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Physica A

journal homepage: www.elsevier.com/locate/physa

Some useful upper bounds for the selection of optimal profiles

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ARTICLE INFO

Article history:
Received 4 January 2012
Received in revised form 19 February 2012
Available online xxxx

Keywords:
Optimal viscous profile
Enhanced chemical oil recovery
Upper bounds
Linear stability

ABSTRACT

In enhanced oil recovery by chemical flooding within tertiary oil recovery, it is often necessary to choose optimal viscous profiles of the injected displacing fluids that reduce growth rates of hydrodynamic instabilities the most thereby substantially reducing the well-known fingering problem and improving oil recovery. Within the three-layer Hele–Shaw model, we show in this paper that selection of the optimal monotonic viscous profile of the middle-layer fluid based on well known theoretical upper bound formula [P. Daripa, G. Pasa, A simple derivation of an upper bound in the presence of a viscosity gradient in three-layer Hele–Shaw flows, Journal of Statistical Mechanics (2006) 11. http://dx.doi.org/10.1088/1742-5468/2006/01/P01014] agrees very well with that based on the computation of maximum growth rate of instabilities from solving the linearized stability problem. Thus, this paper proposes a very simple, fast method for selection of the optimal monotonic viscous profiles of the displacing fluids in multi-layer flows.

1. Introduction

In classical fluid mechanics, one encounters complex problems involving many unstable interfaces and modeling their collective effects can be computationally very intensive. However, if such collective effects can be captured in a statistical sense by a few system parameter dependent data, then such data can be used as a measure to compare such complex systems having different values of one or more parameters. This way, one does not have to go through intensive computational procedure to find changes in certain properties of the complex system as the parameters of the system change. This paper addresses one such problem and shows the promise of the method for the first time on a prototype model problem arising in the context of enhanced oil recovery. Then suggestions are made to further experiment with the proposed method on problems which are extensions of this model problem.

Efficient oil recovery is an important component of energy related research. Initial stages of this recovery process involve a primary recovery stage followed by a secondary displacement process in which the viscous oil in porous media is displaced by the injection of another less viscous fluid, usually water. It is well known that this displacement process is basically unstable (see Saffman and Taylor [1] and Chouke et al. [2]) which causes undesirable fingering problem. The next stage of oil recovery is called enhanced oil recovery (EOR for short). There are many EOR technologies, and one that is of interest here is EOR by chemical flooding which involves a tertiary displacement process. There are many injection policies for EOR by chemical flooding and the simplest one involves injection of one viscous displacing fluid for some time followed by water. This displacement process is essentially unstable and basically involves a three-layer flow: oil followed by the viscous fluid which in turn is followed by water. Oil recovery from this displacement process depends on the viscous profile of the middle layer viscous fluid. Therefore, the selection of an optimal viscous profile that maximizes the oil recovery is a very important problem in this kind chemical flooding scheme. This problem for this displacement process through a homogeneous porous medium is equivalent to minimizing the maximum growth rate of instabilities. Since flow through a porous medium is

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doi:10.1016/j.physa.2012.03.041

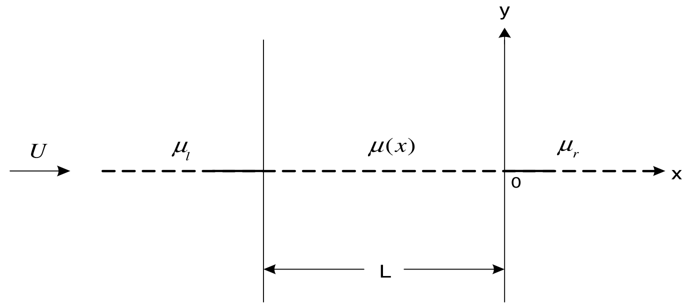

Fig. 1. Three-layer fluid flow.

modeled using Darcy's law, which also holds in the Hele–Shaw cell, it has been common practice to model this EOR process using a three-layer Hele–Shaw model with middle layer fluid having a variable viscous profile with a viscosity gradient that is positive in the direction of basic flow. This is shown in Fig. 1.

