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Nonlinear instability of Hele-Shaw flows with smooth viscous profiles

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Abstract

We rigorously derive nonlinear instability of Hele-Shaw flows moving with a constant velocity in the presence of smooth viscosity profiles where the viscosity upstream is lower than the viscosity downstream. This is a single-layer problem without any material interface. The instability of the basic flow is driven by a viscosity gradient as opposed to conventional interfacial Saffman–Taylor instability where the instability is driven by a viscosity jump across the interface. Existing analytical techniques are used in this paper to establish nonlinear instability.

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1. Introduction

Hele-Shaw flows refer to flows in a Hele-Shaw cell. Such flows of incompressible viscous fluid are governed by linear field equations, namely Darcy's law and incompressibility condition. In two-layer immiscible Hele-Shaw flows of such incompressible viscous fluids, the nonlinear dynamic boundary condition at the interface makes the problem of immiscible displacement nonlinear. Displacement of a low mobility (high viscosity) fluid such as oil by a low viscosity fluid such as water in a Hele-Shaw cell is considered to be a very good approximate model

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of similar displacement processes in porous media for the following reasons. In contrast with Hele-Shaw flows, porous media flows are additionally governed by one more equation, called Buckley–Leverett (B-L for short) equation (see Daripa et al. [2]), which is an evolution equation for saturation (the volume fraction of water in oil at the microscopic level). This is a nonlinear equation which couples the velocity with the saturation. Therefore, the system of field equations for porous media flows is nonlinear as opposed to linear system of field equations for Hele-Shaw flows. A consequence of this is that the front displacing the low mobility fluid in a porous media flow is a shock-wave (Daripa et al. [2]) and that in a Hele-Shaw flow is a contact discontinuity (material interface). The jump in mobility across the front in both types of flows has significant influence on its dynamics. It is useful in this subject to make use of this analogy and use linear field equations of Hele-Shaw flows instead of nonlinear equations of porous media flows to gain insight into some of the fundamental aspects of the dynamics of the front. For example, one of the classical problems in this context has been the interfacial instability in two-layer Hele-Shaw flows in which a constant viscosity fluid is displacing a more viscous fluid having uniform viscosity. This problem has been very well studied since early fifties (see [10]) from which it is well known that this displacement process is linearly unstable. This instability plays an important role in many applied fields including secondary oil recovery (Daripa et al. [2,3]).

There is another aspect of immiscible displacement processes in porous media which is physically important and can be modeled by Hele-Shaw flows. We discuss it here because it has relevance to the problem that we address in this paper. The nonlinear field equations of immiscible displacement of oil by water in porous media admit solutions that usually involve shock front followed by rarefaction waves in saturation. Because of these rarefaction waves, mobility behind the shock front gradually decreases to that of water. Modeling the effect of this graded mobility behind the front using Hele-Shaw flows requires the viscous fluid behind the interface to have a smooth viscous profile with viscosity gradually decreasing away from the interface. This changes the character of the field equations in Hele-Shaw flows from linear to nonlinear (see equations in (2.1)) even without taking into consideration the nonlinear boundary conditions at the front. The implication is that the system of equations governing even single-phase Hele-Shaw flows with graded viscosity is essentially nonlinear in character, as opposed to the uniform viscosity case when the system of equations is linear. Nonlinear stability of this nonlinear single-phase problem has not been addressed to-date which is necessary before similar problems in multi-phase flows with one or more fronts with graded mobility can be properly investigated.

In this paper, we establish linear and nonlinear instability of such single-phase Hele-Shaw flows with smooth viscosity profiles. These are single-layer problems with smooth viscous profiles where the viscosity upstream is lower than the viscosity downstream. The Hele-Shaw equations now form a system of nonlinear partial differential equations. The basic flow, $\mathbf{V}_0 = (V_0, 0)$, of this equation in the presence of viscosity gradient of the type discussed above is linearly unstable. However, it is not obvious that this flow is also nonlinearly unstable to perturbations governed by the full nonlinear equations. In this paper, we establish nonlinear instability of the basic flow in the presence of smooth viscosity profiles using a variational technique which has been recently introduced by Hwang and Guo (see [6]) in the context of nonlinear Rayleigh–Taylor instability.

There are no dissipative mechanisms in the Rayleigh–Taylor case where as the Hele-Shaw system that we consider here has built-in dissipation due to viscosity. The problem of passage from linear to nonlinear instability in a nonlinear PDE system such as ours is, in general, very subtle. Some methods, mostly problem driven, have been devised (for instance [1,6,7,9]) but yet no systematic framework exists for establishing nonlinear instability from linear instability. The proof of nonlinear instability in this paper consists of the following steps.

1. A variational characterization of the spectral problem for the unstable viscosity profile. The hyperbolic type of systems arising in mathematical physics may have continuum spectra (for example linearized Euler equation has a continuum spectrum) which makes it difficult to estimate the complicated spectra and the spectral radii [5]. In particular, only point or discrete spectrum estimate may not be sufficient for the estimate of the spectral radius due to the possible presence of the continuum spectrum. Therefore, a variational characterization of the spectral radius by eigenvalue estimates, not of eigenvalues only, has been used to locate a *dominant* eigenvalue from the spectrum. A similar approach has been used by Hwang and Guo [6] in the context of Rayleigh–Taylor instability.
2. Construction of higher-order approximate solutions by solving an equation for approximate evolution of the dominant growing eigenmode in powers of initial amplitude. This method is similar in spirit to the one introduced by Grenier [8]. This step also has subtlety since one may encounter severe higher-order perturbations, unbounded in L^2 norms for instance. Although it is natural in the formal sense, it is not obvious that we could really construct such approximate solutions if we did not have a dominant eigenvalue. However, a dominant eigenvalue obtained from the analysis in the crucial step 1 (see Theorem 1) allows control of higher-order perturbations in H^s norms for all $s \geqslant 3$.
3. Showing that the actual solution remains close to the exponentially growing approximate solution of step 2 up to a time that scales logarithmically with initial amplitude but for times smaller than possible blow-up time of actual solution via a delicate bootstrap argument which was introduced by Guo and Strauss [9].

2. Formulation and main results

We consider the following Hele-Shaw equations of fluid flows for time $t \geqslant 0$ in the periodic strip $D := \{-\infty < x < \infty, 0 \leqslant y \leqslant 2\pi\} = \mathbb{R} \times \mathbb{T}$:

$$\eta_t + \mathbf{V} \cdot \nabla\eta = 0, \qquad \nabla P = -\eta\,\mathbf{V}, \qquad \nabla\cdot\mathbf{V} = 0. \tag{2.1}$$

Here $\eta(t,x,y)$, $\mathbf{V}(t,x,y)$, $P(t,x,y)$ are respectively the varying viscosity, the velocity, and the pressure. The first equation is the advection equation for viscosity (see [3]), the second equation is the Darcy's law, and the third equation is the continuity equation for incompressible flow. We impose the periodic condition on the boundary. A steady state is given by

$$\mathbf{V}_0 = (V_0, 0), \qquad \eta_0(t,x) = \eta_0(x - V_0 t), \qquad \nabla P_0(t,x) = -\eta_0\mathbf{V}_0, \tag{2.2}$$

where $\eta_0(t,\cdot) \in H^s(\mathbb{R})$, $s \geqslant 3$, is a smooth viscosity profile satisfying

$$0 < c \leqslant \eta_0 \leqslant C < \infty, \qquad \lim_{|x|\to\infty} \eta_{0x} = 0, \tag{2.3}$$

where c and C are constants. Note that the above condition (2.3) includes profiles which assume constant values outside a fixed interval. Such profiles are special cases of viscous profiles in three-layer Hele-Shaw flows often used for control of instabilities (see [3]). The criterion for instability is that there exists $-\infty < x_0 < \infty$ such that

$$V_0\eta_{0x}(x_0) > 0. \tag{2.4}$$

It means that fluid upstream is less viscous than fluid downstream which is physically relevant. We now consider a perturbation $(\mu, \mathbf{v}, p)$ around such a steady state $(\eta_0, \mathbf{V}_0, P_0)$.

