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On the growth rate for three-layer Hele–Shaw flows: Variable and constant viscosity cases

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Abstract

The linear stability of three-layer Hele–Shaw flows with middle-layer having variable viscosity is considered. Based on application of the Gerschgorin's theorem on finite-difference approximation of the linearized disturbance equations, an upper bound of the growth rate is given and its limiting case for the case of constant viscosity middle-layer is considered. A weak formulation of this equation, we obtained after some analysis. The upper bound in this case has also been derived here by analyzing an weak formulation of the problem.
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1. Introduction

This paper concerns displacement of a more viscous fluid by one or more fluids in succession each having less effective viscosity than the one being pushed by it. In traditional two-layer Hele–Shaw flows, the planar interface is unstable when a more viscous fluid is displaced by a less viscous fluid. The disturbances grow at a rate which depends on the viscosity difference at

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the interface. In three-layer Hele–Shaw flows, growth rate of disturbances depends on the viscosities of fluids in all three layers and growth rate of the leading interface, i.e. the interface in contact with the most viscous fluid in our above set-up, depends on viscosity profiles of all three layers. The set-up where the extreme layers have constant viscosities and the middle-layer has varying viscous profile has been of some interest because this can optimally provide control of the disturbances at the leading interface. This study also has relevance in the context of enhanced oil recovery.

In oil recovery by secondary displacement processes, a viscous oil in a porous medium is displaced by the injection of another less viscous immiscible fluid, usually water. The sharp interface (a contact discontinuity), within Hele–Shaw model approximation, separating oil and water suffers from the Saffman–Taylor instability [1] which is one of the sources of poor oil recovery as the moving unstable interface fails to sweep the oil completely before interface breaks at the production well, thereby producing water instead of oil at breakthrough. The surface tension reduces the instability only to some extent which is not sufficient enough for enhancing oil recovery. In order to contain this instability to a meaningful level for improving oil recovery process before breakthrough, various tertiary displacement processes are used. One of these processes involves flooding the reservoir with a fluid having variable viscosity followed by pure water. This recovery process, thus, involves fluid flow involving three regions each containing fluid having different viscosity properties. Because there is some analogy between flows in porous media and Hele–Shaw flows, we consider the three-layer Hele–Shaw flows as it retains the main instability mechanism due to viscosity difference in porous media though the problem here is somewhat simpler due to absence of other effects usually present in porous media.

The paper is laid out as follows. In Section 2, we lay out the disturbance equations of Hele–Shaw flows and its discrete approximations. Using these discrete approximations with Gerschgorin's localization theorem we arrive at a theoretical upper bound on the growth rate when the middle-layer has variable viscosity. This is produced in Section 3. Section 4 analyzes theoretical upper bound for slowly varying viscous profile. In Section 5, using a weak formulation of the disturbance equations valid only for the constant viscosity case, we obtain a theoretical upper bound of the growth rate for the constant viscosity case which is also recovered from the results valid for the slowly varying viscous case. Finally, we conclude in Section 6.

2. The disturbance equation and its discrete approximation

Consider the three-layer Hele–Shaw cell where the fluid in the left layer with viscosity μ_l extends up to $x = -\infty$ and fluid in the right layer with viscosity μ_r extends up to $x = \infty$ and the in-between middle-layer of length L contains a fluid of variable viscosity $\mu(x)$. The fluid at upstream $x = -\infty$ has velocity $(U, 0)$. The underlying equations of this problem admits a simple solution which is that the whole system and thus, the two planar interfaces also move with speed U in the x direction. The system is considered to be infinite in the y direction.

In a moving frame where the frame moves with the velocity $(U, 0)$, the above system is stationary along with the two planar interfaces separating these three fluid layers. In linearized stability analysis of these disturbances in the moving frame, the amplitude of the modal disturbances is assumed to be proportional to a function $f(x)$ and the viscosity of the middle-layer is $\mu(x)$ where, with slight abuse of notation without losing any substance, the same variable x is used in the moving frame. As discussed in detail in [2–4], the linear stability analysis then gives rise to the following equation for the perturbation

$$\left.\begin{array}{l} -(\mu f_x)_x + k^2\mu f = \lambda k^2 U\sigma\mu_x f, \quad x\in(-L,0), \\ f_x(0) = (\lambda p + q)f(0), \quad f_x(-L) = (\lambda r + s)f(-L), \end{array}\right\}, \tag{1}$$

where $\lambda = 1/\sigma$, and p, q, r, s are defined by

$$\left.\begin{array}{l} p = \{(\mu_r - \mu(0))Uk^2 - Tk^4\}/\mu(0), \quad q = -\mu_r k/\mu(0) \leqslant 0, \\ r = \{(\mu_l - \mu(-L))Uk^2 + Sk^4\}/\mu(-L), \quad s = \mu_l k/\mu(-L) \geqslant 0. \end{array}\right\}. \tag{2}$$