For our purposes, an optimal viscous profile of the middle layer fluid in this three-layer Hele–Shaw flow is then defined by the viscous profile that minimizes the maximum growth rate. We have recently numerically solved this problem. An exhaustive detail of our numerical method and results have been presented in Daripa and Ding [3]. In Daripa and Ding [3], the eigenvalue problem resulting from the linear stability analysis of the system of partial differential equations modeling the three-layer Hele–Shaw flow is solved. The effect of various viscous profiles from four different monotonic families on the stabilization of flow is investigated by computing the maximum growth rate for each viscous profile. The profile with most stabilization capacity is found to be an exponential profile that supports viscosity jumps at both the two interfaces. It is found that viscosity jump driven interfacial instabilities at both the interfaces as well as the fingering instability of the middle layer participate almost equally in providing maximum stabilization or equivalently in minimizing the maximal growth rate.

In this paper, we show the promise of a different, much easier and extremely fast approach for selection optimal profiles based on theoretical upper bounds on the maximum growth rate obtained earlier by Daripa and Pasa [4]. In particular, both approaches are used in this paper to determine the optimal viscous profile for flows that support only pure viscosity gradient driven instability. It is found that selections based on these two approaches are in excellent agreement with each other. This conclusion from this case study should have a bearing on the selection of optimal profiles for multi-layer flows involving many viscous profiles.

In Section 2, we briefly review from our earlier works (see Daripa and Pasa [5], Daripa [6]) the model, the formulation of the eigenvalue problem arising from linear stability analysis of the uniform flow and the formula for the upper bound on the maximum growth rate. We also list here four families of viscous profiles and the upper bounds for these viscous profiles. In Section 3, we show and compare results on optimal profiles determined by the two methods: one based on numerical computation of the maximum growth rate $\sigma_{\max}$ and the other based on upper bound formulas. Finally, we conclude in Section 4.

2. Preliminaries

The model is a three-layer flow with two fronts taken to be at $x = 0$ and $x = -L$. Fluid upstream $x = -\infty$ is assumed to have a velocity $(U, 0)$. In the left-most layer, $-\infty < x \leq -L$, viscosity of the fluid (water usually) is a constant denoted by μ_l and that of the fluid (oil usually) in the right-most layer, $x > 0$, is denoted by a constant μ_r. The middle layer, with length L, contains a fluid (to be called polysolution henceforth) of variable viscosity $\mu(x)(\mu_l < \mu(x) < \mu_r)$. The interfacial tension at the leading front separating oil from polysolution is denoted by T_0, and that at the trailing front separating water from polysolution is denoted by T_1. The fluid flow in this three-layer model is described by the following governing equations in each of the three-layers.

$$\nabla \cdot \mathbf{u} = 0, \qquad \nabla p = -\mu\, \mathbf{u}, \qquad \frac{D\mu}{Dt} = 0, \tag{1}$$

where $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ and $\frac{D}{Dt}$ is the material derivative. The first Eq. $(1)_1$ is the continuity equation for incompressible flow, the second Eq. $(1)_2$ is Darcy's law (Darcy [7]), and the third Eq. $(1)_3$ is the advection equation for viscosity. The above system (1) admits a simple basic solution, namely the whole fluid set-up moves with speed U in the x direction and the two interfaces, namely the one separating the left layer from the middle-layer and the other separating the right layer from the middle-layer, are planar, i.e. parallel to the y-axis. The pressure corresponding to this basic solution is obtained by integrating $(1)_2$. In a frame moving with velocity $(U, 0)$, the above system is stationary along with two planar interfaces separating these three fluid layers, and the smooth viscous profile $\mu(x)$ of the middle-layer fluid satisfies $\mu_l < \mu(x) < \mu_r$. Here and below, with slight abuse of notation, the same variable x is used in the moving reference frame. In linearized stability analysis by