In a moving frame $(\mathbf{x}' = \mathbf{x} - \mathbf{V}_0 t, t' = t)$, equations for perturbed quantities take the following form where, with slight abuse of notations, we have used the same variable $\mathbf{x}$ for $\mathbf{x}'$ and t for t':

$$\mu_t + \mathbf{v}\cdot\nabla(\eta_0 + \mu) = 0, \qquad (\eta_0 + \mu)\mathbf{v} = -\nabla p - \mu\mathbf{V}_0, \qquad \nabla\cdot\mathbf{v} = 0. \tag{2.5}$$

From (2.5), we obtain with $\mathbf{v} = (v_1, v_2)$ the linearized system

$$\mu_t + \eta_{0x}v_1 = 0, \qquad \eta_0\mathbf{v} = -\nabla p - \mu\mathbf{V}_0, \qquad \nabla\cdot\mathbf{v} = 0. \tag{2.6}$$

Note that $\eta_{0t} = 0$ in the moving frame. A key step in the passage from linear instability to nonlinear instability lies in a variational characterization of the spectral radius of the whole linear operator by a discrete set of eigenvalues. This involves two steps.

1. We make the following variational formulation in Section 3 for the eigenvalues λ_k for any fixed $k \in \mathbb{N}$:

$$\lambda_k = \sup_{u\in H^1(\mathbb{R})} \frac{\int V_0\eta_{0x}u^2\,dx}{\int \eta_0[u^2 + \frac{u_x^2}{k^2}]\,dx} > 0. \tag{2.7}$$

   Notice that (2.4) ensures positivity of λ_k in (2.7), i.e. linear instability and that from the regularity (2.3) for η_0, all integrals in (2.7) (i.e. $\int \eta_{0x}u^2\,dx$, $\int \eta_0 u^2\,dx$, $\int \eta_0 u_x^2\,dx$) are finite.
2. We then derive in Section 4 (see Theorem 1) the exact spectral radius Λ of the linearized Hele-Shaw system as the limit of λ_k:

$$\Lambda = \sup_{v\in L^2(D)} \frac{\iint V_0\eta_{0x}|\mathbf{v}|^2\,dx\,dy}{\iint \eta_0|\mathbf{v}|^2\,dx\,dy}, \qquad \lim_{k\to\infty}\lambda_k = \Lambda. \tag{2.8}$$

This enables one to locate from the spectrum so-called a dominant eigenvalue which plays a very important role in the construction of higher-order approximate solutions in Section 5. In Section 6, we present an energy estimate for the difference between the approximate solution and the exact solution to the full system followed by an application of a delicate bootstrap argument to the estimate. In Section 7, everything is brought together to establish nonlinear instability. The estimate on the spectral radius which is sharper and simpler than the Rayleigh–Taylor case (see [6]) is obtained using a variational approach commonly used for such purposes.

Before we state our main theorems, we introduce an equivalent linear system which will be used in the variational formulation. Taking t-derivative of $(2.6)_2$ and plugging in $(2.6)_1$ into the resulting equation of $\mathbf{v}$ leads to the first-order linearized equation in $\mathbf{v}$, namely,

$$\eta_0\mathbf{v}_t = -\nabla p_t + \eta_{0x}v_1\mathbf{V}_0 =: L(\mathbf{v}), \qquad \nabla\cdot\mathbf{v} = 0, \tag{2.9}$$

which is equivalent to the linear system (2.6) subject to the following qualifications. It is easy to see that a solution of system (2.6) also solves (2.9). To address the reverse, first note that system (2.6) is four equations for four unknowns: (μ, v_1, v_2, p). System (2.9) is three equations for three unknowns (v_1, v_2, p). Assume that one has a solution to system (2.9). Then one can define μ which will be a solution of Eq. $(2.6)_1$:

$$\mu(x,t) = C - \int_{t_0}^{t} \eta_{0x} v_1(x,s)\,ds.$$

Integrating Eq. $(2.9)_1$ in time, one then finds

$$\eta_0(x)\mathbf{v}(x,t) + \nabla p(x,t) + \mu(x,t)\mathbf{V}_0 = \eta_0(x)\mathbf{v}(x,t_0) + \nabla p(x,t_0) + C\mathbf{V}_0.$$

One needs to find an integration constant C and a time t_0 such that the right-hand side is zero. Assuming that one can do this, (μ, v_1, v_2, p) is a solution of (2.6) and the two systems (2.6) and (2.9) are equivalent.

For purposes below, we introduce the following notations.

Notation 1. $\|u\| = (\iint \eta_0 u^2\,dx\,dy)^{1/2}$ and $\|\mathbf{v}\| = (\|v_1\|^2 + \|v_2\|^2)^{1/2}$ for $\mathbf{v} = (v_1, v_2)$. Notice that this is an equivalent norm to the usual L^2-norm due to (2.3).

Notation 2. $\|u\|_s = \sum_{|\alpha|\leqslant s} \|\partial_\alpha u\|$ and $\|\mathbf{v}\|_s = (\|v_1\|_s^2 + \|v_2\|_s^2)^{1/2}$ for $\mathbf{v} = (v_1, v_2)$.

We obtain the following theorem on the sharp growth rate Λ for the linearized system (2.6).

Theorem 1. *Let* $(\mu(t,x,y), v_1(t,x,y), v_2(t,x,y)) \in [C([0,T]; H^s(D))]^3$ *be a solution to the linearized system* (2.6) *and let* η_0 *be a steady state satisfying* (2.3) *and* (2.4). *Then, for any fixed integer* $s \geqslant 0$*, there exists a constant* $C = C(s, \Lambda, \|\eta_0\|_{s+1})$ *such that*

$$\big\|\mu(t,\cdot), \mathbf{v}(t,\cdot)\big\|_s \leqslant Ce^{\Lambda t}\big\|\mu(0,\cdot), \mathbf{v}(0,\cdot)\big\|_s, \tag{2.10}$$

where Λ *is defined in* (2.8).

In general, a spectral radius may not be same as a growth bound of the linear operator of interest. Theorem 1 shows that it is the case for the linear operator (2.6) which is established using a variational method. This theorem is central to locating a dominant eigenvalue of the spectrum of the linearized system. It enables us to establish the following instability result in a fully nonlinear setting.

Theorem 2. *The steady state* $(\eta_0(t,x), \mathbf{V}_0)$ *of* (2.1) *defined in* (2.2) *is nonlinearly unstable. For any integer* $s \geqslant 0$ *large, there exists* $\varepsilon_0 > 0$ *such that for any small* $\delta > 0$ *there exists a family of classical solutions* $(\eta^\delta(t,x,y), \mathbf{V}^\delta(t,x,y))$ *of* (2.1) *satisfying*

$$\big\|\eta^\delta(0,\cdot) - \eta_0(0,\cdot)\big\|_{H^s(\times)} + \big\|\mathbf{V}^\delta(0,\cdot) - \mathbf{V}_0\big\|_{H^s(\times)} \leqslant \delta,$$

but for $T^\delta = O(|\ln\delta|)$,

$$\sup_{0\leqslant t\leqslant T^\delta}\left\{\left\|\eta^\delta(t,\cdot)-\eta_0(t,\cdot)\right\|_{L^2(\times)}+\left\|\mathbf{V}^\delta(t,\cdot)-\mathbf{V}_0\right\|_{L^2(\times)}\right\}\geqslant\varepsilon_0.$$

Remark 1. The instability time T^δ occurs before the possible blow-up time which is shown in the proof.

3. Linear growing modes with λ_k

The existence of smooth linear growing normal modes to the linear system (2.6) or equivalently (2.9) is established in this section using the method of normal modes. Hence we use the following ansatz for the temporal evolution of the wave components of arbitrary perturbations $\mu(t,x,y)$, $v_1(t,x,y)$, $v_2(t,x,y)$ and $p(t,x,y)$:

$$\left.\begin{aligned}\mu(t,x,y)&=\tilde{\mu}(x)\cos(ky)\exp(\lambda_k t),\\ v_1(t,x,y)&=\tilde{v}_1(x)\cos(ky)\exp(\lambda_k t),\\ v_2(t,x,y)&=\tilde{v}_2(x)\sin(ky)\exp(\lambda_k t),\\ p(t,x,y)&=\tilde{p}(x)\cos(ky)\exp(\lambda_k t),\end{aligned}\right\}\tag{3.1}$$

where $(\tilde{\mu}(x),\tilde{v}_1(x),\tilde{v}_2(x),\tilde{p}(x))\in C^\infty(\mathbb{R})$ and $\tilde{\mu}(x),\tilde{v}_1(x),\tilde{v}_2(x),\tilde{p}(x)\to 0$ as $|x|\to\infty$. Plugging (3.1) into (2.9) yields

$$\begin{aligned}\lambda_k\eta_0\tilde{v}_1&=-\lambda_k\tilde{p}_x+\eta_{0x}\tilde{v}_1V_0,\\ \lambda_k\eta_0\tilde{v}_2&=k\lambda_k p,\\ \tilde{v}_{1x}+k\tilde{v}_2&=0.\end{aligned}$$