It is worth noting that

$$p \geqslant 0 \text{ for } k^2 \leqslant k_1^2 = (\mu_r - \mu(0))U/T, \; r \leqslant 0 \text{ for } k^2 \leqslant k_2^2 = (\mu(-L) - \mu_l)U/S. \tag{3}$$

All these equations are in dimensional form.

We discretize the problem (1) using $(M-1)$ equidistant interior points in $(-L,0)$: $x_M = -l < x_{M-1} < x_{M-2} < \cdots x_1 < x_0 = 0$, with $d = (x_i - x_{i+1})$. We use the first order approximation for the end points derivatives and second order approximation for the interior point derivatives namely,

$$\left.\begin{array}{l} f_x(-L) = (f_{M-1} - f_M)/d, \quad f_x(0) = (f_0 - f_1)/d, \\ f_x(y) = [f(y+d/2) - f(y-d/2)]/d, \quad f_{xx}(y) = [f(y+d) - 2f(y) + f(y-d)]/d^2, \end{array}\right\}, \tag{4}$$

where y is any one of the interior discretization points. Using these finite difference approximations (4) in the boundary conditions given in (1) leads to

$$(f_{M-1} - f_M)/d = (\lambda r + s)f_M, \quad (f_0 - f_1)/d = (\lambda p + q)f_0, \tag{5}$$

which are rewritten as

$$\frac{1}{rd}f_{M-1} - \left(\frac{1}{rd} + \frac{s}{r}\right)f_M = \lambda f_M \quad \text{and} \quad \left(\frac{1}{dp} - \frac{q}{p}\right)f_0 - \frac{1}{dp}f_1 = \lambda f_0. \tag{6}$$

Using (4)–(6) the discrete analog of the pde (1) in a compact form is written as

$$A\overline{f} = \lambda B\overline{f}, \qquad \overline{f} = (f_0, f_1, f_2, \ldots, f_M). \tag{7}$$

As an example, for the case of three interior points matrix A and B are given by

$$\begin{pmatrix} \left(\frac{1}{dp} - \frac{q}{p}\right) & -\frac{1}{dp} & 0 & 0 & 0 \\ \frac{-\mu_{1/2}}{d^2} & \left(\frac{\mu_{1/2}+\mu_{3/2}}{d^2} + \mu_1 k^2\right) & -\frac{\mu_{3/2}}{d^2} & 0 & 0 \\ 0 & -\frac{\mu_{3/2}}{d^2} & \left(\frac{\mu_{3/2}+\mu_{5/2}}{d^2} + \mu_2 k^2\right) & -\frac{\mu_{5/2}}{d^2} & 0 \\ 0 & 0 & -\frac{\mu_{5/2}}{d^2} & \left(\frac{\mu_{5/2}+\mu_{7/2}}{d^2} + \mu_3 k^2\right) & -\frac{\mu_{7/2}}{d^2} \\ 0 & 0 & 0 & \frac{1}{rd} & -\left(\frac{1}{dr} + \frac{s}{r}\right) \end{pmatrix}, \tag{8}$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & k^2 U \mu_1' & 0 & 0 & 0 \\ 0 & 0 & k^2 U \mu_2' & 0 & 0 \\ 0 & 0 & 0 & k^2 U \mu_3' & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{9}$$

where μ_i' denotes first derivative of $\mu(x)$ at point $x = x_i$. Similarly, $\mu_{i/2}$ stands for values of $\mu(x)$ at mid-point of the subinterval $[x_{i-1}, x_i]$. We rewrite (7) as

$$C\overline{f} = \lambda \overline{f}, \quad C = (C_{ij}), \tag{10}$$

which for the case of three-interior points is equivalent to multiplying Eqs. (2)–(4) in system (7) by $(k^2 U \mu_l')^{-1}$, $(k^2 U \mu_r')^{-1}$, $(k^2 U \mu_3')^{-1}$ respectively.