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normal modes, disturbances (denoted by tilde variables below) in the moving reference frame are written in the form

$$(\widetilde{u}, \widetilde{v}, \widetilde{p}, \widetilde{\mu}) = (f(x), \psi(x), \phi(x), h(x))\, \mathrm{e}^{(iky+\sigma t)}, \tag{2}$$

where k is the wave number and σ is the growth rate. We then insert this disturbance form into the linearized disturbance equations obtained from (1) and also into the linearized dynamic and kinematic interfacial conditions (see Ref. [5]). After some algebraic manipulation, we obtain the following differential equation in terms of the eigenfunction $f(x)$ and the first derivative of viscosity μ_x in the middle layer:

$$-(\mu f_x)_x + k^2 \mu f = \lambda k^2 U \mu_x f, \quad x \in (-L, 0) \tag{3}$$

with boundary conditions

$$f_x(0) = (\lambda e + q) f(0), \qquad f_x(-L) = (\lambda r + s) f(-L), \tag{4}$$

where $\lambda = 1/\sigma$ and e, q, r, s are defined by

$$\left.\begin{array}{ll} e = \{(\mu_r - \mu(0))Uk^2 - T_0 k^4\}/\mu(0), & q = -\mu_r k/\mu(0) \le 0, \\ r = \{(\mu_l - \mu(-L))Uk^2 + T_1 k^4\}/\mu(-L), & s = -\mu_l k/\mu(-L) \ge 0. \end{array}\right\} \tag{5}$$

For a given choice of values of $T_0, T_1, U, L, \mu_l, \mu_r$ and the middle layer viscosity profile $\mu(x)$, the eigenvalue problem defined by (3)–(5) is solved numerically on a finite difference grid for the dispersion relation $\sigma(k)$ from which $\sigma_{\max} = \max_k \sigma(k)$ is obtained for a given viscous profile $\mu(x)$.

In Daripa and Pasa [4] (see also [6]), we have derived the following formula for the upper bound σ_{u} on the maximum growth rate $\sigma_{\max}$ in a strict inequality form.

$$\sigma < \sigma_{\mathrm{u}} = \max \left\{ \frac{2T_0}{\mu_r} \left(\frac{U(\mu_r - \mu(0))}{3T_0} \right)^{3/2}, \frac{2T_1}{\mu_l} \left(\frac{U(\mu(-L) - \mu_l)}{3T_1} \right)^{3/2}, U \max_x \left(\frac{\mu_x(x)}{\mu(x)} \right) \right\}. \tag{6}$$

Notice that the upper bound σ_{u} is independent of the length L of the middle layer whereas the exact maximum growth rate $\sigma_{\max}$ depends on L. In spite of this, we will see that selection based on the upper bound is good for all values of L because $\sigma_{\max}$ for various L are in tandem (see Section 3) with each other as well in tandem with σ_{u}.

In Daripa [8], we have validated the formula (6) for σ_{u}. Computation of $\sigma_{\max}$ involves numerically solving the eigenvalue problem which is relatively more time consuming than evaluating σ_{u} from the formula (6). In this paper, we will show that values of σ_{u} and $\sigma_{\max}$ are in tandem with each other for all viscous profiles used. Thus this property can be used for the purpose of selection of optimal viscous profile, i.e., a viscous profile that gives the lowest value of the maximum growth rate $\sigma_{\max}$ over a given class of viscous profiles. This approach is certainly computationally very advantageous over the direct method based on computing $\sigma_{\max}$ from solving the eigenvalue problem numerically. We justify this contention in this paper by comparing selections by these two methods of the optimal profiles from following four families of monotonic viscous profiles. These profiles are defined so that they have two end point viscosities $\mu(0)$ at $x = 0$ and $\mu(-L)$ at $x = -L$. Below we use the notation $\hat{\mu}_r := \mu_r/\mu_l$.

(i) *Linear viscosity profiles:*

$$\mu(x) = \frac{\mu(0) - \mu(-L)}{L} x + \mu(0), \quad -L < x < 0. \tag{7}$$

For this viscous profile (7), it follows from (6) that

$$\sigma < \sigma_u^{\mathrm{lin}} = \max \left\{ \frac{2T_0}{\mu_r} \left(\frac{U(\mu_r - \mu(0))}{3T_0} \right)^{3/2}, \frac{2T_1}{\mu_l} \left(\frac{U(\mu(-L) - \mu_l)}{3T_1} \right)^{3/2}, \frac{U}{L} \left(\hat{\mu}_r - 1 \right) \right\}. \tag{8}$$

The superscript 'lin' on σ_u means the bound is for linear profile. Similarly below 'exp', 'poly' and 'sin' denote exponential, polynomial, and sinusoidal profiles respectively.