By eliminating $\tilde{p}$, $\tilde{p}_x$ and $\tilde{v}_2$ in the above equations, we obtain the following second-order ordinary differential equation with smooth coefficients

$$\eta_0\tilde{v}_1-\frac{1}{k^2}(\eta_0\tilde{v}_{1x})_x=\frac{V_0\eta_{0x}}{\lambda_k}\tilde{v}_1,\tag{3.2}$$

where $\tilde{v}_1(x)\to 0$ as $|x|\to\infty$. Next we show existence of a smooth growing normal mode using a variational method which implies linear instability. Note that existence of a smooth growing mode is equivalent to existence of $(\tilde{\mu}(x),\tilde{\mathbf{v}}(x),\tilde{p}(x))\in[C^\infty(\mathbb{R})]^4$ as above. For a notational simplicity, we use $(\mu(x),\mathbf{v}(x),p(x))$ for $(\tilde{\mu}(x),\tilde{\mathbf{v}}(x),\tilde{p}(x))$ without tilde.

Lemma 1. *For any fixed wave number $k\in\mathbb{N}$, there exists a smooth linear growing mode; that is, there exists $(\mu(x),\mathbf{v}(x),p(x))\in[C^\infty(\mathbb{R})]^4$ in (3.1) with $\mu(x),\mathbf{v}(x),p(x)\to 0$ as $|x|\to\infty$ and with the growth rate λ_k in (2.7).*

Proof. We solve the following variational problem:

$$\inf_{u\in H^1(\mathbb{R})}\int_{-\infty}^{\infty}\eta_0\left[u^2+\frac{u_x^2}{k^2}\right]dx\tag{3.3}$$

with the constraint

$$\int_{-\infty}^{\infty} V_0 \eta_{0x} u^2 \, dx = 1. \tag{3.4}$$

Since u up to a multiplicative constant in the definition of λ_k (see (2.7)) does not change λ_k (note u^2 appears in the numerator as well as in the denominator in (2.7)), (3.4) is not a new assumption. Therefore, without any loss of generality (3.4) holds due to the definition (2.7) of λ_k.

Notice that (2.4) ensures existence of a $u \in H^1(\mathbb{R})$ satisfying (3.4). By the definition of λ_k in (2.7), there exists a minimizing sequence u_n such that

$$\int_{-\infty}^{\infty} \eta_0 \left[u_n^2 + \frac{(u_n)_x^2}{k^2} \right] dx \to \frac{1}{\lambda_k}$$

satisfying

$$\int_{-\infty}^{\infty} V_0 \eta_{0x} u_n^2 \, dx = 1. \tag{3.5}$$

Since $\{u_n\}$ is bounded in $H^1(\mathbb{R})$, uniformly in n, there exists $u_0 \in H^1(\mathbb{R})$ such that $u_n \rightharpoonup u_0$ weakly in $H^1(\mathbb{R})$ up to a subsequence. Our viscosity profile η_0 in (2.3) satisfies $\eta_{0x}(x) \to 0$ as $x \to \pm\infty$. We then deduce that u_0 fulfills the constraint (3.5) as follows. If η_{0x} has a compact support in $\mathbb{R}$, then it follows from Rellich's theorem. In the general case when $\eta_{0x} \to 0$ at infinity as in (2.6), we choose a family of smooth compactly supported functions $\{\eta_K\}_{K\to\infty}$, which approximates η_{0x} such that supp $\eta_K \subset (-K, K)$ for $K > 0$ and $\|\eta_K - \eta_{0x}\|_{L^\infty} \to 0$ as $K \to \infty$. Then we have

$$\begin{aligned}
&\left| \int \eta_{0x} u_n^2 \, dx - \int \eta_{0x} u_0^2 \, dx \right| \\
&\quad \leqslant \left| \int \left[\eta_{0x} u_n^2 - \eta_K u_n^2 \right] dx \right| + \left| \int \left[\eta_K u_n^2 - \eta_K u_0^2 \right] dx \right| + \left| \int \left[\eta_K u_0^2 - \eta_{0x} u_0^2 \right] dx \right| \\
&\quad \leqslant \|\eta_K - \eta_{0x}\|_{L^\infty} \int u_n^2 \, dx + \left| \int \left[\eta_K u_n^2 - \eta_K u_0^2 \right] dx \right| + \|\eta_K - \eta_{0x}\|_{L^\infty} \int u_0^2 \, dx \\
&\quad \to 0 \quad \text{as } K \to \infty.
\end{aligned}$$

Thus u_0 satisfies (3.5). By lower semi-continuity of L^p-norm, we have

$$\int_{-\infty}^{\infty} \eta_0 \left[u_0^2 + \frac{(u_0)_x^2}{k^2} \right] dx \leqslant \liminf_n \int_{-\infty}^{\infty} \eta_0 \left[u_n^2 + \frac{(u_n)_x^2}{k^2} \right] dx = \frac{1}{\lambda_k}.$$

Hence u_0 is a minimizer of the variational problem (3.3).

Next we show that u_0 satisfies Euler–Lagrange equation (3.2) associated with the above variational problem (3.3). Any perturbed function $W(\tau) = u_0 + \tau u$ around a minimizer u_0, for any $\tau \in \mathbb{R}$ and any $u \in H^1(\mathbb{R})$, obeys

$$0 \geqslant \int_{-\infty}^{\infty} \left[\eta_0 \left(\frac{W_x^2(\tau)}{k^2} + W^2(\tau) \right) - \frac{1}{\lambda_k} V_0 \eta_{0x} W^2(\tau) \right] dx =: J(\tau).$$

Thus we have, for any $u \in H^1(\mathbb{R})$,

$$0 = J'(0) = 2 \int_{-\infty}^{\infty} \left[\frac{\eta_0}{k^2} u_{0x} u_x + \eta_0 u_0 u - \frac{1}{\lambda_k} V_0 \eta_{0x} u_0 u \right] dx.$$

By integration by parts, we easily deduce (3.2). Finally, smoothness of a minimizer u_0 follows from the second-order uniformly elliptic equation (3.2) since η_0 is smooth and $\min \eta_0 > 0$. To see this, first elliptic regularity theory gives $v_1 \in H^3(\mathbb{R})$ from $v_1 \in H^1(\mathbb{R})$ and so by repeating this argument, we obtain $v_1 \in H^s$ for all $s \geqslant 1$. Hence $v_1 \in C^\infty(\mathbb{R})$ by Sobolev inequality. Then the regularity of v_2, p, μ follows. Since $v_1, v_2, p, \mu \in H^s$ for all $s \geqslant 1$, they vanish at infinity as desired. Therefore, the proof is complete. □

Next we show that $\{\lambda_k\}$ has a limit Λ as $k \to \infty$ which will be shown to be the optimal growth bound of solutions to the linear operator in the following section.

Lemma 2. *Let λ_k and Λ be as in* (2.7) *and* (2.8)*, respectively. Then λ_k is increasing with k and it satisfies*

$$\lim_{k \to \infty} \lambda_k = \Lambda.$$

Proof. Let $u \in H^1(D)$. Then we have

$$\int_0^{2\pi} \left(\int_{-\infty}^{\infty} \left[u^2(x,y) + u_x^2(x,y) + u_y^2(x,y) \right] dx \right) dy < \infty.$$

It means that

$$\int_{-\infty}^{\infty} \left[u^2(x,y) + u_x^2(x,y) \right] dx < \infty, \quad \text{almost everywhere } y \in [0, 2\pi].$$

In other words, we have $u(\cdot, y) \in H^1(\mathbb{R})$ almost everywhere $y \in [0, 2\pi]$ and thus using the definition λ_k in (2.7) yields

$$\int_{-\infty}^{\infty} V_0 \eta_{0x} u^2(x,y)\, dx \leqslant \lambda_k \int_{-\infty}^{\infty} \eta_0 \left[u^2(x,y) + \frac{u_x^2(x,y)}{k^2} \right] dx, \quad \textit{almost everywhere } y \in [0, 2\pi].$$

By integrating the inequality above in y, we obtain

$$\iint_D V_0\eta_{0x}u^2(x,y)\,dx\,dy \leqslant \lambda_k \iint_D \eta_0\left[u^2(x,y)+\frac{u_x^2(x,y)}{k^2}\right]dx\,dy.$$

