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An optimal viscosity profile in enhanced oil recovery by polymer flooding

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Abstract

Forced displacement of oil by polymer flooding in oil reservoir is one of the effective methods of enhanced (tertiary) oil recovery. A classical model of this process within Hele-Shaw approximation involves three-layer fluid in a Hele-Shaw cell having a variable viscosity fluid in the intermediate layer between oil and water. The goal here is to find an optimal viscosity profile of the intermediate layer that almost eliminates the growth of the interfacial disturbances induced by mild perturbation of the permeability field. We derive the dispersion relation and sharp bounds on the growth rate of the interfacial disturbances for an optimal viscosity profile of the intermediate layer. We also discuss how and why an appropriate choice of variable viscous profile in the intermediate layer can mitigate not only the Saffman–Taylor instability but also the tendency of preferential channeling of flow through high permeable region in the heterogeneous case.
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1. Introduction

One of the factors that has been known to degrade oil recovery is the preferential channeling of flow largely caused by viscosity driven instability and variable permeability (heterogeneity). Viscosity driven instability, also known as the Saffman–Taylor instability, is the cause of what is known as the fingering phenomenon which leads to early breakthrough thereby degrading oil recovery even in homogeneous reservoir. This phenomenon has been studied considerably since the early fifties works of Saffman and Taylor [8], and Chouke et al. [2].

Viscosity driven instability has dreadful short-wave instability and containing this instability has direct positive implications for improved oil recovery. It is more so for heterogeneous reservoirs because this phenomenon can be significantly pronounced in the presence of strong heterogeneities if high permeable regions happen to be perfectly aligned with the direction of growth of the fingers. These various factors have been the motivation behind studies of control (either mitigating or completely eliminating) of viscosity driven instability by various means. Enhanced oil recovery (abbreviated as 'EOR' below) processes are designed to achieve this.

One of the strategies used in EOR is to use polymer flooding, i.e. flood the reservoir first with polymer-thickened-water (abbreviated as 'polysolution' below) followed by water. There are variations on this theme that can be used such as polysolution-alternating-water EOR. Since the polysolution is more viscous than water, the interfacial instability that causes the fingering phenomenon does not grow as rapidly thereby improving the oil recovery process. This has been known since early sixties and there have been several studies (see [5–7,10,11]) of this process which investigate various viscosity profiles in the intermediate layer that may suppress or completely eliminate the viscosity driven instability. There are other ways to enhance secondary oil recovery process a detailed description of some of which can be found in Shah and Schecter [9].

In Daripa et al. [3,4], results of exhaustive numerical studies of this polymer flooding process using saturation model in homogeneous and heterogeneous porous mediums have been reported and discussed in detail. The subjects of some of these studies were displacement processes where oil is displaced by polysolution having uniform concentration of polymer and hence uniform viscosity, which in turn is displaced by water. Even though this polymer flooding improves oil recovery as reported in Daripa et al. [4], the choice of constant viscosity of the polysolution in the intermediate region (between oil and water) may not be optimal in the sense that same amount of polymer can be distributed in the intermediate layer nonuniformly so that the polysolution in intermediate layer has a variable viscosity profile that gives the best improvement in oil recovery. Several studies (see [5–7,10,11]) of this process points in this direction. In mid-eighties, Gorell and Homsy [5] numerically studied the three-layer case with variable viscosity in the middle layer using Hele-Shaw model where the interfaces are taken to be contact discontinuities. At about the same time, Daripa et al. [4] extensively studied instability at dynamic fronts in the initial three-layer fluid set-up using saturation model where the fronts can be shocks with jumps in saturation at the fronts.

