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ON A BOUNDARY CONTROL APPROACH TO DOMAIN EMBEDDING METHODS*

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Abstract. In this paper, we propose a domain embedding method associated with an optimal boundary control problem with boundary observations to solve elliptic problems. We prove that the optimal boundary control problem has a unique solution if the controls are taken in a finite dimensional subspace of the space of the boundary conditions on the auxiliary domain.

Using a controllability theorem due to J. L. Lions, we prove that the solutions of Dirichlet (or Neumann) problems can be approximated within any prescribed error, however small, by solutions of Dirichlet (or Neumann) problems in the auxiliary domain taking an appropriate subspace for such an optimal control problem. We also prove that the results obtained for the interior problems hold for the exterior problems. Some numerical examples are given for both the interior and the exterior Dirichlet problems.

Key words. domain embedding methods, optimal control

AMS subject classifications. 93B05, 93B07, 93B40, 65N30, 65P05, 65R20

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1. Introduction.

The embedding or fictitious domain methods, which were developed especially in the seventies (see [6], [2], [34], [35], [28], or [14]), have been a very active area of research in recent years because of their appeal and potential for applications in solving problems in complicated domains very efficiently. In these methods, complicated domains ω, where solutions of problems may be sought, are embedded into larger domains Ω with simple enough boundaries so that solutions in these embedded domains can be constructed more efficiently. The use of these embedding methods is now commonplace for solving complicated problems arising in science and engineering. To this end, it is worth mentioning the domain embedding methods for Stokes equations (Borgers [5]), for fluid dynamics and electromagnetics (Dinh et al. [12]), and for the transonic flow calculation (Young et al. [36]).

In [3], an embedding method is associated with a distributed optimal control problem. There the problem is solved in an auxiliary domain Ω using a finite element method on a fairly structured mesh which allows the use of fast solvers. The auxiliary domain Ω contains the domain ω, and the solution in Ω is found as a solution of a distributed optimal control problem such that it satisfies the prescribed boundary conditions of the problem in the domain ω. The same idea is also used in [10], where a least squares method is used. In [13], an embedding method is proposed in which a combination of Fourier approximations and boundary integral equations is used. Essentially, there a Fourier approximation for a particular solution of the inhomogeneous equation in Ω is found, and then the solution in ω for the homogeneous equation is sought using the boundary integral methods.

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In recent years, progress in this field has been substantial, especially in the use of the Lagrange multiplier techniques. In this connection, the works of Girault, Glowinski, Hesla, Joseph, Kuznetsov, Lopez, Pan, and Périaux (see [15], [16], [17], [18], and [19]) should be cited.

There are many problems for which an exact solution on some particular domains may be known or computed numerically within a good approximation very efficiently. In these cases, an embedding domain method associated with a boundary optimal control problem allows one to find solutions of the problems very efficiently in complicated domains. Specifically, the particular solution of the inhomogeneous equation can be used to reduce the problem to solving a homogeneous equation in ω subject to appropriate conditions on the boundary of the domain ω. This solution in the complicated domain ω can be obtained via an optimal boundary control problem where one finds the solution of the same homogeneous problem in the auxiliary domain Ω that would satisfy appropriate boundary conditions on the domain ω. We mention that the boundary control approach already has been used by Mäkinen, Neittaanmäki, and Tiba for optimal shape design and two-phase Stefan-type problems (see [29], [32]). Moreover, recently there has been enormous progress in shape optimization using the fictitious domain approaches. We can cite here, for instance, the works of Daňková, Haslinger, Klarbring, Makinen, Neittaanmäki, and Tiba (see [9], [22], [23], and [33]) among many others.

In section 2, an optimal boundary control problem involving an elliptic equation is formulated. In this formulation, the solution on the auxiliary domain Ω is sought such that it satisfies the boundary conditions on the domain ω. In general, such an optimal control problem leads to an ill posed problem, and, consequently, it may not have a solution.

Using a controllability theorem of J. L. Lions, it is proved here that the solutions of the problems in ω can be approximated within any specified error, however small, by the solutions of the problems in Ω for appropriate values of the boundary conditions. In section 3, it is shown that the optimal control problem has a unique solution in a finite dimensional space. Consequently, considering a family of finite dimensional subspaces with their union dense in the whole space of controls, we can approximate the solution of the problem in ω with the solutions of the problems in Ω using finite dimensional optimal boundary control problems. Since the values of the solutions in Ω are approximately calculated on the boundary of the domain ω, we study the optimal control problem with boundary observations in a finite dimensional subspace in section 4. In section 5, we extend the results obtained for the interior problems to the exterior problems. In section 6, we give some numerical examples for both bounded and unbounded domains. The numerical results are presented to show the validity and high accuracy of the method. Finally, in section 7 we provide some concluding remarks. There is still a large room for further improvement and numerical tests. In future works, we will apply this method in conjunction with fast algorithms (see [4], [7], [8]) to solve other elliptic problems in complicated domains.

2. Controllability. Let $\omega,\ \Omega \in \mathcal{N}^{(1),1}$ (i.e., the maps defining the boundaries of the domains and their derivatives are Lipschitz continuous) be two bounded domains in $\mathbf{R}^N$ such that $\bar{\omega} \subset \Omega$. Their boundaries are denoted by γ and Γ, respectively.

In this paper, we use domain embedding and the optimal boundary control approach to solve the elliptic equation

$$Ay = f \quad \text{in } \omega, \tag{2.1}$$

subject to either Dirichlet boundary conditions

$$y = g_\gamma \quad \text{on } \gamma \tag{2.2}$$

or Neumann boundary conditions

$$\frac{\partial y}{\partial n_{A(\omega)}} = h_\gamma \quad \text{on } \gamma, \tag{2.3}$$

where $\frac{\partial}{\partial n_{A(\omega)}}$ is the outward conormal derivative associated with A.

We assume that the operator A is of the form

$$A = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) a_0$$

with $a_{ij} \in C^{(1),1}(\bar{\Omega})$, $a_0 \in C^{(0),1}(\bar{\Omega})$, $a_0 \geq 0$ in Ω, and there exists a constant $c > 0$ such that $\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geq c(\xi_1^2 + \cdots + \xi_N^2)$ in Ω for any $(\xi_1, \ldots, \xi_N) \in \mathbf{R}^N$. Also, we assume that $f \in L^2(\Omega)$, $g_\gamma \in L^2(\gamma)$, and $h_\gamma \in H^{-1}(\gamma)$.

For later use, we define the following. A function $y \in H^{1/2}(\omega)$ is called a solution of the Dirichlet problem (2.1)–(2.2) if it satisfies (2.1) in the sense of distributions and the boundary conditions (2.2) in the sense of traces in $L^2(\gamma)$. A function $y \in H^{1/2}(\omega)$ is called a solution of the Neumann problem (2.1), (2.3) if it satisfies (2.1) in the sense of distributions and the boundary conditions (2.3) in the sense of traces in $H^{-1}(\gamma)$ (see [27, Chap. 2, section 7]).

The Dirichlet problem (2.1)–(2.2) has a unique solution which depends continuously on the data

$$|y|_{H^{1/2}(\omega)} \leq C\{|f|_{L^2(\omega)} + |g_\gamma|_{L^2(\gamma)}\}. \tag{2.4}$$

If there exists a constant $c_0 > 0$ such that $a_0 \geq c_0$ in ω, then the Neumann problem (2.1), (2.3) has a unique solution which depends continuously on the data

$$|y|_{H^{1/2}(\omega)} \leq C\{|f|_{L^2(\omega)} + |h_\gamma|_{H^{-1}(\gamma)}\}. \tag{2.5}$$

If $a_0 = 0$ in ω, then the Neumann problem (2.1), (2.3) has a solution if

$$\int_\omega f + \int_\gamma h_\gamma = 0. \tag{2.6}$$

In this case, the problem has a unique solution in $H^{1/2}(\omega)/\mathbf{R}$ and

$$\inf_{r\in\mathbf{R}} |y + r|_{H^{1/2}(\omega)} \leq C\{|f|_{L^2(\omega)} + |h_\gamma|_{H^{-1}(\gamma)}\}. \tag{2.7}$$

We also remark that the solution of problem (2.1)–(2.2) can be viewed (see [27, Chap. 2, section 6]) as the solution of the problem

$$\begin{aligned} &y \in H^{1/2}(\omega) : \ \textstyle\int_\omega yA^*\psi = \int_\omega f\psi - \int_\gamma g_\gamma \frac{\partial\psi}{\partial n_{A^*(\omega)}} \\ &\text{for any } \psi \in H^2(\omega),\ \psi = 0 \text{ on } \gamma, \end{aligned} \tag{2.8}$$

and that a solution of problem (2.1), (2.3) is also solution of the problem

$$\begin{aligned} &y \in H^{1/2}(\omega) : \ \textstyle\int_\omega yA^*\psi = \int_\omega f\psi + \int_\gamma h_\gamma\psi \\ &\text{for any } \psi \in H^2(\omega),\ \frac{\partial\psi}{\partial n_{A^*(\omega)}} = 0 \text{ on } \gamma, \end{aligned} \tag{2.9}$$

where A^* is the adjoint operator of A given by

$$A^* = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ji}\frac{\partial}{\partial x_j}\right) + a_0.$$

Evidently, the above results also hold for problems in the domain Ω.

We consider in the following only the cases in which the above problems have unique solutions, i.e., the Dirichlet problems, and we assume in the case of the Neumann problems that there exists a constant $c_0 > 0$ such that $a_0 \geq c_0$ in Ω.

Below we use the notations and the notions of optimal control from Lions [26]. First, we study the controllability of the solutions of the above two problems (defined by (2.1)–(2.3)) in ω with the solutions of a Dirichlet problem in Ω. Let

$$\mathcal{U} = L^2(\Gamma) \tag{2.10}$$

be the space of controls. The state of the system for a control $v \in L^2(\Gamma)$ is given by the solution $y(v) \in H^{1/2}(\Omega)$ of the following Dirichlet problem:

$$\begin{aligned} Ay(v) &= f \quad \text{in} \quad \Omega,\\ y(v) &= v \quad \text{on} \quad \Gamma. \end{aligned} \tag{2.11}$$

In the case of the Dirichlet problem (2.1)–(2.2), the space of observations is taken to be

$$\mathcal{H} = L^2(\gamma), \tag{2.12}$$

and the cost function is given by

$$J(v) = \frac{1}{2}|y(v) - g_\gamma|^2_{L^2(\gamma)}, \tag{2.13}$$

where $v \in L^2(\Gamma)$ and $y(v)$ is the solution of problem (2.11). For the Neumann problem given by (2.1) and (2.3), the space of observations is taken to be

$$\mathcal{H} = H^{-1}(\gamma), \tag{2.14}$$

and the cost function is given by

$$J(v) = \frac{1}{2}\left|\frac{\partial y(v)}{\partial n_A(\omega)} - h_\gamma\right|^2_{H^{-1}(\gamma)}. \tag{2.15}$$

Remark 2.1. Since $y(v) \in H^{1/2}(\Omega)$ and $Ay(u) = f \in L^2(\Omega)$, we have $y(v) \in H^2(D)$ for any domain D which satisfies $\bar{\omega} \subset D \subset \bar{D} \subset \Omega$ (see [30, Chap. 4, section 1.2, Theorem 1.3], for instance). Therefore, $y(v) \in H^{3/2}(\gamma)$ with the same values on both the sides of γ. Also, $\frac{\partial y(v)}{\partial n_A(\omega)} \in H^{1/2}(\gamma)$, $\frac{\partial y(v)}{\partial n_A(\Omega-\bar{\omega})} \in H^{1/2}(\gamma)$, and $\frac{\partial y(v)}{\partial n_A(\omega)} + \frac{\partial y(v)}{\partial n_A(\Omega-\bar{\omega})} = 0$. Consequently, the above two cost functions make sense.