We first note that $\lambda_k \leqslant \Lambda$. To see this precisely, pick a $u(x) \in H^1(\mathbb{R})$ and extend it as a constant in the y-direction. Then with the extension $u(x,y) \in H^1(D)$, we have

$$\frac{\int_{-\infty}^{\infty} V_0\eta_{0x}u^2\,dx}{\int_{-\infty}^{\infty}\eta_0[u^2+\frac{u_x^2}{k^2}]\,dx} = \frac{\iint_D V_0\eta_{0x}u^2\,dx\,dy}{\iint_D \eta_0[u^2+\frac{u_x^2}{k^2}]\,dx\,dy}.$$

Then

$$\begin{aligned}\lambda_k &= \sup_{u\in H^1(\mathbb{R})}\frac{\int_{-\infty}^{\infty} V_0\eta_{0x}u^2\,dx}{\int_{-\infty}^{\infty}\eta_0[u^2+\frac{u_x^2}{k^2}]\,dx} = \sup_{u\in H^1(\mathbb{R})}\frac{\iint_D V_0\eta_{0x}u^2\,dx\,dy}{\iint_D \eta_0[u^2+\frac{u_x^2}{k^2}]\,dx\,dy}\\ &\leqslant \sup_{w\in L^2(D)}\frac{\iint_D V_0\eta_{0x}w^2\,dx\,dy}{\iint_D \eta_0 w^2\,dx\,dy} = \Lambda.\end{aligned}$$

Next we show that Λ is indeed the limit of λ_k. For any $\varepsilon > 0$ such that $\varepsilon < \lambda_1\ (> 0)$, we fix $w \in L^2(D)$ such that

$$\Lambda - \varepsilon < \frac{\iint_D V_0\eta_{0x}w^2\,dx\,dy}{\iint_D \eta_0 w^2\,dx\,dy}.$$

Then we choose k large enough such that

$$\frac{\iint_D V_0\eta_{0x}w^2\,dx\,dy}{\iint_D \eta_0 w^2\,dx\,dy} < \frac{\iint_D V_0\eta_{0x}w^2\,dx\,dy}{\iint_D \eta_0[\frac{w_x^2}{k^2}+w^2]\,dx\,dy} + \varepsilon \leqslant \lambda_k + \varepsilon.$$

Therefore $\Lambda \leqslant \lambda_k + 2\varepsilon$. This completes the proof. □

4. Λ as the spectral radius of the linear operator

In this section, for the linearized system we derive a sharp growth rate $\Lambda > 0$ which is obtained as the limit of $\lambda_k > 0$ in (2.7). This means that this number Λ serves as the spectral radius for the whole linear spectrum which is crucial in constructing approximate solutions in the next section. Before presenting the analysis of Λ, we address the issue of global existence of solutions to the linearized system (2.6).

Lemma 3. *Let $T > 0$ and $s \geqslant 0$ be an integer. Then, for any given initial data $(\mu_0(x,y), \mathbf{v}_0(x,y), p(x,y))$, there exists a unique solution $(\mu(t,x,y), \mathbf{v}(t,x,y), p(x,y)) \in [C([0,T]; H^s(D))]^4$ to the linearized system* (2.6).

Proof. We give a brief sketch. From (2.6), we get

$$\|\mathbf{v}\|^2 = \iint -V_0 \mu v_1 \leqslant \frac{1}{2}\|\mathbf{v}\|^2 + C\|\mu\|^2,$$

which implies

$$\|\mathbf{v}\|^2 \leqslant C\|\mu\|^2,$$

where C is a generic constant which depends only on V_0 and η_0 and varies from line to line.

Multiplying Eq. $(2.6)_1$ by μ and integrating in x and y, we get

$$\frac{1}{2}\frac{d}{dt}\|\mu\|^2 \leqslant C_1\|\mu\|^2 + C_2\|\mathbf{v}\|^2 \leqslant C\|\mu\|^2,$$

where C_1, C_2, C are constants which depend on V_0, η_0, and η_{0x}. Using Gronwall inequality, we obtain a priori estimates for global bound in $L^2(D)$ and similarly in $H^s(D)$ for $s > 1$. Applying a standard contraction mapping theorem to the linearized system (2.6) yields local existence as follows:

$$\mu(t) = \mu_0 - \int_0^t \eta_{0x} v_1 \, d\tau,$$

$$\left\|\left(\mu^1 - \mu^2\right)(t)\right\| \leqslant t\|\eta_{0x}\|_{L^\infty}\left\|v_1^1 - v_1^2\right\|, \qquad \left\|\mathbf{v}^1 - \mathbf{v}^2\right\| \leqslant |V_0|\left\|\mu^1 - \mu^2\right\|,$$

which implies

$$\left\|\left(\mu^1 - \mu^2\right)(t)\right\| \leqslant t|V_0|\|\eta_{0x}\|_{L^\infty}\left\|\mu^1 - \mu^2\right\|.$$

Then for short time, we can apply contraction mapping theorem to obtain the existence. Once we get μ, we obtain p and $\mathbf{v}$ from (2.6). Combined with the above estimates, we obtain the lemma. □

For purposes below, we let $\mathbf{w}(t,x,y) = (\mu(t,x,y), v_1(t,x,y), v_2(t,x,y))$. Now we prove Theorem 1 which has been stated in Section 2.

Proof of Theorem 1. The proof below is by induction on $\bar{s}$ (= the number of x-derivatives). When $\bar{s} = 0$, we multiply (2.9) by $\mathbf{v}$ and integrate over x and y to obtain

$$\frac{1}{2}\frac{d}{dt}\iint_D \eta_0|\mathbf{v}|^2 \, dx \, dy = \iint_D V_0 \eta_{0x} v_1^2 \, dx \, dy \leqslant \Lambda \iint_D \eta_0 |\mathbf{v}|^2 \, dx \, dy,$$

where we have used incompressibility condition for $\mathbf{v}$ and the definition of Λ in (2.8). Thus applying Gronwall inequality yields the growth rate Λ for v:

$$\left\|\mathbf{v}(t,\cdot)\right\| \leqslant e^{\Lambda t}\left\|\mathbf{v}(0,\cdot)\right\|.$$

From (2.6), we have

$$\left\|\mu(t,\cdot)\right\| \leqslant Ce^{\Lambda t}\left\|\mathbf{w}(0,\cdot)\right\|,$$

where $C = C(\Lambda, \|\eta_0\|_{C^1})$. In order to treat $\mathbf{v}_y$, we use the observation that this variational structure is preserved by taking y-derivative due to the independence of η_0 on the y variable. To see this precisely, we take y-derivative of (2.9). Then we have

$$\eta_0 \partial_y \mathbf{v}_t = -\nabla \partial_y p_t + \eta_{0x} \partial_y v_1 \mathbf{V}_0.$$

We then multiply the above equation by $\partial_y \mathbf{v}$ and integrate to obtain

$$\left\|\partial_y \mathbf{v}(t,\cdot)\right\| \leqslant e^{\Lambda t}\left\|\partial_y \mathbf{v}(0,\cdot)\right\|, \qquad \left\|\partial_y \mu(t,\cdot)\right\| \leqslant Ce^{\Lambda t}\left\|\partial_y \mathbf{v}(0,\cdot)\right\|.$$

Notice that this works for any order of y-derivatives of $\mathbf{v}$ and μ. For $\bar{s} = 1$, we take the curl of ∂_y^m of $(2.6)_2$ to have

$$\eta_0 \operatorname{scalar\,curl} \partial_y^m \mathbf{v} = -\eta_{0x} \partial_y^m v_2 + V_0 \partial_y^m \partial_y \mu,$$

where ∂_y^m is a pure y-derivative of order m and the scalar curl refers to the only non-zero component of the curl (e.g., a vector $\mathbf{w} = (w_1, w_2) \in \mathbb{R}^2$ is a vector $(w_1, w_2, 0) \in \mathbb{R}^3$. Then $\operatorname{curl} \mathbf{w}$ has only the last component non-zero in $\mathbb{R}^3$, by which the scalar curl $\mathbf{w}$ is defined above). Clearly the right-hand side has the growth rate Λ by the induction hypothesis, i.e.

$$\left\|\operatorname{scalar\,curl} \partial_y^m \mathbf{v}\right\| \leqslant C\big(m, \Lambda, \|\eta_0\|_{C^1}\big)e^{\Lambda t}\left\|\mathbf{w}(0,\cdot)\right\|_{m+1}.$$

Using the divergence-free condition for $\mathbf{v}$ and $(2.6)_1$, we obtain

$$\left\|\partial_y^m \mathbf{w}\right\| \leqslant C\big(m, \Lambda, \|\eta_0\|_{C^1}\big)e^{\Lambda t}\left\|\mathbf{w}(0,\cdot)\right\|_{m+1}.$$

Thus we obtain the growth rate Λ for $\bar{s} = 1$.