3. An upper bound

Using Gerschgorin's localization theorem for the eigenvalues λ of (10), which are contained in the *union* of the following circles, we obtain

$$|\lambda - C_{11}| \leqslant |C_{12}| \tag{11}$$

$$|\lambda - C_{22}| \leqslant |C_{21}| + |C_{23}| \tag{12}$$

$$|\lambda - C_{33}| \leqslant |C_{32}| + |C_{34}| \tag{13}$$

$$|\lambda - C_{44}| \leqslant |C_{43}| + |C_{45}| \tag{14}$$

$$|\lambda - C_{55}| \leqslant |C_{54}|. \tag{15}$$

From inequalities (12)–(14) we obtain

$$\frac{\mu_i}{U\mu_i'} \leqslant \lambda \Rightarrow \sigma \leqslant \frac{U\mu_i'}{\mu_i}. \tag{16}$$

From inequality (11), we obtain

$$\left(\frac{1}{pd} + \frac{-q}{p} - \left|-\frac{1}{pd}\right|\right) \leqslant \lambda \leqslant \left(\frac{1}{pd} + \frac{-q}{p} + \left|-\frac{1}{pd}\right|\right), \tag{17}$$

and from inequality (15), we obtain

$$\left(-\frac{1}{rd} - \frac{s}{r} - \left|\frac{1}{rd}\right|\right) \leqslant \lambda \leqslant \left(-\frac{1}{rd} - \frac{s}{r} + \left|\frac{1}{rd}\right|\right). \tag{18}$$

If $p \geqslant 0$, then it follows from (17) that

$$-\frac{q}{p} \leqslant \lambda \leqslant -\frac{q}{p} + \frac{2}{pd}. \tag{19}$$

If $p \leqslant 0$, then it follows from (17) that

$$\frac{2}{pd} - \frac{q}{p} \leqslant \lambda \leqslant -\frac{q}{p}. \tag{20}$$

If $r \geqslant 0$, then it follows from (18) that

$$-\frac{2}{rd}-\frac{s}{r}\leqslant\lambda\leqslant-\frac{s}{r}. \tag{21}$$

If $r \leqslant 0$, then it follows from (18) that

$$-\frac{s}{r}\leqslant\lambda\leqslant-\frac{s}{r}-\frac{2}{rd}. \tag{22}$$

It is worth recalling here that $q<0$ and $s>0$. We have the following two inequalities, the first one from (19) and (20), and the second one from (21) and (22).

$$\lambda>0 \quad \text{ONLY if} \quad p\geqslant 0, \text{ and also } \lambda>0 \;\; \text{ONLY if} \quad r\leqslant 0. \tag{23}$$

The inequalities (19) and (22) give the most dangerous situation for the wave numbers. Therefore, we have the following estimates which also includes the inequality (16).

$$0\leqslant\sigma\leqslant\frac{U\mu_i'}{\mu_i} \quad \text{or} \quad \sigma\leqslant-\frac{p}{q} \quad \text{or} \quad \sigma\leqslant-\frac{r}{s}. \tag{24}$$

Since

$$p\geqslant 0 \Longleftrightarrow k^2\leqslant\frac{U[\mu_r-\mu(0)]}{T}, \qquad r\leqslant 0 \Longleftrightarrow k^2\leqslant\frac{U[\mu(-L)-\mu_l]}{S}, \tag{25}$$

inequalities (19) and (22) holds iff

$$k^2\leqslant \text{Max}\left\{\frac{U[\mu(-L)-\mu_l]}{S},\frac{U[\mu_r-\mu(0)]}{T}\right\}. \tag{26}$$

Therefore, the inequality (26) is a necessary and sufficient condition for instability of a mode with wave number k. If (26) is true, then it follows that

$$\sigma\leqslant \text{Max}\left\{-\frac{p}{q},-\frac{r}{s},\frac{U\mu_i'}{\mu_i}\right\}. \tag{27}$$