PROPOSITION 2.1. *A control $u \in L^2(\Gamma)$ satisfies $J(u) = 0$, where the control function is given by* (2.13)*, if and only if the solution of* (2.11) *for $v = u$, $y(u) \in H^{1/2}(\Omega)$ satisfies*

$$\begin{aligned} &Ay(u) = f && \text{in } \Omega - \bar{\omega},\\ &y(u) = y && \text{on } \gamma,\\ &\frac{\partial y(u)}{\partial n_A(\Omega-\bar{\omega})} + \frac{\partial y}{\partial n_A(\omega)} = 0 && \text{on } \gamma, \end{aligned} \tag{2.16}$$

and

$$y(u) = y \quad \text{in } \omega, \tag{2.17}$$

where y is the solution of the Dirichlet problem defined by (2.1) *and* (2.2) *in the domain ω. The same result holds if the control function is given by* (2.15) *and y is the solution of the Neumann problem* (2.1) *and* (2.3).

Proof. Let $y(u) \in H^{1/2}(\Omega)$ be the solution of problem (2.11) corresponding to an $u \in L^2(\Gamma)$ such that $J(u) = 0$ with the control function given by (2.13). Consequently, $y(u)$ verifies (2.1) in the sense of distributions and the boundary condition (2.2) in the sense of traces. It gives $y(u) = y$ in ω. Since $y(u)$ satisfies (2.11) in $\Omega - \bar{\omega}$ in the sense of distributions, then, evidently, $y(u)$ is a solution of the equation in (2.16). From (2.17) and Remark 2.1, we obtain that $y(u)$ also satisfies the two boundary conditions of (2.16). The reverse implication is evident.

The same arguments also hold for the Neumann problem defined by (2.1) and (2.3) and the control function given by (2.15). ∎

Since (2.16) is not a properly posed problem, it follows from the above proposition that the optimal control might not exist. However, J. L. Lions proves in [26, Chap. 2, section 5.3, Theorem 5.1] a controllability theorem which can be directly applied to problem (2.11). We mention this theorem below.

LIONS'S CONTROLLABILITY THEOREM. *The set $\{\frac{\partial z_0(v)}{\partial n_A(\Omega-\bar{\omega})} \in H^{-1}(\gamma) : v \in L^2(\Gamma)\}$ is dense in $H^{-1}(\gamma)$, where $z_0(v) \in H^{1/2}(\Omega - \bar{\omega})$ is the solution of the problem*

$$\begin{aligned} Az_0(v) &= 0 && \text{in } \Omega - \bar{\omega}, \\ z_0(v) &= v && \text{on } \Gamma, \\ z_0(v) &= 0 && \text{on } \gamma. \end{aligned}$$

Now, we can easily prove the following lemma.

LEMMA 2.2. *For any $g \in L^2(\gamma)$, the set $\{\frac{\partial z(v)}{\partial n_A(\Omega-\bar{\omega})} \in H^{-1}(\gamma) : v \in L^2(\Gamma)\}$ is dense in $H^{-1}(\gamma)$, where $z(v) \in H^{1/2}(\Omega - \bar{\omega})$ is the solution of the problem*

$$\begin{aligned} Az(v) &= f && \text{in } \Omega - \bar{\omega}, \\ z(v) &= v && \text{on } \Gamma, \\ z(v) &= g && \text{on } \gamma. \end{aligned} \tag{2.18}$$

Proof. Let $z \in H^{1/2}(\Omega - \bar{\omega})$ be the solution of the problem

$$\begin{aligned} Az &= f && \text{in } \Omega - \bar{\omega}, \\ z &= 0 && \text{on } \Gamma, \\ z &= g && \text{on } \gamma. \end{aligned}$$

Using $z_0(v) = z(v) - z$ in the Lions controllability theorem, we get that the set $\{\frac{\partial(z(v)-z)}{\partial n_A(\Omega-\bar{\omega})} \in H^{-1}(\gamma) : v \in L^2(\Gamma)\}$ is dense in $H^{-1}(\gamma)$. Hence the lemma follows. ∎

The following theorem proves controllability of the solutions of problems in ω by the solutions of Dirichlet problems in Ω. In the proof of this theorem below, we use the spaces Ξ^s introduced in Lions and Magenes [27, Chap. 2, section 6.3]. For the sake of completeness, we give definitions of these spaces Ξ^s.

Let $\rho(x)$ be a function in $\mathcal{D}(\bar{\Omega})$ which is positive in Ω and vanishes on Γ. We also assume that for any $x_0 \in \Gamma$, the limit

$$\lim_{x \to x_0 \in \Gamma} \frac{\rho(x)}{d(x, \Gamma)}$$

exists and is positive, where $d(x,\Gamma)$ is the distance from $x \in \Omega$ to the boundary Γ. Then, for $s = 0, 1, 2, \ldots$, the space Ξ^s is defined by

$$\Xi^s(\Omega) = \{u : \rho^{|\alpha|} D^\alpha u \in L^2(\Omega), |\alpha| \le s\}.$$

With the norm

$$||u||_{\Xi^s(\Omega)} = \sum_{|\alpha| \le s} ||\rho^{|\alpha|} D^\alpha u||_{L^2(\Omega)},$$

the space $\Xi^s(\Omega)$ is a Hilbert space, and

$$\Xi^0(\Omega) = L^2(\Omega), \quad H^s(\Omega) \subset \Xi^s(\Omega) \subset L^2(\Omega), \quad \text{and} \quad \mathcal{D}(\Omega) \text{ is dense in } \Xi^s(\Omega).$$

Now, for a positive noninteger real $s = k+\theta$ with k the integer part of s and $0 < \theta < 1$, the space Ξ^s is, as in the case of the spaces H^s, the intermediate space

$$\Xi^s(\Omega) = [\Xi^{k+1}(\Omega), \Xi^k(\Omega)]_{1-\theta}.$$

Finally, for negative real values $-s$, $s > 0$, the space $\Xi^{-s}(\Omega)$ is the dual space of $\Xi^s(\Omega)$, $(\Xi^s(\Omega))'$.

THEOREM 2.3. *The set* $\{y(v)_{|\omega} : v \in L^2(\Gamma)\}$ *is dense, using the norm of* $H^{1/2}(\omega)$, *in* $\{y \in H^{1/2}(\omega) : \ Ay = f \text{ in } \omega\}$, *where* $y(v) \in H^{1/2}(\Omega)$ *is the solution of the Dirichlet problem* (2.11) *for a given* $v \in L^2(\Gamma)$.

Proof. Let us consider $y \in H^{1/2}(\omega)$ such that $Ay = f$ in ω, and a real number $\varepsilon > 0$. We denote the traces of y on γ by $y = g \in L^2(\gamma)$ and $\frac{\partial y}{\partial n_A(\omega)} = h \in H^{-1}(\gamma)$. From the previous lemma, it follows that there exists $v_\varepsilon \in L^2(\Gamma)$ such that the solution $z(v_\varepsilon) \in H^{1/2}(\Omega - \bar{\omega})$ of problem (2.18) satisfies

$$\left| \frac{\partial z(v_\varepsilon)}{\partial n_A(\Omega - \bar{\omega})} + h \right|_{H^{-1}(\gamma)} < \varepsilon.$$

Let $y(v_\varepsilon)$ be the solution of the Dirichlet problem (2.11) corresponding to v_ε, and let us define

$$y_\varepsilon = \begin{cases} y & \text{on } \omega, \\ z(v_\varepsilon) & \text{on } \Omega - \bar{\omega}. \end{cases}$$

Then $(y(v_\varepsilon) - y_\varepsilon) \in H^{1/2}(\Omega)$ and satisfies in the sense of distributions the equation

$$A(y(v_\varepsilon) - y_\varepsilon) = \tfrac{\partial z(v_\varepsilon)}{\partial n_A(\Omega - \bar{\omega})} + h \quad \text{in } \Omega$$

and the boundary conditions

$$y(v_\varepsilon) - y_\varepsilon = 0 \quad \text{on } \Gamma.$$

Consider, as in Remark 2.1, a fixed domain D such that $\bar{\omega} \subset D \subset \bar{D} \subset \Omega$. Then, for any $\psi \in \mathcal{D}(\Omega)$, we have $\int_\Omega A(y(v_\varepsilon) - y_\varepsilon)\psi = \int_\gamma (\frac{\partial z(v_\varepsilon)}{\partial n_A(\Omega - \bar{\omega})} + h)\psi \le |\frac{\partial z(v_\varepsilon)}{\partial n_A(\Omega - \bar{\omega})} + h|_{H^{-1}(\gamma)} |\psi|_{H^1(\gamma)} \le C(D)|\psi|_{H^{3/2}(D)}\varepsilon \le C(D)|\psi|_{\Xi^{3/2}(\Omega)}\varepsilon$, where $C(D)$ depends only on the domain D. Therefore,

$$|A(y(v_\varepsilon) - y_\varepsilon)|_{\Xi^{-3/2}(\Omega)} \le C(D)\varepsilon.$$

Taking into account the continuity of the solution on the data (see Lions and Magenes [27, Chap. 2, section 7.3, Theorem 7.4]), we get

$$|y(v_\varepsilon) - y_\varepsilon|_{H^{1/2}(\Omega)} \le C(D)\varepsilon. \qquad \square$$

Below, the controllability of the solutions of the Dirichlet and the Neumann problems (given by (2.1), (2.2), and (2.1), (2.3), respectively) in ω by Neumann problems in Ω is discussed.

Now as a set of controls we can take the space

$$\mathcal{U} = H^{-1}(\Gamma), \tag{2.19}$$

and for a $v \in H^{-1}(\Gamma)$, the state of the system is the solution $y(v) \in H^{1/2}(\Omega)$ of the problem

$$\begin{aligned} Ay(v) &= f \quad \text{in} \quad \Omega, \\ \tfrac{\partial y(v)}{\partial n_A(\Omega)} &= v \quad \text{on} \quad \Gamma. \end{aligned} \tag{2.20}$$

We remark that

$$\begin{aligned} &i : \{y(v) \in H^{1/2}(\Omega) : v \in L^2(\Gamma),\ y(v) \text{ solution of problem (2.11)}\} \to \\ &\qquad \{y(w) \in H^{1/2}(\Omega) : w \in H^{-1}(\Gamma),\ y(w) \text{ solution of problem (2.20)}\}, \\ &i(y(v)) = y(w) \Leftrightarrow y(v) = y(w) \text{ in } \Omega \end{aligned} \tag{2.21}$$

establish a bijective correspondence. Consequently, Proposition 2.1 also holds if the space of controls there is changed to $H^{-1}(\Gamma)$ and the states $y(v)$ of the system are solutions of problem (2.20). Theorem 2.3 in this case becomes the following theorem.