Now we suppose that any derivative whose order of x is less than $\bar{s}$ ($\geqslant 2$) has growth rate Λ. Let ∂_α be a derivative of a multi-index α whose order of x is $\bar{s} - 1$, then we take curl of ∂_α of $(2.6)_2$ to get

$$\eta_0 \operatorname{scalar\,curl} \partial_\alpha \mathbf{v} = -\operatorname{scalar\,curl} \partial_\alpha(\eta_0 \mathbf{v}) + \eta_0 \operatorname{scalar\,curl} \partial_\alpha \mathbf{v} + V_0 \partial_\alpha \partial_y \mu.$$

The terms on the right-hand side have derivatives whose order of x is still less than $\bar{s}$. Therefore, we apply our induction hypotheses and use divergence-free condition for $\mathbf{v}$ to obtain

$$\|\partial_x \partial_\alpha \mathbf{v}\| \leqslant C\big(\bar{s}, \Lambda, \|\eta_0\|_{C^{\bar{s}}}\big)e^{\Lambda t}\left\|\mathbf{w}(0,\cdot)\right\|_{|\alpha|+1}.$$

Using the equation for μ yields

$$\|\partial_x \partial_\alpha \mu\| \leqslant C\big(\bar{s}, \Lambda, \|\eta_0\|_{C^{\bar{s}+1}}\big)e^{\Lambda t}\left\|\mathbf{w}(0,\cdot)\right\|_{|\alpha|+1}.$$

Therefore we conclude (2.10) and complete the proof. $\square$

5. Approximate solutions

In this section we construct approximate solutions based on a dominant eigenvalue and a method introduced by Grenier [8]. For our purposes here, we fix a dominant eigenvalue $\lambda = \lambda_{k_0}$ corresponding to the growing normal mode obtained in Section 3, satisfying $\Lambda < 2\lambda$ which is possible due to Lemma 2. With δ an arbitrary small parameter and θ a small but fixed positive constant, independent of δ, we show below instability at the following scaled time T^δ defined by

$$\theta = \delta \exp\bigl(\lambda T^\delta\bigr), \tag{5.1}$$

or equivalently by $T^\delta = \lambda^{-1} \ln(\theta/\delta)$. We are now ready to construct a higher-order approximate solution via the dominant eigenvalue λ.

Lemma 4. *For any fixed $N > 0$ and a smooth viscosity profile satisfying* (2.3) *and* (2.4)*, there exists an approximate solution to the full system* (2.5) *of the form*

$$\left.\begin{aligned} \mu^a(t,x,y) &= \sum_{j=1}^{N} \delta^j \chi_j(t,x,y), \\ \mathbf{v}^a(t,x,y) &= \sum_{j=1}^{N} \delta^j \boldsymbol{\Psi}_j(t,x,y), \\ p^a(t,x,y) &= \sum_{j=1}^{N} \delta^j q_j(t,x,y), \end{aligned}\right\} \tag{5.2}$$

satisfying

$$\left.\begin{aligned} \mu^a_t + \nabla\bigl(\mu^a + \eta_0\bigr)\cdot \mathbf{v}^a &= R^a_N, \\ \bigl(\eta_0 + \mu^a\bigr)\mathbf{v}^a + \nabla p^a + \mu^a \mathbf{V}_0 &= \mathbf{S}^a_N, \\ \nabla\cdot\boldsymbol{\Psi}_j &= 0 \quad (1 \leqslant j \leqslant N). \end{aligned}\right\} \tag{5.3}$$

Furthermore, for every integer $s \geqslant 0$ there is a $\theta > 0$ sufficiently small such that if $0 \leqslant t \leqslant T^\delta$ as in (5.1)*, then $\chi_j(t,x,y)$, $\boldsymbol{\Psi}_j(t,x,y)$, $q_j(t,x,y)$, $R^a_N(t,x,y)$, $\mathbf{S}^a_N(t,x,y)$ satisfy*

$$\bigl\|\chi_j(t)\bigr\|_{H^s} \leqslant C_{s,N} \exp(j\lambda t), \quad 1 \leqslant j \leqslant N, \tag{5.4}$$

$$\bigl\|\boldsymbol{\Psi}_j(t)\bigr\|_{H^s} \leqslant C_{s,N} \exp(j\lambda t), \quad 1 \leqslant j \leqslant N, \tag{5.5}$$

$$\bigl\|\nabla q_j(t)\bigr\|_{H^s} \leqslant C_{s,N} \exp(j\lambda t), \quad 1 \leqslant j \leqslant N, \tag{5.6}$$

$$\bigl\|R^a_N(t)\bigr\|_{H^s} \leqslant C_{s,N}\, \delta^{N+1} \exp\bigl\{(N+1)\lambda t\bigr\}, \tag{5.7}$$

$$\bigl\|\mathbf{S}^a_N(t)\bigr\|_{H^s} \leqslant C_{s,N} \delta^{N+1} \exp\bigl\{(N+1)\lambda t\bigr\}. \tag{5.8}$$

Proof. We proceed by induction on j to construct χ_j, $\boldsymbol{\Psi}_j$, q_j, R^a_j, and $\mathbf{S}^a_N$. For $j = 1$, we take the smooth growing wave solution as constructed in Section 3, corresponding to our dominant eigenvalue $\lambda = \lambda_{k_0}$ of the form

$$\chi_1(t,x,y) = \tilde{\mu}(x)\cos(k_0 y)\exp(\lambda t),$$
$$\boldsymbol{\Psi}_1(t,x,y) = \big(\tilde{v}_1(x)\cos(k_0 y)\exp(\lambda t), \tilde{v}_2(x)\sin(k_0 y)\exp(\lambda t)\big),$$
$$q_1(t,x,y) = \tilde{p}(x)\cos(k_0 y)\exp(\lambda t),$$

where k_0 is the wave number associated with λ. Then it is easy to see that

$$R_1^a = \delta^2 \boldsymbol{\Psi}_1 \cdot \nabla\chi_1, \qquad \mathbf{S}_1^a = \delta^2 \chi_1 \boldsymbol{\Psi}_1.$$

Thus $\chi_1, \boldsymbol{\Psi}_1, q_1, R_1^a, \mathbf{S}_1^a$ fulfill (5.4)–(5.8) since $\chi_1, \boldsymbol{\Psi}_1, q_1 \in C^\infty$ and so the lemma holds true for $j = 1$. Suppose now that $\chi_k, \boldsymbol{\Psi}_k, q_k, R_k^a, \mathbf{S}_k^a$ have been constructed satisfying (5.4)–(5.8) for $1 \leqslant k \leqslant j < N$. Then we will construct χ_{j+1}, $\boldsymbol{\Psi}_{j+1}$, q_{j+1}, R_{j+1}^a, $\mathbf{S}_{j+1}^a$ in the following. Let

$$\mu_j = \sum_{k=1}^{j} \delta^k \chi_k, \qquad \mathbf{u}_j = \sum_{k=1}^{j} \delta^k \boldsymbol{\Psi}_k, \qquad p_j = \sum_{k=1}^{j} \delta^k q_k. \tag{5.9}$$

We then define $F_{j+1}(\delta)$ and $\mathbf{G}_{j+1}(\delta)$ as the nonlinear part of the system at $(\mu_j, \mathbf{u}_j)$

$$F_{j+1}(\delta) := \mathbf{u}_j \cdot \nabla\mu_j, \qquad \mathbf{G}_{j+1}(\delta) := \mu_j \mathbf{u}_j. \tag{5.10}$$