(ii) *Exponential viscosity profiles:*

$$\mu(x) = \mu(-L) \exp\{\alpha(x + L)\}, \quad \text{where } \alpha = \frac{1}{L} \ln \frac{\mu(0)}{\mu(-L)}. \tag{9}$$

For this exponential profile (9), from (6) we have

$$\sigma < \sigma_u^{\exp} = \max \left\{ \frac{2T_0}{\mu_r} \left(\frac{U(\mu_r - \mu(0))}{3T_0} \right)^{3/2}, \frac{2T_1}{\mu_l} \left(\frac{U(\mu(-L) - \mu_l)}{3T_1} \right)^{3/2}, \frac{U}{L} \ln\left(\hat{\mu}_r\right) \right\}. \tag{10}$$

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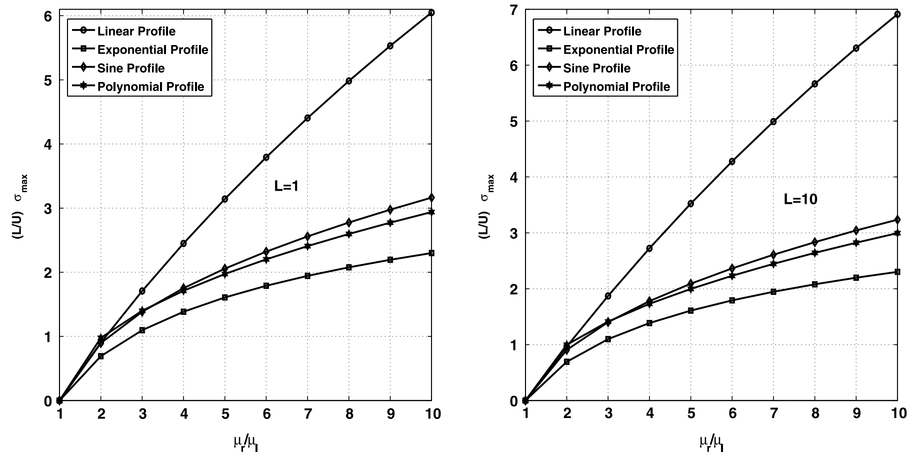

(a) $L = 1$. (b) $L = 10$.

Fig. 2. Plots $(L/U) * \sigma_{\max}$ for four viscosity profiles versus μ_r/μ_l for pure viscosity-gradient driven instability. $\sigma_{\max}$ is the maximum growth rate for the most optimal profile. The other parameter values are $T_0 = T_1 = U = 1$. On screen view of these figures are recommended for clarity of the plots.

(iii) *Polynomial viscosity profiles:*

$$\mu(x) = \frac{(\mu(0) - \mu(-L))}{L^2}(x+L)^2 + \mu(-L). \tag{11}$$

For the polynomial viscosity profile (11),

$$\sigma < \sigma_u^{\text{poly}} = \max\left\{ \frac{2T_0}{\mu_r}\left(\frac{U(\mu_r - \mu(0))}{3T_0}\right)^{3/2}, \frac{2T_1}{\mu_l}\left(\frac{U(\mu(-L) - \mu_l)}{3T_1}\right)^{3/2}, \frac{U}{L}\sqrt{\hat{\mu}_r - 1}\right\}. \tag{12}$$

(iv) *Sine viscosity profiles:*

$$\mu(x) = (\mu(0) - \mu(-L))\sin\left(\frac{\pi x}{2L}\right) + \mu(0). \tag{13}$$

Note that the constant viscosity profile is a spacial case of the above sine profile when $\mu(0) = \mu(L)$, For this sine viscosity profile (13),

$$\sigma < \sigma_u^{\sin} = \max\left\{ \frac{2T_0}{\mu_r}\left(\frac{U(\mu_r - \mu(0))}{3T_0}\right)^{3/2}, \frac{2T_1}{\mu_l}\left(\frac{U(\mu(-L) - \mu_l)}{3T_1}\right)^{3/2}, \frac{U\pi}{2L}\frac{(\hat{\mu}_r - 1)}{\sqrt{2\hat{\mu}_r - 1}}\right\}. \tag{14}$$

We recall the dimensional parameters of the problem: namely μ_l, μ_r, T_0, T_1, U and L. In all our computations below we have used $T_0 = T_1 = U = 1$. Unless otherwise indicated or obvious from presentation of results, we have used mostly $\mu_l = 2$, $\mu_r = 10$, $L = 1$. Maximum growth rate is computed for all the above viscous profiles with $\mu(-L)$ and $\mu(0)$ fixed at μ_l and μ_r respectively so that both interfaces are individually stable.