The problem of finding a continuously varying viscosity profile in the intermediate region that minimizes the growth rate of disturbances at the interface that is sweeping the oil leads to a Sturm–Liouville problem (eigenvalue problem) with eigenvalues in the boundary conditions. Numerically solving this problem, Gorell and Homsy [5] has obtained a numerical optimal viscosity profile in the intermediate region. Carasso and Paşa [1] has revisited this problem and has obtained an analytical formula for the optimal viscosity in the intermediate region. Both of

these studies use Hele-Shaw model in which the interfaces are contact discontinuities and the permeability is constant in space. All these results for Hele-Shaw flows have some relevance for immiscible flows in homogeneous reservoirs. Limitation comes from the fact that the interfaces in Hele-Shaw models are contact discontinuities where as the interfaces in saturation model involving immiscible flows are shock fronts across which there are jumps in the saturation which depend on time as well as on specific location along the fronts.

Extensive numerical studies on the effect of various *constant* viscosities of the intermediate layer on the growth of disturbances on leading (one that is displacing oil) as well as trailing (one that is displacing polysolution) interfaces have been carried out by Daripa et al. [4] under one or both of the following two conditions: (i) the interfaces are shock fronts in saturation which arise in modeling immiscible flows in porous media (see [4]), and (ii) the medium is heterogeneous. However, there are no results to-date of any kind on the optimal viscosity profiles of the intermediate layer under one or both of the above two conditions. Since both of these conditions are prevalent in actual oil reservoirs, it is of interest to assess the effect of each of these conditions separately on the optimal viscosity profile.

Since most reservoirs are heterogeneous, it will be useful first to consider the problem of interfacial instability in the presence of the second condition, i.e. variable permeability, in a three-layer set-up with middle layer having variable viscosity. However, this problem as stated is not analytically tractable because there are no known simple steady basic solutions of the underlying equations involving planar interfaces in the presence of arbitrary variable permeability. On the other hand, in the case of constant permeability such simple solutions with planar interfaces exist. This makes it possible to study the effect of mild perturbation of this constant permeability on the growth rate of the interfacial disturbances. Therefore, we consider this modified problem based on the hypothesis that the effect of mild perturbation of the constant permeability on the growth rate of the interfacial disturbances closely approximates the evolution of such interfacial disturbances as an initial value problem for the heterogeneous case having variable permeability given by the above perturbed permeability. This way we can gain precise estimate of the effect of mild heterogeneity on the growth rate of the interfacial disturbances in the three-layer case with the middle layer having variable viscosity and even provide optimal viscosity profiles of the middle layer that can make the growth rates as small as desired. We undertake such a study in this paper.

The paper is laid out as follows. In Section 2, we present relevant equations and linear stability analysis for planar interfaces in a two-layer fluid set-up in a Hele-Shaw cell and derive well known Saffman–Taylor result using a framework that is often referenced and is also useful in the follow-up Section 3. In essence, the way Section 2 is presented also reduces the length of Section 3. In Section 3, the three-layer fluid model with middle layer having a fluid with variable viscosity is considered and stability analysis of the planar interfaces under mild perturbation in permeability is presented and results on the growth rate of the disturbance and an optimal viscosity profile are derived. Finally, we conclude in Section 4.

2. Two-layer fluid

Here we recall the standard two-layer Saffman–Taylor result (see [8]) concerning the flow of two immiscible fluids in a Hele-Shaw cell. A Hele-Shaw cell consists primarily of two transparent

plates with a narrow gap of width, say d, in between. If a fluid is encapsulated in the cell (which can be propped up at a slant or mounted vertically so that gravity and buoyancy move the liquids), then the motion of fluid inside the cell is quasi-two-dimensional and the governing equations are same as that of Darcy flow in porous media with effective permeability $d^2/12$.

In the case of two immiscible fluids in a Hele-Shaw cell with water displacing oil (as studied by Saffman and Taylor), there exists a "sharp" interface between displacing fluid (water) and oil. We use the continuity equation for the velocity (u,v) and the Darcy law for pressure p, in the fixed coordinate system $x_1 O y$ where x_1 is the direction of flow far upstream, direction y is orthogonal to it in the plane of the flow in the usual Cartesian coordinate setting, and 'O' is the origin of the system. The relevant equations are

$$u_{x_1} + v_y = 0, \quad p_{x_1} = -\mu k u, \quad p_y = -\mu k v, \qquad \forall\ (x_1, y) \in R^2 \tag{1}$$

where μ is the viscosity, and k is the inverse permeability. We have two constant viscosities: water viscosity μ_1 and oil viscosity μ_2. The water velocity far upstream is in the x_1-direction and is constant, denoted by U.