THEOREM 2.4. *The set $\{y(v)_{|\omega} : v \in H^{-1}(\Gamma)\}$ is dense, using the norm of $H^{1/2}(\omega)$, in $\{y \in H^{1/2}(\omega) : Ay = f \text{ in } \omega\}$, where $y(v) \in H^{1/2}(\Omega)$ is a solution of the Neumann problem* (2.20) *for a given $v \in H^{-1}(\Gamma)$.*

3. Controllability with finite dimensional spaces. Let $\{U_\lambda\}_\lambda$ be a family of finite dimensional subspaces of the space $L^2(\Gamma)$ such that, given (2.10) as a space of controls with the Dirichlet problems, we have

$$\bigcup_\lambda U_\lambda \text{ is dense in } \mathcal{U} = L^2(\Gamma). \tag{3.1}$$

For a $v \in L^2(\Gamma)$ we consider the solution $y'(v) \in H^{1/2}(\Omega)$ of the problem

$$\begin{aligned} Ay'(v) &= 0 \quad \text{in } \Omega, \\ y'(v) &= v \quad \text{on } \Gamma. \end{aligned} \tag{3.2}$$

We fix a U_λ. The cost functions J defined by (2.13) and (2.15) are differentiable and convex. Consequently, an optimal control

$$u_\lambda \in U_\lambda : J(u_\lambda) = \inf_{v \in U_\lambda} J(v) \tag{3.3}$$

exists if and only if it is a solution of the equation

$$u_\lambda \in U_\lambda : (y(u_\lambda), y'(v))_{L^2(\gamma)} = (g_\gamma, y'(v))_{L^2(\gamma)} \text{ for any } v \in U_\lambda, \tag{3.4}$$

when the control function is (2.13), and

$$u_\lambda \in U_\lambda : \left(\frac{\partial y(u_\lambda)}{\partial n_A(\omega)}, \frac{\partial y'(v)}{\partial n_A(\omega)}\right)_{H^{-1}(\gamma)} = \left(h_\gamma, \frac{\partial y'(v)}{\partial n_A(\omega)}\right)_{H^{-1}(\gamma)} \text{ for any } v \in U_\lambda, \tag{3.5}$$

when the control function is (2.15). Above, $y(u_\lambda)$ is the solution of problem (2.11) corresponding to u_λ, and $y'(v)$ is the solution of problem (3.2) corresponding to v. If $y_f \in H^2(\Omega)$ is the solution of the problem

$$\begin{aligned} Ay_f &= f \quad &\text{in } \Omega,\\ y_f &= 0 \quad &\text{on } \Gamma, \end{aligned} \tag{3.6}$$

then, for a $v \in L^2(\Gamma)$, we have

$$y(v) = y'(v) + y_f, \tag{3.7}$$

where $y(v)$ and $y'(v)$ are the solutions of problems (2.11) and (3.2), respectively. Therefore, we can rewrite problems (3.4) and (3.5) as

$$u_\lambda \in U_\lambda : \ (y'(u_\lambda), y'(v))_{L^2(\gamma)} = (g_\gamma - y_f, y'(v))_{L^2(\gamma)} \tag{3.8}$$

for any $v \in U_\lambda$, and

$$u_\lambda \in U_\lambda : \ \left(\frac{\partial y'(u_\lambda)}{\partial n_A(\omega)}, \frac{\partial y'(v)}{\partial n_A(\omega)}\right)_{H^{-1}(\gamma)} = \left(h_\gamma - \frac{\partial y_f}{\partial n_A(\omega)}, \frac{\partial y'(v)}{\partial n_A(\omega)}\right)_{H^{-1}(\gamma)} \tag{3.9}$$

for any $v \in U_\lambda$, respectively. Next, we prove the following lemma.

LEMMA 3.1. *For a fixed λ, let $\varphi_1, \ldots, \varphi_{n_\lambda}$, $n_\lambda \in \mathbf{N}$, be a basis of U_λ, and let $y'(\varphi_i)$ be the solution of problem* (3.2) *for $v = \varphi_i$, $i = 1, \ldots, n_\lambda$. Then $\{y'(\varphi_1)_{|\gamma}, \ldots, y'(\varphi_{n_\lambda})_{|\gamma}\}$ and $\{\frac{\partial y'(\varphi_1)}{\partial n_A(\omega)}|\gamma, \ldots, \frac{\partial y'(\varphi_{n_\lambda})}{\partial n_A(\omega)}|\gamma\}$ are linearly independent sets.*

Proof. From Remark 2.1, we have $y'(v) \in H^2(D)$ for any domain D which satisfies $\bar\omega \subset D \subset \bar D \subset \Omega$, and, consequently, $y'(v) \in H^{3/2}(\gamma)$ for any $v \in L^2(\Gamma)$. Assume that for $\xi_1, \ldots, \xi_{n_\lambda} \in \mathbf{R}$ we have $\xi_1 y'(\varphi_1) + \cdots + \xi_{n_\lambda} y'(\varphi_{n_\lambda}) = 0$ on γ. Then

$$y'(\xi_1\varphi_1 + \cdots + \xi_{n_\lambda}\varphi_{n_\lambda}) = 0 \quad \text{on } \gamma, \tag{3.10}$$

and therefore, $y'(\xi_1\varphi_1 + \cdots + \xi_{n_\lambda}\varphi_{n_\lambda}) = 0$ on ω. This implies that

$$\tfrac{\partial y'(\xi_1\varphi_1 + \cdots + \xi_{n_\lambda}\varphi_{n_\lambda})}{\partial n_A(\Omega - \bar\omega)} = 0 \quad \text{on } \gamma. \tag{3.11}$$

From (3.10) and (3.11), we get $y'(\xi_1\varphi_1 + \cdots + \xi_{n_\lambda}\varphi_{n_\lambda}) = 0$ on $\Omega - \bar\omega$, and therefore, $\xi_1\varphi_1 + \cdots + \xi_{n_\lambda}\varphi_{n_\lambda} = y'(\xi_1\varphi_1 + \cdots + \xi_{n_\lambda}\varphi_{n_\lambda}) = 0$ on Γ, or $\xi_1 = \cdots = \xi_{n_\lambda} = 0$. The second part of the statement can be proved using similar arguments. $\square$

The following proposition proves the existence and uniqueness of the optimal control when the states of the system are the solutions of the Dirichlet problems.

PROPOSITION 3.2. *Let us consider a fixed U_λ. Then problems* (3.8) *and* (3.9) *have unique solutions. Consequently, if the boundary conditions of Dirichlet problems* (2.11) *lie in the finite dimensional space U_λ, then there exists a unique optimal control of problem* (3.3) *corresponding to either the Dirichlet problem* (2.1), (2.2) *or the Neumann problem* (2.1), (2.3).

Proof. For a given λ, let V_λ denote the subspace of $L^2(\gamma)$ generated by $\{y'(\varphi_i)_{|\gamma}\}_{1\le i\le n_\lambda}$, where $\{\varphi_i\}_{1\le i\le n_\lambda}$ is a basis of U_λ, and $y'(\varphi_i)$ is the solution of problem (3.2) with $v = \varphi_i$. Since the norms $|\xi_1\varphi_1 + \cdots + \xi_{n_\lambda}\varphi_{n_\lambda}|_{L^2(\Gamma)}$ in U_λ, and $|\xi_1 y'(\varphi_1) + \cdots + \xi_{n_\lambda} y'(\varphi_{n_\lambda})|_{L^2(\gamma)}$ in V_λ are equivalent to the norm $(\xi_1^2 + \cdots + \xi_{n_\lambda}^2)^{1/2}$, the above lemma then implies that there exist two positive constants c and C such that

$$c|v|_{L^2(\Gamma)} \le |y'(v)|_{L^2(\gamma)} \le C|v|_{L^2(\Gamma)} \text{ for any } v \in U_\lambda.$$

Consequently, from the Lax–Milgram lemma we get that (3.8) has a unique solution. A similar reasoning proves that (3.9) also has a unique solution. This time we use the norm equivalence

$$c|v|_{L^2(\Gamma)} \leq \left| \frac{\partial y'(v)}{\partial n_A(\Omega - \bar{\omega})} \right|_{H^{-1}(\gamma)} \leq C|v|_{L^2(\Gamma)} \text{ for any } v \in U_\lambda$$

in the Lax–Milgram lemma. □

The following theorem proves the controllability of the solutions of the Dirichlet and Neumann problems in ω by the solutions of the Dirichlet problems in Ω.

THEOREM 3.3. *Let $\{U_\lambda\}_\lambda$ be a family of finite dimensional spaces satisfying* (3.1)*. We associate the solution y of the Dirichlet problem* (2.1), (2.2) *in ω with problem* (3.3), *in which the cost function is given by* (2.13)*. Also, the solution y of the Neumann problem* (2.1), (2.3) *is associated with problem* (3.3), *in which the cost function is given by* (2.15)*. In both cases, there exists a positive constant C, and for any given $\varepsilon > 0$ there exists U_{λ_ε} such that*

$$|y(u_{\lambda_\varepsilon})_{|\omega} - y|_{H^{1/2}(\omega)} < C\varepsilon,$$

where $u_{\lambda_\varepsilon} \in U_{\lambda_\varepsilon}$ is the optimal control of the corresponding problem (3.3) *with $\lambda = \lambda_\varepsilon$, and $y(u_{\lambda_\varepsilon})$ is the solution of problem* (2.11) *with $v = u_{\lambda_\varepsilon}$.*

Proof. Let us consider an $\varepsilon > 0$ and $y \in H^{1/2}(\omega)$ as the solution of problem (2.1), (2.2). From Theorem 2.3, there exists $v_\varepsilon \in L^2(\Gamma)$ such that $y(v_\varepsilon) \in H^{1/2}(\Omega)$, the solution of problem (2.11) with $v = v_\varepsilon$, satisfies $|y - y(v_\varepsilon)_{|\omega}|_{H^{1/2}(\omega)} < \varepsilon$. Consequently, there exists a constant C_1 such that

$$|g_\gamma - y(v_\varepsilon)|_{L^2(\gamma)} < C_1\varepsilon. \tag{3.12}$$

Since $\cup_\lambda U_\lambda$ is dense in $L^2(\Gamma)$, there exist λ_ε and $v_{\lambda_\varepsilon} \in U_{\lambda_\varepsilon}$ such that $|v_\varepsilon - v_{\lambda_\varepsilon}|_{L^2(\Gamma)} < \varepsilon$, and then there exists a positive constant C_2 such that

$$|y(v_\varepsilon) - y(v_{\lambda_\varepsilon})|_{L^2(\gamma)} < C_2\varepsilon. \tag{3.13}$$