3. Fast method for selection of optimal viscous profiles

The maximum growth rates $\sigma_{\max}$ for all four monotonic viscous profiles with no viscosity jumps at the interfaces are computed by solving the eigenvalue problem by the finite difference method mentioned after Eq. (5) in Section 2. On the other hand, the upper bounds σ_{u} for these same viscous profiles are evaluated from exact formulas given in the previous section. Plots of normalized $\sigma_{\max}$ versus μ_r/μ_l are shown in Fig. 2(a) when $L = 1$ and in Fig. 2(b) when $L = 10$. Fig. 3 shows plots of normalized σ_{u} versus viscosity ratio μ_r/μ_l.

Notice that the trends and qualitative natures of the plots in Figs. 2 and 3 are in tandem with each other even though the quantitative data are different. Moreover, the trends of the plots in Fig. 2(a) and (b) are same and computations show that it holds for all other values of L. Even though the quantitative data between Figs. 2 and 3 are different, they do not have much bearing since only the trend in the data as the profiles change is important for our purposes. Among other observations that can be made from these figures, we see that both methods select the exponential profile as the optimal profile over all values of viscosity ratio $\hat{\mu} = \mu_r/\mu_l$. Since the method based on the upper bound formula is the easiest one to use, it is recommended that the upper bound formula should be used in the selection of optimal profiles. This method is obviously very fast and accurate.

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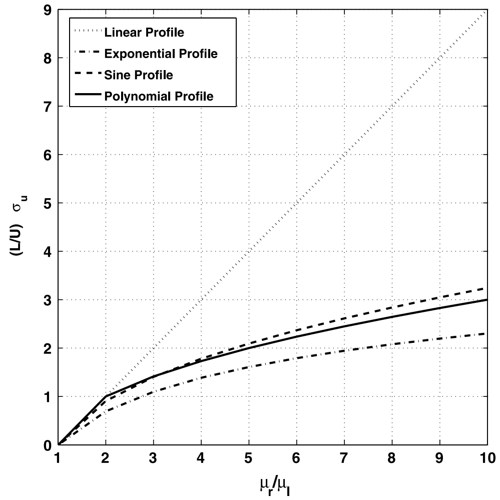

Fig. 3. Plots of normalized upper bounds ($\sigma_u * L/U$) on growth rates for exponential, linear, polynomial and sine viscous profiles versus the viscosity ratio μ_r/μ_l for pure viscosity-gradient driven instability.

4. Conclusions

In this paper, we have shown using a three-layer case study that selection of the optimal profile of the middle-layer fluid based on the theoretical upper bound formula agrees with that based on the exact computation of minimized maximum growth rates. Even though the results presented are for cases with no jumps in viscosity at interfaces, experiments show that they hold when the viscosity jumps at interfaces are allowed. The success of the method based on the upper bound formula in selecting the optimal profile for the three-layer case is a proof of the principle that the collective effects of interfacial and layer (fingering) instabilities are very well captured by the upper bound formulas for their application to the problem of selection of optimal profiles. Upper bound formulas for multi-layer flows with arbitrary number of displacing fluids which are generalizations of the three-layer formula (6) have been presented and discussed in Daripa [6]. These formulas essentially are compact statistical descriptions of the cumulative effects of many interfacial and layer interactions in such flows on hydrodynamic instabilities. The three-layer case study presented here for the selection of optimal profile should be a guide to others to experiment with the proposed method based on upper bound formulas for the selection of optimal profiles of many internal-layer fluids in multi-layer flows.

Acknowledgments

I would like to thank one of the two reviewers for helpful suggestions. It is a pleasure to thank Xueru Ding for help with the figures. This paper was made possible by NPRP grant # 08-777-1-141 from the Qatar National Research Fund (a member of the Qatar Foundation). The statements made herein are solely the responsibility of the author.

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