In the mobile coordinate system $x = x_1 - Ut$, we consider the following basic solution with the planar interface at $x = 0$: $u = 0$, $v = 0$, and $p = P(x)$ where U and $P(x)$ satisfy

$$\partial P / \partial x = -\mu k U; \quad \underline{x < 0 : \mu = \mu_1}; \quad \underline{x > 0 : \mu = \mu_2}. \tag{2}$$

We consider the perturbations $\tilde{u}, \tilde{v}, \tilde{p}$ and study the *linear* perturbations of the basic solution (2). We obtain the following perturbations system:

$$(\tilde{u})_x + (\tilde{v})_y = 0, \quad x \neq 0, \tag{3}$$

$$(\tilde{p})_x = -\mu k \tilde{u}, \quad x \neq 0, \tag{4}$$

$$(\tilde{p})_y = -\mu k \tilde{v}, \quad x \neq 0. \tag{5}$$

We consider normal mode analysis using the following form of the perturbation in the x-component of the velocity,

$$\tilde{u}(x, y, t) = f(x) \exp(i\alpha y + \sigma t), \tag{6}$$

where σ is the growth rate and the wavenumber α is taken to be greater than zero without any loss of generality. The perturbations $\tilde{v}$ and $\tilde{p}$ are obtained using Eqs. (3) and (5):

$$\tilde{v} = -(f_x / i\alpha) \exp(i\alpha y + \sigma t), \quad \tilde{p} = -(\mu k f_x / \alpha^2) \exp(i\alpha y + \sigma t). \tag{7}$$

If the planar surface is disturbed slightly such that its equation becomes $x = \tilde{g}(y, t)$, then the kinematic condition that each particle remains there gives

$$\tilde{g}_t = \tilde{u}(0, y, t), \quad \text{on } x = \tilde{g}(y, t), \tag{8}$$

within linear approximation (assuming perturbation is small). It then follows from (6) and (8) that

$$\tilde{g}(y, t) = (f(0) / \sigma) \exp(i\alpha y + \sigma t). \tag{9}$$

The dynamic boundary condition for the free interface, within linear approximation, is given by

$$p^+(0) - p^-(0) = T \tilde{g}_{yy}, \tag{10}$$

where the superscripts '+' and '−' are used to denote the "right" and "left" limit values, T is the surface tension and $\tilde{g}_{yy}$ is the approximate curvature of the perturbed interface. Above and below, h_y denotes the derivative of an arbitrary function $h(y)$ with respect to y.

We have the following expression for the pressure near the point $x = 0$:

$$p(0 + \tilde{g}(y,t)) = P(0 + \tilde{g}(y,t)) + \tilde{p}(0 + \tilde{g}(y,t)). \tag{11}$$

The basic pressure P is continuous because the planar interface of the basic solution has zero curvature. First-order Taylor approximation of P gives

$$P(0 + \tilde{g}(y,t)) = P(0) + \tilde{g}(y,t) \cdot \partial P/\partial x(0). \tag{12}$$

The right limit values of $\partial P/\partial x$ and $\tilde{p}$ are the following:

$$\partial P^+/\partial x(0) = -Uk\mu^+(0); \quad \tilde{p}^+(0) = -(\mu^+(0)kf_x^+(0)/\alpha^2)\exp(i\alpha y + \sigma t), \tag{13}$$

and similar expressions can be obtained for corresponding left limit values.