From (3.12) and (3.13) we get

$$|g_\gamma - y(v_{\lambda_\varepsilon})|_{L^2(\gamma)} < C_3\varepsilon$$

and, consequently,

$$|g_\gamma - y(u_{\lambda_\varepsilon})|_{L^2(\gamma)} < C_4\varepsilon,$$

where $u_{\lambda_\varepsilon} \in L^2(\Gamma)$ is the unique optimal control of problem (3.3) on U_{λ_ε} with the cost function given by (2.13). Therefore,

$$|y(u_{\lambda_\varepsilon})_{|\omega} - y|_{H^{1/2}(\omega)} < C\varepsilon.$$

A similar reasoning can be made for the solution $y \in H^{1/2}(\omega)$ of problem (2.1), (2.3). □

Using the basis $\varphi_1, \ldots, \varphi_{n_\lambda}$ of the space U_λ, we define the matrix

$$\Pi_\lambda = ((y'(\varphi_i), y'(\varphi_j))_{L^2(\gamma)})_{1 \leq i,j \leq n_\lambda} \tag{3.14}$$

and the vector

$$l_\lambda = ((g_\gamma - y_f, y'(\varphi_i))_{L^2(\gamma)})_{1 \leq i \leq n_\lambda}. \tag{3.15}$$

Then problem (3.8) can be written as

$$\xi_\lambda = (\xi_{\lambda,1}, \dots, \xi_{\lambda,n_\lambda}) \in \mathbf{R}^{n_\lambda} : \Pi_\lambda \xi_\lambda = l_\lambda. \tag{3.16}$$

Consequently, using Theorem 3.3, the solution y of problem (2.1), (2.2) can be obtained within any prescribed error by setting the restriction to ω of

$$y(u_\lambda) = \xi_{\lambda,1} y'(\varphi_1) + \cdots + \xi_{\lambda,n_\lambda} y'(\varphi_{n_\lambda}) + y_f, \tag{3.17}$$

where $\xi_\lambda = (\xi_{\lambda,1}, \dots, \xi_{\lambda,n_\lambda})$ is the solution of algebraic system (3.16). Above, y_f is the solution of problem (3.6), and $y'(\varphi_i)$ are the solutions of problems (3.2) with $v = \varphi_i$, $i = 1, \dots, n_\lambda$.

An algebraic system (3.16) is also obtained in the case of problem (3.9). This time the matrix of the system is given by

$$\Pi_\lambda = \left(\left(\frac{\partial y'(\varphi_i)}{\partial n_A(\omega)}, \frac{\partial y'(\varphi_j)}{\partial n_A(\omega)} \right)_{H^{-1}(\gamma)} \right)_{1 \le i,j \le n_\lambda}, \tag{3.18}$$

and the free term is

$$l_\lambda = \left(\left(h_\gamma - \frac{\partial y_f}{\partial n_A(\omega)}, \frac{\partial y'(\varphi_i)}{\partial n_A(\omega)} \right)_{H^{-1}(\gamma)} \right)_{1 \le i \le n_\lambda}. \tag{3.19}$$

Therefore, using Theorem 3.3, the solution y of problem (2.1), (2.3) can be estimated by (3.17). Also, y_f is the solution of problem (3.6), and $y'(\varphi_i)$ are the solutions of problems (3.2) with $v = \varphi_i$, $i = 1, \dots, n_\lambda$.

The case of the controllability with finite dimensional optimal controls for states of the system given by the solution of a Neumann problem is treated in a similar way. As in the previous section, the space of the controls is $\mathcal{U}$, given in (2.19), and the state of the system $y(v) \in H^{1/2}(\Omega)$ is given by the solution of Neumann problem (2.20) for a $v \in H^{-1}(\Gamma)$.

Let $\{U_\lambda\}_\lambda$ be a family of finite dimensional subspaces of the space $H^{-1}(\Gamma)$ such that

$$\bigcup_\lambda U_\lambda \text{ is dense in } \mathcal{U} = H^{-1}(\Gamma). \tag{3.20}$$

This time, the function $y'(v) \in H^{1/2}(\Omega)$ appearing in (3.4), (3.5), (3.8), and (3.9) is the solution of the problem

$$\begin{aligned} Ay'(v) &= 0 && \text{in } \Omega, \\ \tfrac{\partial y'(v)}{\partial n_A(\Omega)} &= v && \text{on } \Gamma \end{aligned} \tag{3.21}$$

for a $v \in H^{-1}(\Gamma)$. Also, $y_f \in H^2(\Omega)$ appearing in (3.7), (3.8), and (3.9) is the solution of the problem

$$\begin{aligned} Ay_f &= f && \text{in } \Omega, \\ \tfrac{\partial y_f}{\partial n_A(\Omega)} &= 0 && \text{on } \Gamma. \end{aligned} \tag{3.22}$$

With these changes, Lemma 3.1 also holds in this case, and the proof of the following proposition is similar to that of Proposition 3.2.

PROPOSITION 3.4. *For a given U_λ, the problems* (3.8) *and* (3.9) *have unique solutions. Consequently, if the boundary conditions of Neumann problems* (2.20) *lie in the finite dimensional space U_λ, then there exists a unique optimal control of problem* (3.3)*, corresponding to either Dirichlet problem* (2.1), (2.2) *or Neumann problem* (2.1), (2.3)*.*

A proof similar to that given for Theorem 3.3 can also be given for the following theorem.

THEOREM 3.5. *Let $\{U_\lambda\}_\lambda$ be a family of finite dimensional spaces satisfying* (3.20)*. We associate the solution $y \in H^{1/2}(\omega)$ of problem* (2.1), (2.2) *with problem* (3.3), *in which the cost function is given by* (2.13)*. Also, the solution y of problem* (2.1), (2.3) *is associated with problem* (3.3), *in which the cost function is given by* (2.15)*. In both cases, there exists a positive constant C, and for any given $\varepsilon > 0$ there exists λ_ε such that*

$$|y(u_{\lambda_\varepsilon})_{|\omega} - y|_{H^{1/2}(\omega)} < C\varepsilon,$$

where $u_{\lambda_\varepsilon} \in U_{\lambda_\varepsilon}$ is the optimal control of the corresponding problem (3.3) *with $\lambda = \lambda_\varepsilon$, and $y(u_{\lambda_\varepsilon})$ is the solution of problem* (2.20) *with $v = u_{\lambda_\varepsilon}$.*

Evidently, in the case of the controllability with solutions of Neumann problem (2.20) we can also write algebraic systems (3.16) using a basis $\varphi_1, \dots, \varphi_{n_\lambda}$ of a given subspace U_λ of the space $\mathcal{U} = H^{-1}(\Gamma)$. As in the case of the controllability with solutions of the Dirichlet problem (2.11), these algebraic systems have unique solutions.

Theorems 3.3 and 3.5 prove the convergence of the embedding method associated with the optimal boundary control. An error analysis would be desirable, but it would go beyond the scope of this paper.

Remark 3.1. We have defined y_f as a solution of problems (3.6) or (3.22) in order to have $y(v) = y'(v) + y_f$ or $\frac{\partial y(v)}{\partial n_A(\Omega)} = \frac{\partial y'(v)}{\partial n_A(\Omega)} + \frac{\partial y_f}{\partial n_A(\Omega)}$, respectively, on the boundary Γ. In fact, we can replace $y(v)$ by $y'(v) + y_f$ in the cost functions (2.13) and (2.15) with $y_f \in H^2(\Omega)$ satisfying only

$$Ay_f = f \text{ in } \Omega, \tag{3.23}$$

and the results obtained in this section still hold.

Indeed, the two sets $\{y(v) = y'(v) + y_f \in H^{1/2}(\Omega) : v \in L^2(\Gamma)\}$ corresponding to y_f given by (3.23) and (3.6), $y'(v)$ being the solution of (3.2), are identical to the set $\{y(v) \in H^{1/2}(\Omega) : v \in L^2(\Gamma)\}$, $y(v)$ being the solution of (2.11). Also, the two sets $\{y(v) = y'(v) + y_f \in H^{1/2}(\Omega) : v \in H^{-1}(\Gamma)\}$ corresponding to y_f given by (3.23) and (3.22), $y'(v)$ being the solution of (3.21), are identical to the set $\{y(v) \in H^{1/2}(\Omega) : v \in H^{-1}(\Gamma)\}$, $y(v)$ being the solution of (2.20).

4. Approximate observations in finite dimensional spaces. In solving problems (3.8), (3.9), we require an appropriate interpolation which makes use of the values of $y'(v)$ computed only at some points on the boundary γ. We show below that using these interpolations, i.e., observations in finite dimensional subspaces, we can obtain the approximate solutions of problems (2.1), (2.2) and (2.1), (2.3).

As in the previous sections, we first deal with the case when the states of the system are given by the Dirichlet problem (2.11). Let U_λ be a fixed finite dimensional subspace of $\mathcal{U} = L^2(\Gamma)$ with the basis $\varphi_1, \dots, \varphi_{n_\lambda}$.

Let us assume that for problem (2.1), (2.2), we choose a family of finite dimensional spaces $\{H_\mu\}_\mu$ such that

$$\bigcup_\mu H_\mu \text{ is dense in } \mathcal{H} = L^2(\gamma). \tag{4.1}$$

Similarly, for problem (2.1), (2.3) we choose the finite dimensional spaces $\{H_\mu\}_\mu$ such that

$$\bigcup_\mu H_\mu \text{ is dense in } \mathcal{H} = H^{-1}(\gamma). \tag{4.2}$$

The subspace H_μ given in (4.1) and (4.2) is a subspace of $\mathcal{H}$ given in (2.12) and (2.14), respectively.

An appropriate choice of H_μ is made based on the problem to be solved as discussed above. For a given φ_i, $i = 1, \ldots, n_\lambda$, we consider below the solution $y'(\varphi_i)$ of problem (3.2) corresponding to $v = \varphi_i$, and we approximate its trace on γ by $y'_{\mu,i}$. Also, the approximation of $\frac{\partial y'(\varphi_i)}{\partial n_A(\omega)}$ on γ is denoted by $\frac{\partial y'_{\mu,i}}{\partial n_A(\omega)}$.