Plugging (5.9) into (2.5) and matching the $(j+1)$th coefficients of δ defines $(j+1)$th-order coefficients χ_{j+1}, $\boldsymbol{\Psi}_{j+1}$, and q_{j+1} as solutions of the following inhomogeneous linear system

$$\partial_t \chi_{j+1} + \boldsymbol{\Psi}_{j+1} \cdot \nabla\eta_0 = -\frac{F_{j+1}^{(j+1)}(0)}{(j+1)!},$$
$$\eta_0 \boldsymbol{\Psi}_{j+1} + \nabla q_{j+1} + \chi_{j+1}\mathbf{V}_0 = -\frac{\mathbf{G}_{j+1}^{(j+1)}(0)}{(j+1)!},$$
$$\nabla\cdot\boldsymbol{\Psi}_{j+1} = 0,$$

with initial data $\chi_{j+1}(0,x,y) = 0$, $\boldsymbol{\Psi}_{j+1}(0,x,y) = (0,0)$. For $0 \leqslant t \leqslant T^\delta$ and θ small, we have

$$\frac{F_{j+1}^{(j+1)}(0)}{(j+1)!} = \sum_{j_1+j_2=j+1} A_{j_1 j_2} \boldsymbol{\Psi}_{j_1} \cdot \nabla\chi_{j_2}, \qquad \frac{\mathbf{G}_{j+1}^{(j+1)}(0)}{(j+1)!} = \sum_{j_1+j_2=j+1} B_{j_1 j_2} \chi_{j_1} \boldsymbol{\Psi}_{j_2},$$

where $1 \leqslant j_k \leqslant j$ and $A_{j_1 j_2}, B_{j_1 j_2}$ depend on η_0 and $\mathbf{V}_0$. Applying induction hypotheses (5.4)–(5.6) yields, for every $s \geqslant 0$,

$$\left\| \frac{F_{j+1}^{(j+1)}(0)}{(j+1)!} \right\|_s \leqslant C_{s,N} \exp\big\{(j_1+j_2)\lambda t\big\} = C_{s,N} \exp\big\{(j+1)\lambda t\big\}, \tag{5.11}$$

$$\left\| \frac{\mathbf{G}_{j+1}^{(j+1)}(0)}{(j+1)!} \right\|_s \leqslant C_{s,N} \exp\big\{(j_1+j_2)\lambda t\big\} = C_{s,N} \exp\big\{(j+1)\lambda t\big\}. \tag{5.12}$$

By applying Duhamel's principle, Theorem 1, (5.11) and (5.12), we obtain

$$\|\chi_{j+1}\|_s \leqslant C\int_0^t e^{\Lambda(t-\tau)}\left\|\frac{F_{j+1}^{(j+1)}(0)}{(j+1)!}(\tau)\right\|_s d\tau \leqslant C_{s,N}\int_0^t e^{\Lambda(t-\tau)}e^{(j+1)\lambda\tau}\,d\tau \leqslant C_{s,N}e^{(j+1)\lambda t},$$

$$\|\mathbf{\Psi}_{j+1}\|_s \leqslant C\int_0^t e^{\Lambda(t-\tau)}\left\|\frac{\mathbf{G}_{j+1}^{(j+1)}(0)}{(j+1)!}(\tau)\right\|_s d\tau \leqslant C_{s,N}\int_0^t e^{\Lambda(t-\tau)}e^{(j+1)\lambda\tau}\,d\tau \leqslant C_{s,N}e^{(j+1)\lambda t},$$

$$\|\nabla q_{j+1}\|_s \leqslant C\left(\|\chi_{j+1}\|_s + \|\mathbf{\Psi}_{j+1}\|_s + \left\|\frac{\mathbf{G}_{j+1}^{(j+1)}(0)}{(j+1)!}\right\|_s\right) \leqslant C_{s,N}e^{(j+1)\lambda t},$$

due to $j+1 \geqslant 2$ and $\Lambda < 2\lambda$. Notice that our construction is possible since we were able to choose a dominant eigenvalue λ from the spectrum. Thus (5.4)–(5.6) hold true for $j+1$. Having constructed all χ_j, $\mathbf{\Psi}_j$, q_j for $1 \leqslant j \leqslant N$, we define

$$\mu^a = \sum_{j=1}^N \delta^j \chi_j, \qquad \mathbf{v}^a = \sum_{j=1}^N \delta^j \mathbf{\Psi}_j, \qquad p^a = \sum_{j=1}^N \delta^j q_j.$$

Then we have

$$\mu_t^a + \mathbf{v}^a \cdot \nabla\eta_0 = -\sum_{j=1}^N \delta^{j+1}\frac{F_{j+1}^{(j+1)}(0)}{(j+1)!},$$

$$\eta_0\mathbf{v}^a + \nabla p^a + \mu^a\mathbf{V}_0 = -\sum_{j=1}^N \delta^{j+1}\frac{\mathbf{G}_{j+1}^{(j+1)}(0)}{(j+1)!}.$$

Hence $(\mu^a, \mathbf{v}^a, p^a)$ satisfies (5.3) with the remainder

$$R_N^a = \mathbf{v}^a \cdot \nabla\mu^a - \sum_{j=1}^N \delta^{j+1}\frac{F_{j+1}^{(j+1)}(0)}{(j+1)!},$$

$$\mathbf{S}_N^a = \mu^a\mathbf{v}^a - \sum_{j=1}^N \delta^{j+1}\frac{\mathbf{G}_{j+1}^{(j+1)}(0)}{(j+1)!}.$$

Notice that the $(j+1)$th-order terms in δ of the first and the second terms of the right-hand sides in the remainders R_N^a, $\mathbf{S}_N^a$ are the same for $1 \leqslant j+1 \leqslant N$ by our construction. To see this, we note that only the first j terms of μ^a, $\mathbf{v}^a$ contribute to the $(j+1)$th-order terms of the nonlinear parts $\mathbf{v}^a \cdot \nabla\mu^a$ and $\mu^a\mathbf{v}^a$. Furthermore the contributions of the first j terms of μ^a, $\mathbf{v}^a$ in $\mathbf{v}^a \cdot \nabla\mu^a$ and $\mu^a\mathbf{v}^a$ were defined above (5.10) as $F_{j+1}(\delta)$ and $G_{j+1}(\delta)$. Hence by the Taylor expansion in δ, we see the first N terms in R_N^a, $\mathbf{S}_N^a$ vanish. Thus we deduce (5.7)–(5.8) and complete the lemma. $\square$

6. Bootstrap argument

In this section, we estimate the difference of an exact solution to (2.5) and an approximate solution to (5.3) based on a bootstrap argument which was introduced by Guo and Strauss [9]. Before the energy estimate for the difference we state local existence to the full system (2.5), which can be obtained by a straight-forward method (see [4]).

Lemma 5. *For every integer $s \geqslant 3$ and any given initial data $(\mu_0, \mathbf{v}_0, p_0) \in [H^s(D)]^4$ with $\eta(0) := \eta_0(x) + \mu_0(x,y) \geqslant m > 0$, there is a time $T > 0$ such that there exists a unique solution $(\mu, \mathbf{v}, p) \in [C([0,T]; H^s(D))]^4$ to (2.5) with $\eta(t) = \eta_0(x) + \mu(t,x,y) > 0$.*

Proof. Using the incompressibility condition $\nabla \cdot \mathbf{v} = 0$, we have, as long as $\eta_0(x) + \mu(x,y) > 0$,

$$\min_{(x,y)\in D}\big(\eta_0(x) + \mu(x,y)\big)\int |\mathbf{v}|^2\,dx\,dy \leqslant \int (\eta_0 + \mu)|\mathbf{v}|^2\,dx\,dy = -\int V_0 \mu v_1\,dx\,dy,$$

$$\frac{1}{2}\frac{d}{dt}\|\mu\|^2 = \int -\eta_{0x} v_1 \mu\,dx\,dy.$$

Since $\eta_0(x) + \mu_0(x,y) \geqslant m > 0$, it is easy to see that there exists a time T such that $\|\mathbf{v}(t)\|_{L^2}$, $\|\mu(t)\|_{L^2}$ are bounded for $0 \leqslant t \leqslant T$. Similarly, we can argue for $\|\mathbf{v}(t)\|_{H^s}$, $\|\mu(t)\|_{H^s}$. Combined with a standard contraction mapping theorem as in Lemma 3 and noting we can recover p from the second equation in (2.5), we deduce the lemma. □

Let $(\mu, \mathbf{v}, p) \in C([0,T]; H^s(D))$ be a local-in-time solution as in Lemma 5 and $(\mu^a, \mathbf{v}^a, p^a)$ be an approximate solution as constructed in Lemma 4. Set the difference to be

$$\mu^d = \mu - \mu^a, \qquad \mathbf{v}^d = \mathbf{v} - \mathbf{v}^a, \qquad p^d = p - p^a.$$

Then it satisfies

$$\mu_t^d + \mathbf{v}\cdot\nabla\mu^d + \mathbf{v}^d\cdot\nabla\big(\mu^a + \eta_0\big) = -R_N^a, \tag{6.1}$$

$$\big(\eta_0 + \mu^a\big)\mathbf{v}^d + \nabla p^d + \mu^d(\mathbf{V}_0 + \mathbf{v}) = -\mathbf{S}_N^a, \tag{6.2}$$

$$\nabla\cdot\mathbf{v}^d = 0. \tag{6.3}$$

We obtain the following energy estimate for the difference $(\mu^d, \mathbf{v}^d, p^d)$.