Since this is true for any arbitrary number of grid points, we conclude from applying this inequality in the limit of zero mesh size that

$$\sigma\leqslant \text{Max}\left\{-\frac{p}{q},-\frac{r}{s},\underset{x}{\text{Max}}\left(\frac{U\mu'}{\mu}\right)\right\}. \tag{28}$$

(a) Case I

$$\frac{U[\mu(-L)-\mu_l]}{S}<\frac{U[\mu_r-\mu(0)]}{T} \tag{29}$$

- Case I.a: If $k^2<[\mu(-L)-\mu_l]U/S$, then $r<0$, $p>0$, and we have (27).
- Case I.b: If $[\mu(-L)-\mu_l]U/S<k^2<[\mu_r-\mu(0)]U/T$, then $r>0$, $p>0$, and we have $\sigma\leqslant \text{Max}\left\{-(p/q),\underset{x}{\text{Max}}\left(\frac{U\mu'}{\mu}\right)\right\}$.
- Case I.c: If $[\mu_r-\mu(0)]U/T<k^2$, then $r>0$, $p<0$, and we have from (23) and (16), the inequality $\sigma\leqslant\underset{x}{\text{Max}}\left(\frac{U\mu'}{\mu}\right)$.

$$\underset{k}{\mathrm{Max}}\left(-\frac{p}{q}\right)=\frac{2T}{\mu_r}\left(\frac{U(\mu_r-\mu(0))}{3T}\right)^{3/2}. \tag{30}$$

(b) Case II

$$\frac{U[\mu_r-\mu(0)]}{T}<\frac{U[\mu(-L)-\mu_l]}{S} \tag{31}$$

- Case II.a: If $k^2<[\mu_r-\mu(0)]U/T$, then $r<0$, $p>0$, and we have (27).
- Case II.b: If $[\mu_r-\mu(0)]U/T<k^2<[\mu(-L)-\mu_l]U/S$, then $r<0$, $p<0$, and we have $\sigma\leqslant \mathrm{Max}\left\{-(r/s),\underset{x}{\mathrm{Max}}\left(\frac{U\mu'}{\mu}\right)\right\}$.
- Case II.c: If $[\mu(-L)-\mu_l]U/S<k^2$, then $r>0$, $p<0$, and we have $\sigma\leqslant\underset{x}{\mathrm{Max}}\left(\frac{U\mu'}{\mu}\right)$.

$$\underset{k}{\mathrm{Max}}\left(-\frac{r}{s}\right)=\frac{2S}{\mu_l}\left(\frac{U(\mu(-L)-\mu_l)}{3S}\right)^{3/2}. \tag{32}$$

4. Case of slowly varying viscosity

For the viscosity profiles satisfying

$$\underset{x}{\mathrm{Max}}\left(\frac{U\mu'}{\mu}\right)\leqslant \mathrm{Max}\{p/(-q),r/(-s)\}, \tag{33}$$

or equivalently,

$$\underset{x}{\mathrm{Max}}\left(\frac{\mu'}{\mu}\right)\leqslant \mathrm{Max}\left\{\frac{2}{3\sqrt{3}\mu_r}\sqrt{\frac{U}{T}}(\mu_r-\mu(0))^{3/2},\frac{2}{3\sqrt{3}\mu_l}\sqrt{\frac{U}{S}}(\mu(-L)-\mu_l)^{3/2}\right\}, \tag{34}$$

we then obtain from the above consideration

$$\sigma\leqslant \mathrm{Max}\left\{-\frac{p}{q},-\frac{r}{s}\right\}. \tag{35}$$

Instead, to have an effective constant viscosity, we consider a *linear, very slowly increasing* viscosity $\mu_c(x)$ in the middle-layer.

$$\mu_c(x)=\epsilon\cdot x+\mu_c(0), \tag{36}$$

where $\epsilon=(\mu_c(0)-\mu_c(-L))/L$ and $1<\mu_c(-L)<\mu_c(0)<\alpha$. The condition (33) is satisfied in our case, if ϵ given in (36) satisfies the following condition.

$$U\frac{\mu_c(0)-\mu_c(-L)}{L\mu_c(0)}\leqslant \mathrm{Max}\left\{\frac{2S}{\mu_l}\left(\frac{U(\mu(-L)-\mu_l)}{3S}\right)^{3/2},\frac{2T}{\mu_r}\left(\frac{U(\mu_r-\mu(0))}{3T}\right)^{3/2}\right\}, \tag{37}$$

where we have used relations (31) and (33). This condition is satisfied if $(\mu_c(0) - \mu_c(-L))$ is small enough, that is if we have a very slowly increasing viscosity in the intermediate region.