Since the system (3.16) has a unique solution, the determinants of the matrices Π_λ given in (3.14) and (3.18) are nonzero. Consequently, if $|y'(\varphi_i) - y'_{\mu,i}|_{L^2(\gamma)}$ or $|\frac{\partial y'(\varphi_i)}{\partial n_A(\omega)} - \frac{\partial y'_{\mu,i}}{\partial n_A(\omega)}|_{H^{-1}(\gamma)}$ are small enough, then the matrices

$$\Pi_{\lambda\mu} = ((y'_{\mu,i}, y'_{\mu,j})_{L^2(\gamma)})_{1\le i,j\le n_\lambda} \tag{4.3}$$

and

$$\Pi_{\lambda\mu} = \left(\left(\frac{\partial y'_{\mu,i}}{\partial n_A(\omega)}, \frac{\partial y'_{\mu,j}}{\partial n_A(\omega)}\right)_{H^{-1}(\gamma)}\right)_{1\le i,j\le n_\lambda} \tag{4.4}$$

have nonzero determinants. In this case, each of the algebraic systems

$$\xi_{\lambda\mu} = (\xi_{\lambda\mu,1}, \ldots, \xi_{\lambda\mu,n_\lambda}) \in \mathbf{R}^{n_\lambda} : \Pi_{\lambda\mu}\xi_{\lambda\mu} = l_{\lambda\mu} \tag{4.5}$$

has a unique solution. In this system, the free term is

$$l_{\lambda\mu} = ((g_{\gamma\mu} - y_{f\mu}, y'_{\mu,i})_{L^2(\gamma)})_{1\le i\le n_\lambda} \tag{4.6}$$

if the matrix $\Pi_{\lambda\mu}$ is given by (4.3) and

$$l_{\lambda\mu} = \left(\left(h_{\gamma\mu} - \frac{\partial y_{f\mu}}{\partial n_A(\omega)}, \frac{\partial y'_{\mu,i}}{\partial n_A(\omega)}\right)_{H^{-1}(\gamma)}\right)_{1\le i\le n_\lambda} \tag{4.7}$$

if the matrix $\Pi_{\lambda\mu}$ is given by (4.4). Above, we have denoted by $g_{\gamma\mu}$ and $h_{\gamma\mu}$ some approximations in H_μ of g_γ and h_γ, respectively. Also, $y_{f\mu}$ and $\frac{\partial y_{f\mu}}{\partial n_A(\omega)}$ are some approximations of y_f and $\frac{\partial y_f}{\partial n_A(\omega)}$ in the corresponding H_μ of $L^2(\gamma)$ and $H^{-1}(\gamma)$, respectively, with $y_f \in H^2(\Omega)$ satisfying (3.23).

The solution y of problems (2.1), (2.2) and (2.1), (2.3) can be approximated with the restriction to ω of

$$y(u_{\lambda\mu}) = \xi_{\lambda\mu,1} y'(\varphi_1) + \cdots + \xi_{\lambda\mu,n_\lambda} y'(\varphi_{n_\lambda}) + y_f, \tag{4.8}$$

where $\xi_\lambda = (\xi_{\lambda\mu,1}, \ldots, \xi_{\lambda\mu,n_\lambda})$ is the solution of appropriate algebraic system (4.5).

For a vector, $\xi = (\xi_1, \ldots, \xi_{n_\lambda})$, we use the norm $|\xi| = \max_{1\le i\le n_\lambda} |\xi_i|$, and the corresponding matrix norm is denoted by $||\cdot||$. From (3.17) and (4.8) we have

$$|y(u_\lambda) - y(u_{\lambda\mu})|_{H^{1/2}(\omega)} \le C_\lambda |\xi_\lambda - \xi_{\lambda\mu}|, \tag{4.9}$$

where C_λ depends only on the basis in U_λ. From

$$||\Pi_\lambda^{-1} - \Pi_{\lambda\mu}^{-1}|| \leq \frac{||\Pi_\lambda^{-1}|| ||\Pi_\lambda - \Pi_{\lambda\mu}||}{1/||\Pi_\lambda^{-1}|| - ||\Pi_\lambda - \Pi_{\lambda\mu}||}$$

and algebraic systems (3.16) and (4.5), we have $\xi_\lambda = \Pi_\lambda^{-1} l_\lambda$ and $\xi_{\lambda\mu} = \Pi_{\lambda\mu}^{-1} l_{\lambda\mu}$ and we get that there exists $C_\lambda > 0$, depending on the basis in U_λ, such that

$$|\xi_\lambda - \xi_{\lambda\mu}| \leq C_\lambda \left(||\Pi_\lambda - \Pi_{\lambda\mu}|| + |l_\lambda - l_{\lambda\mu}| \right). \tag{4.10}$$

In the case of matrices (3.14) and (4.3) and the free terms (3.15) and (4.6), we have

$$\begin{aligned} ||\Pi_\lambda - \Pi_{\lambda\mu}|| &\leq C_\lambda \max_{1\leq i\leq n_\lambda} |y'(\varphi_i) - y'_{\mu,i}|_{L^2(\gamma)}, \\ |l_\lambda - l_{\lambda\mu}| &\leq C_\lambda \big(|g_\gamma - g_{\gamma\mu}|_{L^2(\gamma)} \\ &\quad + |y_f - y_{f\mu}|_{L^2(\gamma)} \big) + C \max_{1\leq i\leq n_\lambda} |y'(\varphi_i) - y'_{\mu,i}|_{L^2(\gamma)}. \end{aligned} \tag{4.11}$$

Instead, if we take matrices (3.18) and (4.4) and the free terms (3.19) and (4.7), then we get

$$\begin{aligned} ||\Pi_\lambda - \Pi_{\lambda\mu}|| &\leq C_\lambda \max_{1\leq i\leq n_\lambda} \left| \frac{\partial y'(\varphi_i)}{\partial n_A(\omega)} - \frac{\partial y'_{\mu,i}}{\partial n_A(\omega)} \right|_{H^{-1}(\gamma)}, \\ |l_\lambda - l_{\lambda\mu}| &\leq C_\lambda \left(|h_\gamma - h_{\gamma\mu}|_{H^{-1}(\gamma)} + \left| \frac{\partial y_f}{\partial n_A(\omega)} - \frac{\partial y_{f\mu}}{\partial n_A(\omega)} \right|_{H^{-1}(\gamma)} \right) \\ &\quad + C \max_{1\leq i\leq n_\lambda} |\frac{\partial y'(\varphi_i)}{\partial n_A(\omega)} - \frac{\partial y'_{\mu,i}}{\partial n_A(\omega)}|_{H^{-1}(\gamma)}, \end{aligned} \tag{4.12}$$

where C is a constant and C_λ depends on the basis in U_λ.

For states of the system given by the Neumann problem (2.20), U_λ is a subspace of $\mathcal{U} = H^{-1}(\Gamma)$. The material presented above for the case of the Dirichlet problems in Ω is applicable to the case of the Neumann problems in Ω, except for the difference that this time $y'(\varphi_i)$ are the solutions of problems (3.21) with $v = \varphi_i$, $i = 1, \ldots, n_\lambda$.

In both cases (i.e., when the control is affected via Dirichlet and Neumann problems), we obtain the following theorem from Theorems 3.3 and 3.5 and (4.9)–(4.12).

THEOREM 4.1. *Let $\{U_\lambda\}_\lambda$ be a family of finite dimensional spaces, either satisfying* (3.1) *if we consider problem* (2.11)*, or satisfying* (3.20) *if we consider problem* (2.20)*. Also, we associate problem* (2.1), (2.2) *or* (2.1), (2.3) *with a family of spaces $\{H_\mu\}_\mu$ satisfying* (4.1) *or* (4.2)*, respectively. Then, for any $\varepsilon > 0$, there exists λ_ε such that the following hold.*

(i) *If the space H_μ is taken such that $|y'(\varphi_i) - y'_{\mu,i}|_{L^2(\gamma)}$, $i = 1, \ldots, n_{\lambda_\varepsilon}$, are small enough, y is the solution of problem* (2.1)–(2.2)*, and $y(u_{\lambda_\varepsilon\mu})$ is given by* (4.8)*, in which $\xi_{\lambda\mu}$ is the solution of the algebraic system* (4.5) *with the matrix given by* (4.3) *and the free term given by* (4.6)*, then the algebraic system* (4.5) *has a unique solution and*

$$\begin{aligned} &|y(u_{\lambda_\varepsilon\mu})_{|\omega} - y|_{H^{1/2}(\omega)} < C\varepsilon \\ &+ C_{\lambda_\varepsilon} \left(|g_\gamma - g_{\gamma\mu}|_{L^2(\gamma)} + |y_f - y_{f\mu}|_{L^2(\gamma)} + \max_{1\leq i\leq n_\lambda} |y'(\varphi_i) - y'_{\mu,i}|_{L^2(\gamma)} \right). \end{aligned}$$

(ii) *If the space H_μ is taken such that $|\frac{\partial y'(\varphi_i)}{\partial n_A(\omega)} - \frac{\partial y'_{\mu,i}}{\partial n_A(\omega)}|_{H^{-1}(\gamma)}$, $i = 1, \ldots, n_{\lambda_\varepsilon}$, are small enough, y is the solution of problem* (2.1)–(2.3)*, and $y(u_{\lambda_\varepsilon\mu})$ is given by* (4.8) *in* *which* $\xi_{\lambda\mu}$ *is the solution of the algebraic system* (4.5) *with the matrix given by* (4.4) *and the free term given by* (4.7), *then the algebraic system* (4.5) *has a unique solution and*

$$\begin{aligned}&|y(u_{\lambda_\varepsilon\mu})_{|\omega} - y|_{H^{1/2}(\omega)} < C\varepsilon \\ &+ C_{\lambda_\varepsilon}\left(|h_\gamma - h_{\gamma\mu}|_{H^{-1}(\gamma)} + \left|\frac{\partial y_f}{\partial n_A(\omega)} - \frac{\partial y_{f\mu}}{\partial n_A(\omega)}\right|_{H^{-1}(\gamma)}\right. \\ &\left.+ \max_{1\le i\le n_\lambda}\left|\frac{\partial y'(\varphi_i)}{\partial n_A(\omega)} - \frac{\partial y'_{\mu,i}}{\partial n_A(\omega)}\right|_{H^{-1}(\gamma)}\right),\end{aligned}$$

where C *is a constant and* C_{λ_ε} *depends on the basis of* U_{λ_ε}.

Remark 4.1. Since the matrices $\Pi_{\lambda\mu}$ given by (4.3) and (4.4) are assumed to be nonsingular, it follows that $\{y'_{\mu,i}\}_{i=1,\dots,n_\lambda}$ and $\{\frac{\partial y'_{\mu,i}}{\partial n_A(\omega)}\}_{i=1,\dots,n_\lambda}$ are some linearly independent sets in $L^2(\gamma)$ and $H^{-1}(\gamma)$, respectively. Consequently, if m_μ is the dimension of the corresponding subspace H_μ, then $n_\lambda \le m_\mu$.

5. Exterior problems. In this section, we consider the domain $\omega \subset \mathbf{R}^N$ of problems (2.1), (2.2) and (2.1), (2.3) as the complement of the closure of a bounded domain, and it lies on only one side of its boundary. The same assumptions are made on the domain Ω of problems (2.11) and (2.20), and, evidently, $\omega \subset \Omega$. In order to retain continuity and to prove that the solutions of the problems in ω can be approximated by the solutions of problems in Ω, we have to specify the spaces in which the problems have solutions and also their correspondence with the trace spaces.

Since the domain $\Omega - \bar{\omega}$ is bounded, Lions's controllability theorem does not need to be extended to unbounded domains. Moreover, we see that the boundaries γ and Γ of the domains ω and Ω are bounded, and, consequently, we can use finite open covers of them (as for the bounded domains) to define the traces.

In order to avoid the use of the fractional spaces of the spaces in ω and Ω, we simply remark that if the controls in the Lions controllability theorem are taken in $H^{1/2}(\Gamma)$ instead of $L^2(\Gamma)$, then a similar proof of it gives the following.