Lemma 6. *For any integer $s \geqslant 3$, let $(\mu^d(t,x,y), \mathbf{v}^d(t,x,y), p^d(t,x,y)) \in [L^\infty_{\mathrm{loc}}(H^s(D))]^4$ as in (6.1)–(6.3), $(\mu^a(t,x,y), \mathbf{v}^a(t,x,y), p^a(t,x,y)) \in [L^\infty_{\mathrm{loc}}(H^s(D))]^4$, and $(R_N^a(t,x,y), \mathbf{S}_N^a(t,x,y)) \in [L^\infty_{\mathrm{loc}}(H^s(D))]^3$ as in Lemma 4. Then there exists a universal constant $C_0 = C_0(s, \|\eta_0\|_{C^s})$ and $C = C(s, |V_0|, \|\eta_0\|_{C^{s+1}})$ such that*

$$\frac{d}{dt}\|\mu^d\|_s^2 \leqslant C\big(\|\mathbf{v}^d\|_s + \|\mathbf{v}^a\|_s + \|\mu^a\|_{s+1}^2 + 1\big)\|\mu^d\|_s^2 + C\|\mathbf{v}^d\|_s^2 + \|R_N^a\|_s^2, \tag{6.4}$$

$$\|\mathbf{v}^d\|_s \leqslant C_0\big(\|\mu^a\|_s + \|\mu^d\|_s\big)\|\mathbf{v}^d\|_s + C\big(\|\mathbf{v}^a\|_s + 1\big)\|\mu^d\|_s + \|\mathbf{S}_N^a\|_s, \tag{6.5}$$

$$\frac{d}{dt}\|\mu^a\|_s^2 \leqslant C(\|\mathbf{v}^a\|_s + 1)\|\mu^a\|_s^2 + C\|\mathbf{v}^a\|_s^2 + \|R_N^a\|_s^2, \tag{6.6}$$

$$\|\mathbf{v}^a\|_s \leqslant C_0\|\mu^a\|_s\|\mathbf{v}^a\|_s + C\|\mu^a\|_s + \|\mathbf{S}_N^a\|_s. \tag{6.7}$$

Proof. We first treat $\mathbf{v}^d$. For $s=0$, using Sobolev imbedding in (6.2) and incompressibility (6.3) yields

$$\begin{aligned}\|\mathbf{v}^d\| &\leqslant C(\|\mu^a\|_2 + \|\mu^d\|_2)\|\mathbf{v}^d\| + (|V_0| + C\|\mathbf{v}^a\|_2)\|\mu^d\| + \|\mathbf{S}_N^a\| \\ &\leqslant C_0(\|\mu^a\|_2 + \|\mu^d\|_2)\|\mathbf{v}^d\| + C(1 + \|\mathbf{v}^a\|_2)\|\mu^d\| + \|\mathbf{S}_N^a\|, \end{aligned} \tag{6.8}$$

where C_0 depends only on Sobolev imbedding constant. For $s>3$, we take ∂_α-derivative of (6.2) with $|\alpha| = s$ to get

$$\eta_0\partial_\alpha \mathbf{v}^d = \partial_\alpha(\eta_0\mathbf{v}^d) - \eta_0\partial_\alpha\mathbf{v}^d - \nabla\partial_\alpha p^d - \partial_\alpha(\mu^a\mathbf{v}^d + \mu^d\mathbf{V}_0 + \mu^d\mathbf{v}) - \partial_\alpha\mathbf{S}_N^a.$$

Then we use (6.3), Gagliardo–Nirenberg–Moser inequality, and Sobolev imbedding for $s \geqslant 3$ to deduce

$$\begin{aligned}\|\mathbf{v}^d\|_s &\leqslant C\|\mathbf{v}^d\|_{s-1} + C(\|\mu^d\|_s + \|\mu^a\|_s)\|\mathbf{v}^d\|_s + C(\|\mathbf{v}^a\|_s + 1)\|\mu^d\|_s + \|\partial_\alpha\mathbf{S}_N^a\| \\ &\leqslant \frac{1}{2}\|\mathbf{v}^d\|_s + C\|\mathbf{v}^d\| + C(\|\mu^d\|_s + \|\mu^a\|_s)\|\mathbf{v}^d\|_s + C(\|\mathbf{v}^a\|_s + 1)\|\mu^d\|_s + \|\partial_\alpha\mathbf{S}_N^a\|,\end{aligned}$$

where we have used interpolation between H^s and L^2 for $\|\mathbf{v}^d\|_{s-1}$. Using (6.8) yields

$$\|\mathbf{v}^d\|_s \leqslant C_0(\|\mu^a\|_s + \|\mu^d\|_s)\|\mathbf{v}^d\|_s + C(\|\mathbf{v}^a\|_s + 1)\|\mu^d\|_s + \|\mathbf{S}_N^a\|_s,$$

where C_0 depends only on s and $\|\eta_0\|_{C^s}$.

We now turn to μ^d. Taking ∂_α-derivative of (6.1) with $|\alpha| = s$ and then multiplying by $\partial_\alpha\mu^d$ we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_\alpha\mu^d\|^2 &= \iint_D \partial_\alpha\mu_t^d\partial_\alpha\mu^d \\ &= -\iint_D \partial_\alpha(\mathbf{v}\cdot\nabla\mu^d)\partial_\alpha\mu^d - \iint_D \partial_\alpha\{\mathbf{v}^d\cdot\nabla(\mu^a + \eta_0)\}\,\partial_\alpha\mu^d - \iint_D \partial_\alpha R_N^a\partial_\alpha\mu^d.\end{aligned}$$

Using (6.3), Gagliardo–Nirenberg–Moser inequality, and Sobolev imbedding for $s \geqslant 3$ yield

$$\begin{aligned}\iint_D \partial_\alpha(\mathbf{v}\cdot\nabla\mu^d)\partial_\alpha\mu^d &= \iint\{\partial_\alpha(\mathbf{v}\cdot\nabla\mu^d)\partial_\alpha\mu^d - \mathbf{v}\cdot\nabla\partial_\alpha\mu^d\partial_\alpha\mu^d\} + \iint \mathbf{v}\cdot\nabla\partial_\alpha\mu^d\partial_\alpha\mu^d \\ &\leqslant C(\|\mathbf{v}^d\|_s + \|\mathbf{v}^a\|_s)\|\mu^d\|_s^2.\end{aligned}$$

Treating the second and third terms in a similar manner we obtain

$$\frac{d}{dt}\|\mu^d\|_s^2 \leqslant C\left(\|\mathbf{v}^d\|_s + \|\mathbf{v}^a\|_s + \|\mu^a\|_{s+1}^2 + 1\right)\|\mu^d\|_s^2 + C\|\mathbf{v}^d\|_s^2 + \|R_N^a\|_s^2.$$

Thus we deduce (6.4). Similarly, we obtain (6.6)–(6.7) which completes the proof. □

We now apply bootstrap argument to Lemma 6 to establish the following lemma which allows one to control the growth of $\|\mu^d\|_s$ and $\|\mathbf{v}^d\|_s$ by the growth of the remainder $\|R_N^a\|_s$ and $\|\mathbf{S}_N^a\|_s$ for sufficiently large $N > 0$ (i.e. $2(N+1)\lambda$).