The above expressions and the relation (11) are used to obtain the right and left limit values of the pressure near the point $x = 0$:

$$p^+(0) = P(0) - \mu^+(0)k\left\{\frac{f_x^+(0)}{\alpha^2} + \frac{U}{\sigma}f(0)\right\}\exp(i\alpha y + \sigma t), \tag{14}$$

$$p^-(0) = P(0) - \mu^-(0)k\left\{\frac{f_x^-(0)}{\alpha^2} + \frac{U}{\sigma}f(0)\right\}\exp(i\alpha y + \sigma t). \tag{15}$$

For the case of zero surface tension ($T = 0$) and $\mu^-(0) = \mu^+(0)$, condition (10) after using (14) and (15) gives

$$f_x^-(0) = f_x^+(0). \tag{16}$$

We consider the case $T \neq 0$, μ discontinuous at $x = 0$, and $\tilde{g}_{yy} \neq 0$. Then using (14) and (15) in condition (10) gives

$$\sigma = \frac{kU\alpha^2(\mu^+(0) - \mu^-(0)) - T\alpha^4}{\mu^-(0)f_x^-(0) - \mu^+(0)f_x^+(0)} \cdot \frac{f(0)}{k}. \tag{17}$$

The following equation of f is obtained by cross differentiating the relations (4) and (5) and then using (6) and (7).

$$f(x)_{xx} - \alpha^2 f(x) = 0, \quad x \neq 0. \tag{18}$$

Note that the perturbation $\tilde{u}$ is continuous across the interface. Hence f is also continuous across the interface (see (6)). For zero perturbations far upstream and downstream, Eq. (18) has following solution.

$$\underline{x < 0}: \ f(x) = ae^{\alpha x}, \ f_x^-(0) = \alpha a; \quad \underline{x > 0}: \ f(x) = ae^{-\alpha x}, \ f_x^+(0) = -\alpha a. \tag{19}$$

We have $\mu^-(0) = \mu_1$ (water viscosity) and $\mu^+(0) = \mu_2$ (oil viscosity). The above solution for f in (17) gives the growth rate

$$\sigma_s = \frac{kU\alpha(\mu^+(0) - \mu^-(0)) - T\alpha^3}{k(\mu^-(0) + \mu^+(0))} = \frac{kU\alpha(\mu_2 - \mu_1) - T\alpha^3}{k(\mu_2 + \mu_1)}, \tag{20}$$

where subscript 's' stands for Saffman–Taylor since this is the well known formula of Saffman and Taylor (see [8]). In the case $\mu^-(0) > \mu^+(0)$, relation (17) gives $\sigma_s < 0$, $\forall \alpha \in R$, and hence the interface is stable. On the contrary, if $\mu_1 = \mu^-(0) < \mu^+(0) = \mu_2$, then the interface is unstable because

$$\sigma_s > 0 \quad \text{for} \quad \alpha < \alpha_{sc} = \sqrt{\frac{kU(\mu_2 - \mu_1)}{T}}, \tag{21}$$

where α_{sc} is the cut-off wave number where the growth rate is zero. The most dangerous wave-number $\alpha_{sm} < \alpha_{sc}$ and the maximum growth rate σ_{sm} are given by

$$\alpha_{sm} = \alpha_{sc}/3, \quad \sigma_{sm} = \frac{2\{kU(\mu_2 - \mu_1)\}^{3/2}}{3k(\mu_2 + \mu_1)\sqrt{3T}}. \tag{22}$$

The interfacial stability as discussed above shows that the interface, when water displacing oil, is unstable and suffers from short-wave instabilities ($\sigma = O(\alpha)$, as $\alpha \to \infty$) for zero surface tension case. These are detrimental to the sweeping efficiency of the interface.

Since most dangerous mode grows as 3/2 power to the viscosity jump across the interface (see (22)), one of the ways to control the interfacial instability and thereby improve sweeping efficiency is then to control the viscosity jump at the interface that displaces the oil. One of the ways to achieve this is to have an intermediate layer of fluid between oil and water which perhaps has viscosity closer to oil than water. This requires a study of the displacement process involving a three-layer fluid.