The set $\{\frac{\partial z_0(v)}{\partial n_A(\Omega-\bar{\omega})} \in H^{-1/2}(\gamma) : v \in H^{1/2}(\Gamma)\}$ *is dense in* $H^{-1/2}(\gamma)$, *where* $z_0(v) \in H^1(\Omega - \bar{\omega})$ *is the solution of the problem*

$$\begin{aligned}&Az_0(v) = 0 \quad && \textit{in } \Omega - \bar{\omega}, \\ &z_0(v) = v && \textit{on } \Gamma, \\ &z_0(v) = 0 && \textit{on } \gamma.\end{aligned}$$

Now we associate to the operator A the symmetric bilinear form

$$a(y,z) = \sum_{i,j=1}^{N} \int_\Omega a_{ij}\frac{\partial y}{\partial x_i}\frac{\partial z}{\partial x_j} + \int_\Omega a_0 yz \quad \text{for} \quad y,\ z \in H^1(\Omega),$$

which is continuous on $H^1(\Omega) \times H^1(\Omega)$. Evidently, a is also continuous on $H^1(\omega) \times H^1(\omega)$. Now if $f \in L^2(\omega)$, taking the boundary data $g_\gamma \in H^{1/2}(\gamma)$ and $h_\gamma \in H^{-1/2}(\gamma)$, then problems (2.1), (2.2) and (2.1), (2.3) can be written in the variational form

$$\begin{aligned}&y \in H^1(\omega) : a(y,z) = \textstyle\int_\omega fz \quad \text{for any} \quad z \in H_0^1(\omega), \\ &y = g_\gamma \text{ on } \gamma,\end{aligned} \tag{5.1}$$

and

$$y \in H^1(\omega) : a(y,z) = \int_\omega fz + \int_\gamma h_\gamma z \quad \text{for any} \quad z \in H^1(\omega), \tag{5.2}$$

respectively. Similar equations can also be written for problems (2.11) and (2.20).

Therefore, if there exists a constant $c_0 > 0$ such that $a_0 \geq c_0$ in Ω, then the bilinear form a is $H^1(\Omega)$-elliptic, i.e., there exists a constant $\alpha > 0$ such that $\alpha |y|^2_{H^1(\Omega)} \leq a(y,y)$ for any $y \in H^1(\Omega)$. It follows from the Lax–Milgram lemma that problems (2.11) and (2.20) have unique weak solutions in $H^1(\Omega)$. Naturally, problems (2.1), (2.2) and (2.1), (2.3) in ω also have unique weak solutions given by the solutions of problems (5.1) and (5.2), respectively.

We know that there exist an isomorphism and homeomorphism of $H^1(\Omega)/H^1_0(\Omega)$ onto $H^{1/2}(\Gamma)$ (see Theorem 7.53, p. 216, in [1], or Theorem 5.5, p. 99, and Theorem 5.7, p. 103, in [30]), i.e., there are two constants $k_1,\ k_2 > 0$ such that we have the following.

- For any $y \in H^1(\Omega)$, there exists $v \in H^{1/2}(\Gamma)$ such that $y = v$ on Γ and $| v |_{H^{1/2}(\Gamma)} \leq k_1 \mid y \mid_{H^1(\Omega)}$.
- For any $v \in H^{1/2}(\Gamma)$, there exists $y \in H^1(\Omega)$ such that $y = v$ on Γ and $| y |_{H^1(\Omega)} \leq k_2 \mid v \mid_{H^{1/2}(\Gamma)}$.

Using this correspondence, we can easily prove the continuous dependence of the solutions on data. For instance, for problems (2.1), (2.2) and (2.1), (2.3) we have

$$|y|_{H^1(\omega)} \leq C\{|f|_{L^2(\omega)} + |g_\gamma|_{H^{1/2}(\gamma)}\}$$

and

$$|y|_{H^1(\omega)} \leq C\{|f|_{L^2(\omega)} + |h_\gamma|_{H^{-1/2}(\gamma)}\},$$

respectively.

Therefore, if there exists a constant $c_0 > 0$ such that $a_0 \geq c_0$ in Ω, then we can proceed in the same manner and obtain similar results for the exterior problems to those obtained in the previous sections for the interior problems. Evidently, in this case we take

$$\mathcal{U} = H^{1/2}(\Gamma) \tag{5.3}$$

as a space of the controls for problem (2.11), in place of that given in (2.10), and the space of controls for problem (2.20) is taken as

$$\mathcal{U} = H^{-1/2}(\Gamma), \tag{5.4}$$

in place of the space given in (2.19).

If $a_0 = 0$ in Ω, the domain being unbounded, then the problems might not have solutions in the classical Sobolev spaces (see [11]), and we have to introduce the weighted spaces which take into account the particular behavior of the solutions at infinity.

For domains in $\mathbf{R}^2$, we use the weighted spaces introduced in [24], [25], specifically,

$$W^1(\Omega) = \{v \in \mathcal{D}'(\Omega) : (1+r^2)^{-1/2}(1+\log\sqrt{1+r^2})^{-1} v \in L^2(\Omega),\ \nabla v \in (L^2(\Omega))^2\},$$

where $\mathcal{D}'(\Omega)$ is the space of the distributions on Ω, and r denotes the distance from the origin. The norm on $W^1(\Omega)$ is given by

$$| v |_{W^1(\Omega)} = \left(| (1+r^2)^{-1/2}(1+\log\sqrt{1+r^2})^{-1} v |^2_{L^2(\Omega)} + | \nabla v |^2_{(L^2(\Omega))^2} \right)^{1/2}.$$

For domains in $\mathbf{R}^N$, $N \geq 3$, appropriate spaces, introduced in [21] and used in [20], [31], are

$$W^1(\Omega) = \{ v \in \mathcal{D}'(\Omega) : (1+r^2)^{-1/2} v \in L^2(\Omega),\ \nabla v \in (L^2(\Omega))^N \}$$

with the norm

$$| v |_{W^1(\Omega)} = \left(| (1+r^2)^{-1/2} v |^2_{L^2(\Omega)} + | \nabla v |^2_{(L^2(\Omega))^N} \right)^{1/2}.$$

We remark that the space $H^1(\Omega)$ is continuously embedded in $W^1(\Omega)$, and the two spaces coincide for the bounded domains. We use $W_0^1(\Omega)$ to denote the closure of $\mathcal{D}(\Omega)$ in $W^1(\Omega)$.

Concerning the space of the traces of the functions in $W^1(\Omega)$, we notice that, the boundary Γ being bounded, these traces lie in $H^{1/2}(\Gamma)$. This fact immediately follows from considering a bounded domain $D \subset \Omega$ such that $\Gamma \subset D$ and from taking into account that $W^1(D)$ and $H^1(D)$ are identical.

Assuming that

$$(1+r^2)^{1/2}(1+\log\sqrt{1+r^2}) f \in L^2(\Omega) \text{ if } N = 2,$$

$$(1+r^2)^{1/2} f \in L^2(\Omega) \text{ if } N \geq 3,$$

and using the spaces W^1 in place of the spaces H^1, we can rewrite the problems (5.1) and (5.2) and also similar equations for problems (2.11) and (2.20).

For $N = 2$, the bilinear form $a(y, z)$ generates on $W_0^1(\Omega)$ an equivalent norm with that induced by $W^1(\Omega)$ (see [24]). Also, the bilinear form $a(y, z)$ generates on $W^1(\Omega)/\mathbf{R}$ a norm which is equivalent to the standard norm.

For $N \geq 3$, the previously introduced norm on $W^1(\mathbf{R}^N)$ is equivalent to that generated by the bilinear form $a(.,.)$ (see [21]). Now if we extend the functions in $W_0^1(\Omega)$ with zero in $\mathbf{R}^N - \Omega$, we get that the bilinear form $a(y, z)$ also generates on $W_0^1(\Omega)$ a norm equivalent to that induced by $W^1(\Omega)$. Moreover, using the fact that the domain Ω is the complement of a bounded set, it can be proved that the bilinear form $a(y, z)$ generates in $W^1(\Omega)$ a norm equivalent to the above introduced norm.

Therefore, we can conclude that, in the case of $a_0 = 0$ on Ω, the exterior problems have unique solutions in the spaces W^1 if $N \geq 3$. If $N = 2$, the Dirichlet problems have unique solutions in W^1, and the Neumann problems have unique solutions in $W^1/\mathbf{R}$.

Using the fact that the spaces $W^1(D)$ and $H^1(D)$ coincide on the bounded domains D, the continuous embedding of $H^1(\Omega)$ in $W^1(\Omega)$, and the homeomorphism and isomorphism between $H^{1/2}(\Gamma)$ and $H^1(\Omega)/H_0^1(\Omega)$, we can easily prove that there exist a homeomorphism and isomorphism between $H^{1/2}(\Gamma)$ and $W^1(\Omega)/W_0^1(\Omega)$. Consequently, we get the following continuous dependence on the data of the solution y of problem (2.1), (2.2):

$$|y|_{W^1(\omega)} \leq C\{ |(1+r^2)^{1/2}(1+\log\sqrt{1+r^2}) f|_{L^2(\omega)} + |g_\gamma|_{H^{1/2}(\gamma)} \} \qquad \text{if } N = 2,$$

and

$$|y|_{W^1(\omega)} \leq C\{|(1+r^2)^{1/2} f|_{L^2(\omega)} + |g_\gamma|_{H^{1/2}(\gamma)}\} \qquad \text{if } N \geq 3.$$

For the problem (2.1), (2.3), we have

$$\inf_{s \in \mathbf{R}} |y+s|_{W^1(\omega)} \leq C\{|(1+r^2)^{1/2}(1+\log\sqrt{1+r^2}) f|_{L^2(\omega)} + |h_\gamma|_{H^{-1/2}(\gamma)}\} \qquad \text{if } N = 2,$$

and

$$|y|_{W^1(\omega)} \leq C\{|(1+r^2)^{1/2} f|_{L^2(\omega)} + |h_\gamma|_{H^{-1/2}(\gamma)}\} \qquad \text{if } N \geq 3.$$

Therefore, we can prove in a manner similar to the previous sections that when $a_0 = 0$ on Ω and $N \geq 3$, the solutions of the Dirichlet and Neumann problems in ω can be approximated with solutions of both the Dirichlet and the Neumann problems in Ω. Naturally, the controls are taken in the appropriate space (5.3) or (5.4). If $a_0 = 0$ on Ω and $N = 2$, the solutions of the Dirichlet problems in ω can be approximated with solutions of the Dirichlet problem in Ω. The Neumann problems do not have unique solutions.