Lemma 7. *We suppose the same assumptions as in Lemma* 6 *and assume that with C_0 in Lemma* 6*,*

$$C_0 \times \left\{\|\mu^a\|_{s+1} + \|\mu^d\|_s\right\} \leqslant \frac{1}{2}. \tag{6.9}$$

Then there exist universal constants $C_1 = C_1(s, \|\eta_0\|_{C^s})$ and $C = C(s, |V_0|, \|\eta_0\|_{C^{s+1}})$ such that

$$\frac{d}{dt}\|\mu^d\|_s^2 \leqslant C_1\|\mu^d\|_s^2 + C\left\{\|\mathbf{S}_N^a\|_s^2 + \|R_N^a\|_s^2\right\}, \tag{6.10}$$

$$\|\mathbf{v}^d\|_s \leqslant C\|\mu^d\|_s + C\|\mathbf{S}_N^a\|_s. \tag{6.11}$$

Proof. Applying (6.9) to (6.5) and (6.7) yields

$$\|\mathbf{v}^d\|_s \leqslant C\left(\|\mathbf{v}^a\|_s + 1\right)\|\mu^d\|_s + C\|\mathbf{S}_N^a\|_s, \tag{6.12}$$

$$\|\mathbf{v}^a\|_s \leqslant C\left(1 + \|\mathbf{S}_N^a\|_s\right). \tag{6.13}$$

We substitute (6.13) in (6.12) and apply (6.9) again to (6.12) to obtain (6.11). Substituting (6.13) and (6.11) for (6.4) and using (6.9) yields

$$\begin{aligned}\frac{d}{dt}\|\mu^d\|_s^2 &\leqslant C\left(\|\mu^d\|_s + \|\mu^a\|_{s+1}^2 + \|\mathbf{S}_N^a\|_s + 1\right)\|\mu^d\|_s^2 + C\|\mathbf{v}^d\|_s^2 + C\|R_N^a\|_s^2\\ &\leqslant C\left(\|\mu^d\|_s + \|\mu^a\|_{s+1}^2 + 1\right)\|\mu^d\|_s^2 + \|\mu^d\|_s^2\|\mathbf{S}_N^a\|_s + C\left\{\|\mathbf{S}_N^a\|_s^2 + \|R_N^a\|_s^2\right\}\\ &\leqslant C_1\|\mu^d\|_s^2 + C\left\{\|\mathbf{S}_N^a\|_s^2 + \|R_N^a\|_s^2\right\}.\end{aligned}$$

Note that (6.9) implies $C(\|\mu^d\|_s + \|\mu^a\|_{s+1}^2 + 1) \leqslant C(1 + \frac{1}{2C_0} + (\frac{1}{2C_0})^2) =: C_1$. Thus we deduce our lemma. □

7. Nonlinear instability by a dominant eigenvalue

In this section we establish nonlinear instability using the dominant eigenvalue λ and bootstrap Lemma 6. We now prove our main Theorem 2 which has already been stated in Section 2.

Proof of Theorem 2. We first choose $N > 0$ large such that

$$N > \frac{C_1}{2\lambda} - 1, \tag{7.1}$$

where $\lambda = \lambda_{k_0}$ is the dominant eigenvalue with which approximate solutions were constructed in Section 5 while $C_1 = C_1(s, \|\eta_0\|_{C^s})$ is the constant in (6.10) of Lemma 7.

Let $(\mu^a(t,x,y), \mathbf{v}^a(t,x,y)) \in [L^\infty_{\mathrm{loc}}(H^s(D))]^3$ be the approximate solution in Lemma 4 with the choice of N in (7.1). Then we construct a family of solutions $(\mu^\delta(t,x,y), \mathbf{v}^\delta(t,x,y))$ to (2.5) which will be shown to be unstable using the approximate solution $(\mu^a(t,x,y), \mathbf{v}^a(t,x,y))$.

For given small $\delta > 0$, there exists a local-in-time solution $(\mu^\delta(t), \mathbf{v}^\delta(t))$ with initial data $(\mu^a(0), \mathbf{v}^a(0))$. Then the difference satisfies $\mu^d(0) = 0$, $\mathbf{v}^d(0) = \mathbf{0}$ initially. Define $T > 0$ by

$$T = \sup\left\{ t \;\middle|\; C_0 \times \left\{ \left\|\mu^a\right\|_{s+1} + \left\|\mu^d\right\|_s \right\} \leqslant \frac{1}{2} \right\}, \tag{7.2}$$

where $C_0 = C_0(s, \|\eta_0\|_{C^s})$ is the constant in Lemma 6. It is easy to see that T is well defined since $\mu^d(0) = 0$ and $\|\mu^a\|_s = O(\delta)$.

We show that instability time T^δ in (5.1) occurs before the possible blow-up time by contradiction, i.e. $T^\delta < T$ if θ is chosen small enough. Suppose $T^\delta \geqslant T$. Then for $t \leqslant T$ $(\leqslant T^\delta)$, we have by (5.2)

$$\begin{aligned} \left\|\mu^a(t)\right\|_{s+1} &\leqslant C \sum_{j=1}^{N} \delta^j \left\|\chi_j(t)\right\|_{s+1} \leqslant \sum_{j=1}^{N} C_j \delta^j \exp(j\lambda t) \\ &\leqslant \sum_{j=1}^{N} C_j \delta^j \exp\bigl(j\lambda T^\delta\bigr) = \sum_{j=1}^{N} C_j \theta^j < \frac{1}{4C_0} \quad \text{for small values of } \theta. \end{aligned}$$

We apply (6.10) in the bootstrap Lemma 7 together with (5.7)–(5.8) to obtain for $t \leqslant T$

$$\begin{aligned} \frac{d}{dt}\left\|\mu^d(t)\right\|_s^2 &\leqslant C_1 \left\|\mu^d(t)\right\|_s^2 + C\left\{ \left\|\mathbf{S}_N^a(t)\right\|_s^2 + \left\|R_N^a(t)\right\|_s^2 \right\} \\ &\leqslant C_1 \left\|\mu^d(t)\right\|_s^2 + C\delta^{2(N+1)} \exp\bigl[2(N+1)\lambda t\bigr]. \end{aligned}$$

Using inequality (7.1) for N and applying Gronwall inequality yields for $t \leqslant T$

$$\begin{aligned} \left\|\mu^d(t)\right\|_s &\leqslant C\delta^{N+1} \exp\bigl[(N+1)\lambda t\bigr] \leqslant C\delta^{N+1} \exp\bigl[(N+1)\lambda T^\delta\bigr] \\ &= C\theta^{N+1} < \frac{1}{4C_0} \quad \text{if } \theta \text{ is small.} \end{aligned} \tag{7.3}$$

Thus we have, for $t = T$

$$\left\|\mu^a(T)\right\|_{s+1} + \left\|\mu^d(T)\right\|_s < \frac{1}{2C_0}.$$

This contradicts the definition of T and therefore we conclude $T^\delta < T$. Notice also that (6.12)–(6.13) implies that $\|\mathbf{v}^d(t)\|_s$ is bounded for all $0 \leqslant t \leqslant T$.

We now show that nonlinear instability occurs at $t = T^\delta$. By our choice of $\boldsymbol{\Psi}_1$ as the growing normal mode with the dominant eigenvalue λ and by (5.2), we have, at $t = T^\delta$

$$\begin{aligned}\|\mathbf{v}^a(T^\delta)\| &\geqslant \delta\|\boldsymbol{\Psi}_1(T^\delta)\| - \sum_{j=2}^{N}\delta^j\|\boldsymbol{\Psi}_j(T^\delta)\| \geqslant C\delta\exp(\lambda T^\delta) - \sum_{j=2}^{N}C_j\delta^j\exp(j\lambda T^\delta)\\ &= C\theta - \sum_{j=2}^{N}C_j\theta^j \geqslant \frac{C}{2}\theta \quad \text{if } \theta \text{ is small.}\end{aligned}$$

By (6.11) in the bootstrap Lemma 7 and by (7.3), we obtain for all $0 \leqslant t \leqslant T^\delta$,

$$\|(\mathbf{v}^\delta - \mathbf{v}^a)(t)\|_s \leqslant C\delta^{N+1}\exp((N+1)\lambda t) \leqslant C\theta^{N+1}. \tag{7.4}$$

By (7.3) and (7.4), we deduce

$$\begin{aligned}\|\mathbf{v}^\delta(T^\delta)\| &\geqslant \|\mathbf{v}^a(T^\delta)\| - \|(\mathbf{v}^\delta - \mathbf{v}^a)(T^\delta)\|\\ &\geqslant \|\mathbf{v}^a(T^\delta)\| - \|(\mathbf{v}^\delta - \mathbf{v}^a)(T^\delta)\|_s\\ &\geqslant \frac{C}{2}\theta - C\theta^{N+1} \geqslant \frac{C}{4}\theta \equiv \varepsilon_0 > 0.\end{aligned}$$

In a similar way, we deduce the same for μ^δ and thus complete the proof. □

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