Since adding polymer in water increases the viscosity of this polymer-thickened-water phase (to be called 'polysolution' henceforth), use of such a polysolution, instead of water, in displacing oil followed by water which in turn displaces the polysolution is likely to suppress the instability of the interface sweeping the oil, assuming validity of the above analysis involving two-layer case to this three-layer case (see [4] for some discussion on such polymer flooding process). Since the viscosity of polysolution depends on the concentration of polymer, a polysolution with variable viscosity whose viscosity gradually decreases along the layer from the oil–polysolution interface to the water–polysolution interface is likely to offer most advantage given a fixed amount of polymer, considering the fact that polymer is very expensive. The goal is to find an optimal viscous profile that can arbitrarily reduce the growth rate of the interfacial instabilities (one that is sweeping the oil) induced by mild perturbation in permeability. This is the subject of study of the next section. Of course, the problem is now somewhat more complicated as the viscous profile of the intermediate layer affects the stability properties of both the interfaces under perturbation in permeability.

3. Three-layer fluid

We consider the "three-layer" fluid in a Hele-Shaw cell with polysolution having variable viscosity in the middle layer sandwiched between water and oil in the extreme layers. Now, there are two interfaces: oil–polysolution interface and oil–water interface. In the displacement process where the water is displacing the intermediate layer which in turn is displacing the oil, the sweeping efficiency of the process, in part, is dictated by the stability characteristics of the oil–polysolution interface since this interface is actually sweeping the oil.

Since the middle layer has polymer thickened aqueous phase in which polymer is passively advected, we need an equation of continuity for polymer concentration. However, since viscosity depends directly on this polymer concentration and the viscosity μ is considered invertible with respect to the polymer concentration, instead we obtain a continuity equation for viscosity μ in the middle layer (see [5]). Therefore, the following basic solution with planar interfaces at $x = -l$ and at $x = 0$ exists in the mobile coordinate system $x = x_1 - Ut$:

$$u = 0, \quad v = 0, \quad p = P(x), \quad \mu = \mu(x), \tag{23}$$

where $P(x)$ and $\mu(x)$ satisfy following equations:

$$x \neq -\{l, 0\} : \ \partial P/\partial x = -\mu k U; \quad \partial P/\partial y = 0, \tag{24}$$

$$x \in (-l, 0) : \ \mu_t + U\mu_x = 0, \tag{25}$$

$$\underline{x < -l} : \ \mu = \mu_1, \quad \underline{x > 0} : \ \mu = \mu_2. \tag{26}$$

We consider the perturbations $\tilde{k}, \tilde{u}, \tilde{v}, \tilde{p}, \tilde{\mu}$, of the permeability, velocity, pressure and viscosity (only in the intermediate region), and obtain the system:

$$(\tilde{u})_x + (\tilde{v})_y = 0, \quad x \neq -l, 0; \tag{27}$$

$$(\tilde{p})_x = -\mu k \tilde{u} - \mu \tilde{k} U - \tilde{\mu} k U, \quad x \neq -l, 0; \tag{28}$$

$$(\tilde{p})_y = -\mu k \tilde{v}, \quad x \neq -l, 0; \tag{29}$$

$$\tilde{\mu}_t + \tilde{u}(\mathrm{d}\mu/\mathrm{d}x) = 0, \quad x \in (-l, 0). \tag{30}$$

Compared with the similar governing equations (see (3)–(5)) of the previous section, we now have the new terms $(-\tilde{\mu}kU - \mu\tilde{k}U)$ in the Eq. (28) for the x-derivative of the pressure perturbation, and a new Eq. (30) for the perturbed viscosity in the middle layer. Note that the Eqs. (27) and (29) coincide with the Eqs. (3) and (5).

We use normal mode analysis involving these perturbations with an ansatz similar to that used in (6) for the x-component of the velocity. We assume that perturbations in the above quantities in the Hele-Shaw cell is primarily introduced by infinitesimal movement of the plates which are somewhat flexible and this disturbance can grow in time within linear theory. This is the rationale for using this ansatz similar to (6) for $\tilde{k}$.