Since $y(v)$ and g_γ lie in $H^{1/2}(\gamma)$ in the case of problem (2.1), (2.2), and $\frac{\partial y(v)}{\partial n_A(\omega)}$ and h_γ lie in $H^{-1/2}(\gamma)$ when we solve (2.1), (2.3), the natural choices for the space of observations are

$$\mathcal{H} = H^{1/2}(\gamma) \tag{5.5}$$

and

$$\mathcal{H} = H^{-1/2}(\gamma), \tag{5.6}$$

respectively. Even if the convergence is assured for these spaces, their norms are numerically estimated with much difficulty. However, noticing that the inclusions $H^{1/2}(\gamma) \subset L^2(\gamma) \subset H^{-1/2}(\gamma) \subset H^{-1}(\gamma)$ are continuous, we can take the spaces of observations, as in the case of the bounded domains, given in (2.12) and (2.14). We mentioned earlier the need to avoid the use of the fractional Sobolev spaces for unbounded domains because of the lack of work on this subject (to the best of our knowledge), especially concerning the continuous dependence of the solution on the data of the problem. In the next section, we give a numerical example where the space of the controls is taken as for the bounded domains and the obtained results are accurate.

6. Numerical results. In this section, we choose some specific U_λ and H_μ. Hence we drop the subscripts λ and μ. First, we summarize the results obtained in the previous sections on the algebraic system we need to solve to obtain solutions, within a prescribed error, of problems (2.1), (2.2) or (2.1), (2.3), using the solutions of problems (2.11) or (2.20).

We recall that if, for both the bounded and unbounded domains, there exists a constant $c_0 > 0$ such that the coefficient a_0 of the operator A satisfies $a_0 \geq c_0$ in Ω, then the solutions of problems (2.1), (2.2) or (2.1), (2.3) can be approximated by the solutions of both problems (2.11) and (2.20). If $a_0 = 0$ in Ω, then the solutions of problems (2.1), (2.2) can be approximated by the solutions of problems (2.11) for both the bounded and the unbounded domains, and if also $N \geq 3$, then by the solutions of problems (2.20) for unbounded domains only. If $a_0 = 0$ in Ω with the domains unbounded, then the solutions of problems (2.1), (2.3) can be obtained from the solutions of problems (2.11) and also from the solutions of (2.20) if $N \geq 3$.

Actually, we have to solve the algebraic system (4.5), which is rewritten as

$$\xi \in \mathbf{R}^n : \Pi\xi = l. \tag{6.1}$$

Some remarks on the computing of the elements of the matrix Π and the free term l are made below.

- Depending on the problem in Ω, we choose the space of controls $\mathcal{U}$ and a finite dimensional subspace of it, $U \subset \mathcal{U}$. Let $\varphi_1, \dots, \varphi_n$, $n \in \mathbf{N}$, be the basis of U, and let $y'(\varphi_i)$, $i = 1, \dots, n$ be the corresponding solutions of problems (3.2) or (3.21) if the problem in Ω is (2.11) or (2.20), respectively.
- If the problem in ω is (2.1), (2.2), then we calculate the values of $y'(\varphi_i)$, $i = 1, \dots, n$, at the mesh points on γ. For the problem (2.1), (2.3) we calculate the values of $\frac{\partial y'(\varphi_i)}{\partial n_a(\omega)}$, $i = 1, \dots, n$, at the mesh points on γ.
- Using the computed values of $y'(\varphi_i)$ or $\frac{\partial y'(\varphi_i)}{\partial n_a(\omega)}$, $i = 1, \dots, n$, at the mesh points on γ, we compute the elements of the matrix Π which are some inner products either in $\mathcal{H} = L^2(\gamma)$ when we solve the problem (2.1), (2.2) or in $\mathcal{H} = H^{-1}(\gamma)$ when we solve the problem (2.1), (2.3). The finite dimensional subspace $H \subset \mathcal{H}$ depends on the numerical integration method that we use. We remark that the matrix Π is symmetric and full.
- The elements of the free term l are also some inner products in the space of observations $\mathcal{H}$. We use a solution y_f of (3.23) and the boundary data of the problem in ω (i.e., g_γ or h_γ if the problem is (2.1), (2.2) or (2.3), (2.1), respectively) in these inner products.
- For problem (2.1), (2.2), the matrix Π and the free term l are given by (4.3) and (4.6), respectively. Also, for problem (2.1), (2.3) the matrix Π and the free term l are given in (4.4) and (4.7), respectively. In these equations, y'_i and $\frac{\partial y'_i}{\partial n_A(\omega)}$ are some approximations in H of $y'(\varphi_i)$ and $\frac{\partial y'(\varphi_i)}{\partial n_A(\omega)}$, respectively. These approximations arise from the use of numerical integration on γ and numerical values of $y'(\varphi_i)$ and $\frac{\partial y'(\varphi_i}{\partial n_A(\omega)}$ at the mesh points on γ. These values can be found either by evaluating an algebraic expression or by interpolation. Indeed, when the finite element method or any other method is used with a mesh over Ω which does not fit with the boundary γ, the values of the functions y_f and $y'(\varphi_i)$, $i = 1, \dots, n$ at some mesh points in γ are found by interpolation.

Finally, if $\xi = (\xi_1, \dots, \xi_n)$ is the solution of algebraic system (6.1) and y is the solution of the problem we solve, then its approximation is the restriction to ω of

$$\xi_1 y'(\varphi_1) + \cdots + \xi_n y'(\varphi_n) + y_f. \tag{6.2}$$

We recall that the matrices Π_λ given in (3.14) and (3.18) are nonsingular, and therefore, each of the problems (3.16) has a unique solution. Also, algebraic systems (6.1) have unique solutions if their matrices and free terms are good approximations in H of the matrix and the free term of the algebraic systems (3.16), respectively. Also, from Remark 4.1 we must take $n \leq m$, n being the dimension of U and m the dimension of H. However, as we recall from section 2, the problem in infinite dimensional space may not have a solution. Consequently, for very large n, we might obtain algebraic systems (3.16) that are almost singular. These algebraic systems can be solved by an iterative method such as the conjugate gradient method. However, we applied the Gauss elimination method in order to find out whether the algebraic system is singular or nonsingular. This is done by checking the diagonal elements during the elimination phase.

In the following two subsections, we give some numerical examples for both interior and exterior problems in which the solutions of the problems in Ω are found either directly by a formula, or by a method using a mesh over Ω.

6.1. Interior problems.

Example 6.1. The first numerical test refers to the Dirichlet problem

$$\begin{aligned} -\Delta y &= f \text{ in } \omega, \\ y &= g_\gamma \text{ on } \gamma, \end{aligned} \tag{6.3}$$

where $\omega \subset \mathbf{R}^2$ is a square centered at the origin with sides parallel to the axes and of length of 2 units. The approximate solution of this problem is given by the solution of the Dirichlet problem

$$\begin{aligned} -\Delta y(v) &= f \text{ in } \Omega, \\ y(v) &= v \text{ on } \Gamma, \end{aligned} \tag{6.4}$$

in which the domain Ω is the disc centered at the origin with radius equal to 2. The solutions of the homogeneous Dirichlet problems in Ω are found by the Poisson formula

$$y(v)(z) = \frac{1}{2\pi r} \int_{|\zeta|=r} v(\zeta) \frac{r^2 - |z|^2}{|z-\zeta|^2} dS_\zeta. \tag{6.5}$$

The circle Γ is discretized with n equidistant points, and $U \subset \mathcal{U} \equiv L^2(\Gamma)$ is taken as the space of the piecewise constant functions. Naturally, an element φ_i in the basis of H is a function defined on Γ which takes the value 1 between the nodes i and $i+1$ and vanishes in the rest of Γ. The square γ is also discretized with m equidistant points, and $H \subset \mathcal{H} \equiv L^2(\gamma)$ is taken as the space of the continuous piecewise linear functions. Evidently, the inclusions in (3.1) and (4.1) are dense because the union of the spaces (over some sequence of mesh size approaching zero) of continuous piecewise linear or piecewise constant functions is dense in L^2.

The values of the integrals in the Poisson formula at the points on γ are calculated using the numerical integration with three nodes. The integrals in the inner products in $L^2(\gamma)$ are calculated using an exact formula when H is the space of the continuous piecewise linear functions. In particular, if we have on γ two continuous piecewise linear functions y_1 and y_2 such that

$$\begin{aligned} y_1(x) &= m_1^k(x - x_k) + y_1^k, \\ y_2(x) &= m_2^k(x - x_k) + y_2^k \end{aligned} \tag{6.6}$$

for $x \in [x_k, x_{k+1}]$, $k = 1 \ldots, m$, then

$$\int_\gamma y_1 y_2 = h \sum_{k=1}^{m} \left[y_1^k y_2^k + \frac{h^2}{3} m_1^k m_2^k + \frac{h}{2}(m_1^k y_2^k + m_2^k y_1^k) \right], \tag{6.7}$$

where $h = x_{k+1} - x_k$ is the mesh size on γ.

TABLE 6.1
Relative errors for the interior Dirichlet problem.

n	$\mathrm{err_d}$	$\mathrm{err_b}$
80	.36692E-07	.15956E-06
72	.46271E-08	.41101E-07
60	.14682E-09	.25103E-08
45	.12475E-08	.54357E-08
40	.64352E-12	.11638E-07
36	.67121E-12	.11648E-06
30	.12371E-05	.33923E-05
24	.39543E-12	.19851E-04
18	.10609E-03	.43901E-03
12	.29916E-10	.54208E-02
10	.94618E-02	.17096E-01

All computations below have been performed in fifteen digit arithmetics (double precision).

In the first example, we choose the exact solution to be $u(x_1, x_2) = x_1^2 + x_2^2$. Hence $g_\gamma(x_1, x_2) = x_1^2 + x_2^2$, and $f = -4$. We have taken $y_f = 2x_1^2$ as a solution of the inhomogeneous equation in Ω. It has been compared with the computed one at 19 equidistant points on a diagonal of the square: (-1.4,-1.4),. . .,(0,0),. . .,(1.4,1.4). Below $\mathrm{err_d}$ denotes the maximum of the relative errors between the exact and the computed solutions at these 19 considered points in the domain ω. A similar error only on the boundary γ is denoted by $\mathrm{err_b}$.

Table 6.1 shows errors $\mathrm{err_d}$ and $\mathrm{err_b}$ against n, the number of the equidistant points on Γ which is the dimension of the finite dimensional space U. Recall that Γ is boundary of the embedding domain Ω. All these computations use a mesh size of 0.1 on γ. It corresponds to $m = 80$, the number of equidistant points on γ, which is the dimension of the finite dimensional space H. The smallest diagonal element during the Gauss elimination method is of the order 10^{-17} for $n = 80$ and $n = 72$, and of the order 10^{-14} for $n = 60$. It is greater than 10^{-10} for $n = 10, \ldots, 45$. We should mention that in the cases when $n > 60$, where the last pivot is very small, we notice an increase in error. In all these cases the error $\mathrm{err_b}$, which can be calculated for any example even when the exact solution is not known, is a good indicator of the computational accuracy.

In the above example, the right-hand side f of the equation in ω is given by an exact algebraic formula, and it was extended in Ω by the same formula. Moreover, we have had for this simple example an exact solution y_f of the inhomogeneous equation in Ω, which could be exactly evaluated at the mesh points of the boundary γ of the domain ω. Also, the solutions of the homogeneous problems in Ω, given by the above Poisson formula, could be evaluated directly at these mesh points. In the following example we study the effect of various extensions of f in Ω on the computed solutions in ω. Therefore, in this example, the solution of the problem in Ω could be computed only at some nodes of a regular mesh over Ω, and their values at the mesh points on γ are calculated by interpolation.