Since, a constant viscosity profile is the limiting case of a slowly varying viscosity profile satisfying (36), the upper bound given by (35) should hold in the case of constant viscosity profile. Below, we show that this is indeed the case by analyzing Eq. (1) for the constant viscosity case.

5. Case of constant viscosity

In this case problem (1) reduces to

$$\left.\begin{array}{l} f_{xx} - k^2 f = 0, x \in (-L, 0), \\ f_x(0) = (\lambda p + q) f(0), \quad f_x(-L) = (\lambda r + s) f(-L), \end{array}\right\} \tag{38}$$

where p, q, r, s are the same as defined in (2) with $\mu(-L) = \mu(0) = \mu$, the constant viscosity of the middle-layer. We multiply the equation in (38) by f and then integrate over $(-L, 0)$ which, after using the boundary conditions defined in problem (38), yields

$$\lambda\{pf^2(0) - rf^2(-L)\} = \int_{-L}^{0} f_x^2 + k^2 \int_{-L}^{0} f^2 + sf^2(-L) - qf^2(0). \tag{39}$$

Therefore,

$$\sigma = \frac{1}{\lambda} = \frac{pf^2(0) - rf^2(-L)}{\int_{-L}^{0} f_x^2 + k^2 \int_{-L}^{0} f^2 + sf^2(-L) - qf^2(0)}, \tag{40}$$

which gives the inequality

$$\sigma \leqslant \frac{pf^2(0) - rf^2(-L)}{-qf^2(0) + sf^2(-L)}. \tag{41}$$

Let $a = pf^2(0) > 0$, $b = -rf^2(-L) > 0$, $c = -qf^2(0) > 0$, $d = sf^2(-L) > 0$. Then we have

$$\sigma \leqslant \frac{a+b}{c+d} \leqslant \frac{a}{c} = \frac{p}{-q} \quad \text{if } bc \leqslant ad \tag{42}$$

and

$$\sigma \leqslant \frac{a+b}{c+d} \leqslant \frac{b}{d} = \frac{-r}{s} \quad \text{if } ad \leqslant bc. \tag{43}$$

It is easy to see that above two cases are the only possibilities and hence we conclude from (42) and (43) that upper bound on the growth rate is the same one as given above by (35).

6. Summary

In this paper, we have derived an upper bound of the growth rate for the constant viscosity middle-layer in two ways: (i) applying the upper bound for the variable viscosity case to very slowly varying linear viscous profiles and assuming that the upper bound for the limiting case $\mu_x \to 0$ is the upper bound for $\mu_x = 0$ case; (ii) using analysis directly on the equations valid for $\mu_x = 0$ case. Both methods yield the same result.

Eq. (1) has eigenvalues λ (which is the inverse of growth rate) as well as the viscosity gradient μ_x in its right hand side. For the constant viscosity case, $\mu_x = 0$ and hence the righthand side of the equation becomes identically zero. This changes the structure of the equation since the equation does not have the eigenvalue in it anymore and eigenvalue appears only in the boundary conditions. So, the analysis of the Sections 2 and 3 which is based on application of Gerschgorin's theorem to the eigenvalue problem for the case $\mu_x \neq 0$ now breaks down for the constant viscosity case.

Since the analysis of Sections 2 and 3 is applicable to very slowly varying linear profiles, we applied upper bound result obtained in Section 3 to these profiles. It turns out (see Eqs. (36) and (37)) that for very slowly varying linear viscous profile, the upper bound does not depend on the viscosity gradient. This is an indication that the same upper bound should also hold for the constant viscosity case.

In Section 5, we derived an upper bound for the constant viscosity case directly from Eq. (38) appropriate for this case. Through a weak formulation of this equation, we obtained after some analysis the same upper bound as the one we obtained in Section 4 for very slowly varying linear profiles. It is perhaps interesting to note that we arrive at the same result by two different methods: Section 4 is essentially based on application of Gerschgorin's theorem to the eigenvalue problem for varying viscosity case where as the Section 5 is based on analysis of a weak formulation of the problem for the constant viscosity case.

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