On carrying out the algebra with the above mentioned ansatz in the above set of equations, we obtain the same previous expressions (7) for $\tilde{v}$ and $\tilde{p}$. We consider the perturbed interface given by (8) at $x = 0$ and the pressure drop given by (10). Therefore we have the same growth rate (17) as a function of f and limit values of the viscosity. The viscosity perturbation is given by (6) and (30):

$$\tilde{\mu} - (\mu_x(x)f(x)/\sigma)\exp(\mathrm{i}\alpha y + \sigma t). \tag{31}$$

Below we use the notation

$$r(x) = \mu_x(x)/\mu(x), \quad x \in I.R. \tag{32}$$

The equation of f is obtained by cross differentiating (28) and (29):

$$(\mu k \tilde{u} + \mu \tilde{k} U + \tilde{\mu} k U)_y = (\mu k \tilde{v})_x. \tag{33}$$

The influence of perturbed permeability occurs only if $\tilde{k}$ depends on y. We obtain the following equations of f:

$$f_{xx}(x) + r(x) f_x(x) - \alpha^2 f(x) = -\frac{1}{\sigma} \alpha^2 f(x) U r(x) + \alpha^2 U \varepsilon, \quad x \in (-l, 0); \tag{34}$$

$$f_{xx}(x) - \alpha^2 f(x) = \alpha^2 U \varepsilon, \quad x \notin (-l, 0). \tag{35}$$

Note that Eq. (35) follows from (34) with $r(x) = 0$. The ε in the above equations is the amplitude of the permeability perturbation. We have to find the solution f of the Eqs. (34) and (35), where μ is an unknown function of x. The boundary conditions are:

$$x = -l : \ \mu^+(-l) = \mu^-(-l) = \mu_1, \quad \text{surface tension } T = 0. \tag{36}$$

$$x = 0 : \ \mu_1 < \mu^-(0) < \mu_2, \quad \text{surface tension } T \neq 0. \tag{37}$$

The relation (16) and condition (36) give

$$f_x^-(l) = f_x^+(-l). \tag{38}$$

Since the function f is continuous and bounded far upstream and downstream, we obtain

$$f^+(-l) = f^-(-l), \quad f^+(0) = f^-(0). \tag{39}$$

Now the problem is to solve the system (34) and (35) subject to the boundary conditions (38) and (39) and find the optimal viscosity μ, in the intermediate layer which gives us the smallest value of the growth rate (17).

We consider a constant value of r in (32)–(34) and search for an Euler solution of the system (34) and (35). We use the last condition given in (39) and obtain

$$x < -l : \ f(x) = c \exp(\alpha x) - U \varepsilon, \tag{40}$$

$$x \in (-l, 0) : \ f(x) = b \exp(qx) - \sigma U \varepsilon / (\sigma - U r), \tag{41}$$

$$x > 0 : \ f(x) = a \exp(-\alpha x) - U \varepsilon, \tag{42}$$

where $a > 0$, a, b, c are complex numbers, and $q^2 + rq - \alpha^2 (1 - U r / \sigma) = 0$. Subject to the condition

$$1 - U r / \sigma > 0, \tag{43}$$

the roots of this equation are real of which only the positive root is of relevance for our solution. Moreover, a consequence of this inequality is $0 < q < \alpha$ which can easily be verified from the above quadratic equation for q. The validity of the above condition (43) is justified towards the end of this section under Remark 1.

The first two conditions given in (39) and the continuity of f at $x = 0$ give

$$b = \frac{U^2 \varepsilon r}{\sigma - U r} \theta, \quad a = \frac{U^2 \varepsilon r}{\sigma - U r} (\theta - 1) = f(0) + U \varepsilon, \quad \theta = \frac{\alpha}{\alpha - q} \exp(ql). \tag{44}$$

The solutions (41) and (42) give

$$\mu^-(0)f_x^-(0) = \mu^-(0)qb, \quad f_x^+(0) = -\alpha a, \tag{45}$$

where constants a and b are given by (44). Substituting (45) in the formula (17) for the growth rate σ, we obtain, after simple algebraic manipulation (note that constants a and b also depend on σ), the following dispersion relation

$$\sigma = \frac{E(\alpha)Ur\alpha\mathbf{e}^{ql}}{(\alpha - q)E(\alpha) + kUr(\mu^-(0)q\alpha\mathbf{e}^{ql} + \mu_2\alpha(\alpha\mathbf{e}^{ql} - \alpha + q))}, \tag{46}$$

where we have used

$$E(\alpha) = \alpha^2 kU(\mu_2 - \mu^-(0)) - \alpha^4 T. \tag{47}$$

Note from relation for $E(\alpha)$ that the σ in (46) stands for the growth rate of the interface displacing the oil. Henceforth, this will be implicit when we use σ.

