_i$ is a union of components of $G - Z_k$ and contains at most $\epsilon^{k+1}|V(G)|$ vertices.

Choose $k = \lceil \log_\epsilon(\frac{\delta - \frac{1}{2}}{2}) \rceil - 1$. So $\epsilon^{k+1} \leq \frac{1}{2}(\delta - \frac{1}{2}) \leq \frac{1}{2}\delta - \frac{1}{4}$. Let $g$ be the function such that $g(x) = \sum_{i=0}^{k} 2^i f(\epsilon^i x)$ for every $x$.

If $|V(\bigcup_{i=1}^{2^{k+1}} H_i)| \leq \frac{1}{2}(\delta + \frac{1}{2})|V(G)|$, then since $\delta \geq \frac{1}{2}$, $\frac{1}{2}(\delta + \frac{1}{2})|V(G)| \leq \delta|V(G)|$, so $(V(\bigcup_{i=1}^{2^{k+1}} H_i) \cup Z_k, Z_k)$ is a $\delta$-balanced separator of $G$ of order $|Z_k| \leq g(|V(G)|)$. So we may assume $|V(\bigcup_{i=1}^{2^{k+1}} H_i)| \geq \frac{1}{2}(\delta + \frac{1}{2})|V(G)|$. Hence there exists a minimum integer $q \in [k]$ such that $|V(\bigcup_{i=1}^{q} H_i)| \geq \frac{1}{2}(\delta + \frac{1}{2})|V(G)|$. Let $A^* = V(\bigcup_{i=1}^{q} H_i)$ and $B^* = V(\bigcup_{i=q+1}^{2^{k+1}} H_i)|$. By the minimality of $q$, $|A^*| = |V(\bigcup_{i=1}^{q} H_i)| = |V(\bigcup_{i=1}^{q-1} H_i)| + |V(H_q)| \leq \frac{1}{2}(\delta + \frac{1}{2})|V(G)| + |V(H_q)| \leq \frac{1}{2}(\delta + \frac{1}{2})|V(G)| + \epsilon^{k+1}|V(G)| \leq \frac{1}{2}(\delta + \frac{1}{2})|V(G)| + (\frac{1}{2}\delta - \frac{1}{4})|V(G)| \leq \delta|V(G)|$. Note that $|A^*| \geq \frac{1}{2}(\delta + \frac{1}{2})|V(G)| \geq \frac{1}{2}|V(G)|$ since $\delta \geq \frac{1}{2}$. So $|B^*| \leq |V(G)| - |A^*| \leq \frac{1}{2}|V(G)| \leq \delta|V(G)|$. Hence $(A^* \cup Z_k, B^* \cup Z_k)$ is a $\delta$-balanced separator of $G$ of order $|Z_k| \leq g(|V(G)|)$.

Now we assume there exist constants $c > 0$ and $0 \leq \beta < 1$ such that $f(x) \leq cx^\beta$. We want to show that there exists a constant $c' > 0$ such that $g(x) \leq c'x^\beta$.

Recall that $g(x) = \sum_{i=0}^{k} 2^i f(\epsilon^i x)$ for every $x$. So $g(x) \leq c \sum_{i=0}^{k} 2^i (\epsilon^i x)^\beta \leq cx^\beta \cdot \sum_{i=0}^{k} (2\epsilon^\beta)^i \leq cx^\beta (k+1)(1 + (2\epsilon^\beta)^k) = c'x^\beta$, where $c' = c(k+1)(1 + (2\epsilon^\beta)^k)$. Note that $k$ only depends on $\epsilon$ and $\delta$. ■