The parameter ε describing the fluctuation of the permeability does not appear *explicitly* in the final dispersion relation (46) but implicitly it does which makes the result (46) mathematically consistent. A constructive way to explain the dependence of the growth rate σ on ϵ is to first note that the growth rate σ of a mode with wave number α can be calculated from (46) for given values of μ_2, $\mu^-(0)$, k (unperturbed permeability), U, and r provided q (see (46)) is known a priori which it is not. But q depends on the growth rate σ itself through (44)$_2$ and (44)$_3$. Thus the dispersion relation (46) is a nonlinear equation in the growth rate which depends on the parameter ε. Formally, one can set up an iterative method to calculate σ which starts with an initial guess of q, then using (46) to calculate σ, followed by (44)$_2$ (i.e. $\frac{U^2\varepsilon r}{\sigma - Ur}(\theta - 1) = f(0) + U\varepsilon$) to calculate θ (which depends on ε, the amplitude of perturbation in permeability) and then using (44)$_3$ (i.e. $\theta = \frac{\alpha}{\alpha - q}\exp(ql)$) for updating the value of q. This implicitly shows the dependence of σ on ε. However, our basic result is not (46) but rather estimate (53) (see below) which follows from it for the optimal viscous profile (48) (see below) and does not require any calculation involving q as above. Below, we obtain a qualitative result estimating this growth constant σ as a function of α, μ, l, and r.

Considering the constant value of r in the intermediate region that we have chosen above, the optimal viscosity profile in the intermediate region consistent with (32) and continuity of viscosity $\mu^+(-l) = \mu_1$ at $x = -l$ interface is then given by

$$\mu(x) = \mu_1 \exp\{r(x + l)\}. \tag{48}$$

The constant r in (48) is still undefined and can be chosen as

$$r \approx \frac{1}{l} \ln \frac{\mu_2 - \eta}{\mu_1}, \tag{49}$$

where η is a small positive number. Note that $\exp(rl) < \infty$, regardless of how large is l. It follows from (48) and (49) that the viscosity $\mu^-(0)$ in the intermediate region at the interface $x = 0$ is given by

$$\mu^-(0) = \mu_1 \mathbf{e}^{rl} \approx \mu_2 - \eta < \mu_2. \tag{50}$$

This η equals the jump in viscosity, $[\mu] = \mu_2 - \mu^-(0)$, at the interface separating oil from the variable viscosity fluid in the intermediate layer.

For large values of l, from the dispersion relation (46) we obtain

$$\sigma \approx \frac{E(\alpha)}{k(\mu^-(0)q + \mu_2\alpha)}. \tag{51}$$

It is worth noting that this result resembles two-layer Saffman–Taylor result (see (17)) with an adjusted viscosity. The adjusted viscosity, of course, is a result of the stability analysis of the three-layer case as presented above. Since (see (47))

$$E(\alpha) > 0 \quad \text{for } \alpha \leqslant \alpha_c = \sqrt{\frac{kU\eta}{T}}, \tag{52}$$

we obtain from (51) after using (50) and $0 < q < \alpha$, that

$$\frac{E(\alpha)}{2k\mu_2\alpha} \leqslant \sigma \leqslant \frac{E(\alpha)}{k\mu_2\alpha}, \tag{53}$$

which gives the upper and lower bounds of the growth rate σ. It is important to note that only the modes with wavenumbers in the small interval $0 < \alpha \leqslant \alpha_c = \sqrt{\frac{kU\eta}{T}}$ are unstable because for α in this range we have $\sigma > 0$.

The upper bound of the growth rate of the most dangerous wave number α_m is obtained by seeking α at which $E(\alpha)/\alpha$ takes a maximum value (see (51)) and then using this value α in the right hand side of (51) which gives

$$\alpha_m = \alpha_c/3, \quad \sigma \leqslant \sigma_m = \frac{2\{kU(\mu_2 - \mu^-(0))\}^{3/2}}{3k\mu_2\sqrt{3T}} = \frac{2\{kU[\mu]\}^{3/2}}{3k\mu_2\sqrt{3T}}. \tag{54}$$

Therefore, the optimal profile gives a maximum growth rate of order $[\mu]^{3/2}$ where $[\mu] = \mu_2 - \mu^-(0)$ is the jump in viscosity at the interface separating oil from the variable viscosity fluid in the intermediate layer.

Remark 1. We need to justify the condition (43). We use the left inequality of (53) to obtain $(1/\sigma) \leqslant \frac{2k\mu_2\alpha}{E(\alpha)}$; in which we use the fact that $E(\alpha) = \alpha^2 kU(\mu_2 - \mu_1 e^{rl}) - \alpha^4 T$ (see (48) and (50)) is bounded as $r \to 0$. Then $(Ur/\sigma) \leqslant Ur\frac{2k\mu_2\alpha}{E(\alpha)} \to 0$ as $r \to 0$ whose consistency with (43) suggests that this condition will hold for values of r away from zero but less than some finite threshold value of r.

4. Discussions and conclusion

When a less viscous fluid displaces a more viscous one, the Saffman–Taylor instability for zero surface tension case suffers from short-wave instability which is completely eliminated by surface tension in the presence of which the cutoff-wavenumber α_{sc} and the growth rate of the most unstable mode are proportional to, respectively, 1/2 and 3/2 power to the jump in viscosity at the interface. In the three-layer case, these same estimates hold for the interface sweeping the oil when the intermediate layer is long and has viscosity profile (48). The advantage of using this optimal viscosity profile is that jump in viscosity can be made as small as we please and thus the maximum

growth rate as well as cut-off wavenumber can be made arbitrarily small, thereby almost eliminating the instability of the interface. The long waves with wavenumbers below the cut-off wavenumber, $\alpha_{\mathrm{c}} = \sqrt{\frac{kU[\mu]}{T}}$, can only grow but the maximum growth rate (see (54)) can be made so small, at least with large l, that its existence may not have much relevance in the context of oil recovery, considering the degree of uncertainty that exist in such oil recovery processes.

In the case of mild heterogeneity, the dispersion relation (46) shows that the growth rate of individual modes depends on the length 'l' of the intermediate layer and the modified measure 'q' of the small perturbation 'ε' in permeability. The relationship between 'q' and 'ε' is complicated involving the growth rate 'σ' (see (44)). Therefore, the dispersion relation (46) is a nonlinear equation in the growth rate itself for individual modes. Remarkably, for the same viscous profile (48) which can diminish the growth of the interfacial disturbance arbitrarily small, upper and lower bounds of the growth rate are given by the simple formulae (53) for large 'l'. These bounds do not depend on the amplitude of perturbation in permeability and, in fact, are same as for the homogeneous case where the interfaces are directly subjected to disturbances. This is primarily due to the fact that the variation in the growth rate due to small perturbation, ε, in permeability is negligibly small for the optimal viscous profile (48). In other words, if we were to write the growth rate as a Taylor series in ε, i.e., $\sigma(\alpha, \varepsilon) = \sigma(\alpha) + \varepsilon \sigma_\varepsilon(\alpha, 0) + O(\varepsilon^2)$, then the $\sigma_\varepsilon(\alpha, 0)$ is very small for the optimal viscous profile (48) and only the leading term in the series really matters for practical purposes. Thus, what our result indirectly shows that an appropriate choice of variable viscous profile in the intermediate layer can mitigate not only the Saffman–Taylor instability but also the tendency of preferential channeling of flow through high permeable regions in the heterogeneous case.

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