Example 6.2. This example concerns the Dirichlet problem

$$\begin{aligned} &\Delta y - \sigma^2 y = f \text{ in } \omega, \\ &y = g_\gamma \text{ on } \gamma, \end{aligned} \tag{6.8}$$

where $\omega \subset \mathbf{R}^2$ is bounded by the straight lines $x_1 = -\pi/2$, $x_1 = \pi/2$, and $x_2 = -1.5$ and the curve $y = 0.5 + \cos(x + \pi/2)$. We approximate the solution of this problem by a solution of the Dirichlet problem

$$\begin{aligned} &\Delta y(v) - \sigma^2 y(v) = f \text{ in } \Omega, \\ &y(v) = v \text{ on } \Gamma, \end{aligned} \tag{6.9}$$

in which the domain Ω is the disc centered at the origin with the radius of 2.3 (see Figure 6.1 (a)). We have taken $\sigma^2 = 0.75$ in numerical computations.

We approximate the functions f and v by the discrete Fourier transforms

$$\begin{aligned} f(r,\theta) &= \sum_{k=-n/2}^{n/2-1} f_k(r) e^{ik\theta}, \\ v(\theta) &= \sum_{k=-n/2}^{n/2-1} v_k e^{ik\theta}. \end{aligned} \tag{6.10}$$

Then the solution of problem (6.9),

$$y(v) = y_f + y'(v), \tag{6.11}$$

can also be written as a discrete Fourier transform

$$\begin{aligned} y_f(r,\theta) &= \sum_{k=-n/2}^{n/2-1} y_k(r) e^{ik\theta}, \\ y'(v)(r,\theta) &= \sum_{k=-n/2}^{n/2-1} y'_k(r) e^{ik\theta}, \end{aligned} \tag{6.12}$$

where the Fourier coefficients $y_k(r)$ and $y'_k(r)$ are given by

$$\begin{aligned} y_k(r) = &- \int_0^r \rho K_k(\sigma r) I_k(\sigma\rho) f_k(\rho) d\rho - \int_r^R \rho I_k(\sigma r) K_k(\sigma\rho) f_k(\rho) d\rho \\ &+ \frac{I_k(\sigma r)}{I_k(\sigma R)} \int_0^R \rho K_k(\sigma R) I_k(\sigma\rho) f_k(\rho) d\rho, \\ y'_k(r) = &\frac{I_k(\sigma r)}{I_k(\sigma R)} v_k. \end{aligned} \tag{6.13}$$

Above, R is the radius of the disc, and I_k and K_k are the modified Bessel functions of the first and second kinds, respectively. We recall that $y'(v)$ and y_f in (6.11) are the solutions of problems (3.2) and (3.6), respectively. A fast algorithm is proposed in [4], which, using (6.13) and the fast Fourier transforms, evaluates y_f and $y'(v)$ in (6.12) at the nodes of a mesh on the disc Ω with n equidistant nodes in tangential direction and l equidistant nodes in the radial direction.

It is worth noting from (6.10) that the finite dimensional space of controls U is the space of real periodic functions defined on $[0, 2\pi]$ which can be written as a Fourier transform with the terms $-n/2, \ldots, 0, \ldots, n/2 - 1$. On the other hand, we have $\mathcal{U} = L^2(\Gamma) = L^2(0, 2\pi)$, and since the functions in $L^2(0, 2\pi)$ can be approximated by discrete Fourier transforms, we get that (3.1) holds with U as the above finite dimensional spaces. Since the controls v are real functions, it follows from (6.10)

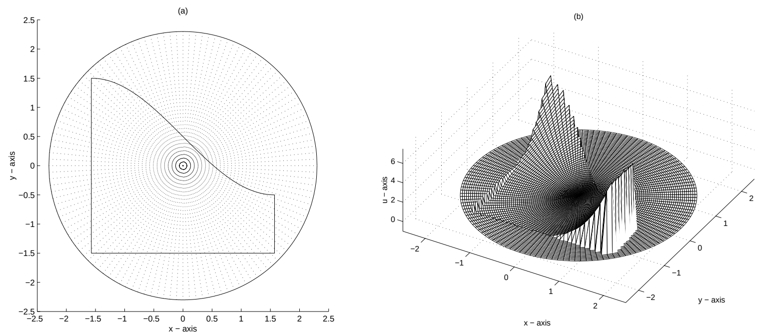

FIG. 6.1. (a) *Domains,* (b) *exact solution.*

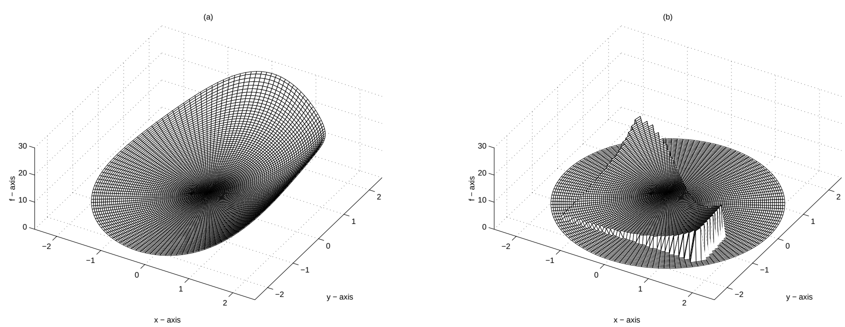

FIG. 6.2. *Extension of f by* (a) *the formula in the domain ω,* (b) *zero.*

that $v_i = \bar{v}_{-i}$ for $i = 1, \ldots, n/2-1$ and v_0 is real provided we choose $v_{-n/2} \in R$. Consequently, a basis of U is given by the functions: φ_0 which has the Fourier coefficient $v_0 = 1$, the other ones being zero, $\varphi_{-n/2}$ which has the Fourier coefficient $v_{-n/2} = 1$, the other ones being zero, and φ_j, $-n/2+1 \le j \le n/2-1$, $j \neq 0$, have the Fourier coefficients $v_j = 1+i$, $v_{-j} = 1-i$ with the rest being zero.

The boundary γ is discretized with m equidistant points, and, as in the previous example, H is taken to be the space of the piecewise linear functions. The integrals in the inner products in $L^2(\gamma)$ are calculated by the same formulae (6.7). We recall that the values of y_f and $y'(\varphi_)$ at the mesh points of the boundary γ were obtained by interpolation of function values at mesh points on Ω. Assuming that the point (r, θ) lies between the four mesh nodes (r_1, θ_1), (r_2, θ_1), (r_1, θ_2), (r_2, θ_2), we have linearly interpolated in radial direction first the values corresponding to (r_1, θ_1) and (r_2, θ_1), and then the values corresponding to (r_1, θ_2), (r_2, θ_2). Using the two obtained values, we have made a linear interpolation in the tangential direction.

For numerical purposes, we have taken $f(x_1, x_2) = (2 + x_1(1-\sigma^2))e^{x_1} + (2 + x_2(1-\sigma^2))e^{x_2}$ and $g_\gamma(x_1, x_2) = x_1 e^{x_1} + x_2 e^{x_2}$ in (6.8). Then problem (6.8) has the exact solution $y(x_1, x_2) = x_1 e^{x_1} + x_2 e^{x_2}$, which is shown in Figure 6.1 (b). In order to assess the effect of various extensions of the function f outside of ω on the numerical results, we have taken for this example only two types of extensions: (i) extending f using the above formula in ω; (ii) extending f by zero (see Figure 6.2).

Tables 6.2 through 6.5 show the arithmetic mean of the absolute errors between the exact and the computed solutions against various values of n (the number of the nodes in tangential direction, i.e., the number of nodes on Γ) and δ_r (the mesh size in radial direction), while keeping the number of mesh points on γ fixed at $m = 360$ for

Table 6.2
Errors on $\gamma - f$ extended with the formula in ω.

n/δ_r	0.1	0.05	0.02	0.01
8	0.15555E+00	0.15571E+00	0.15577E+00	0.15577E+00
16	0.25622E-01	0.25530E-01	0.25505E-01	0.25500E-01
32	0.58700E-02	0.55274E-02	0.55131E-02	0.55146E-02
64	0.26025E-02	0.13450E-02	0.12478E-02	0.12411E-02
128	0.12901E-02	0.56973E-03	0.36080E-03	0.35200E-03

Table 6.3
Errors in $\omega - f$ extended with the formula in ω.

n/δ_r	0.1	0.05	0.02	0.01
8	0.98198E-01	0.92264E-01	0.89501E-01	0.88875E-01
16	0.33058E-01	0.31403E-01	0.30967E-01	0.30862E-01
32	0.83707E-02	0.69124E-02	0.65851E-02	0.65232E-02
64	0.38456E-02	0.18422E-02	0.14402E-02	0.13976E-02
128	0.30019E-02	0.95631E-03	0.40010E-03	0.34864E-03

Table 6.4
Errors on $\gamma - f$ extended by zero.

n/δ_r	0.1	0.05	0.02	0.01
8	0.20670E+00	0.20546E+00	0.20347E+00	0.20331E+00
16	0.32825E-01	0.33941E-01	0.34529E-01	0.35906E-01
32	0.67604E-02	0.69137E-02	0.79452E-02	0.83573E-02
64	0.39507E-02	0.19624E-02	0.24754E-02	0.26836E-02
128	0.14346E-02	0.78505E-03	0.13167E-02	0.13784E-02

Table 6.5
Errors in $\omega - f$ extended by zero.

n/δ_r	0.1	0.05	0.02	0.01
8	0.15520E+00	0.15211E+00	0.15156E+00	0.15206E+00
16	0.30860E-01	0.27219E-01	0.25270E-01	0.25012E-01
32	0.72434E-02	0.54336E-02	0.52230E-02	0.51386E-02
64	0.40250E-02	0.19991E-02	0.15890E-02	0.15415E-02
128	0.33861E-02	0.15554E-02	0.10941E-02	0.10286E-02

all these computations. The results in Tables 6.2 and 6.3 have been obtained with the extension of f in Ω made with the formula in ω, and the results in Tables 6.4 and 6.5 have been obtained with the extension made by zero. In Tables 6.2 and 6.4, we show the errors computed on the boundary γ by taking the average over $m = 360$ boundary points. On the other hand, we show in Tables 6.3 and 6.5 the errors computed in the domain ω by taking the average over all mesh points in ω.

We notice in these tables that errors on the boundary γ are of the same order as in the domain ω, and the extension of the function f outside of ω with the formula in ω gives smaller errors than the extension by zero. It may be worth noting here that the errors for this example are higher than those for the previous example (see Table 6.1) because the values of y_f and $y(\varphi_i)$ on the boundary γ were found by interpolation in this example and by an exact algebraic expression in the previous example. Thus interpolation error is one of the possible sources of larger error in these tables for this example. Figure 6.3 shows absolute errors at the mesh nodes in the domain ω when $n = 128$ and $\delta_r = 0.01$ for the two extensions of f.

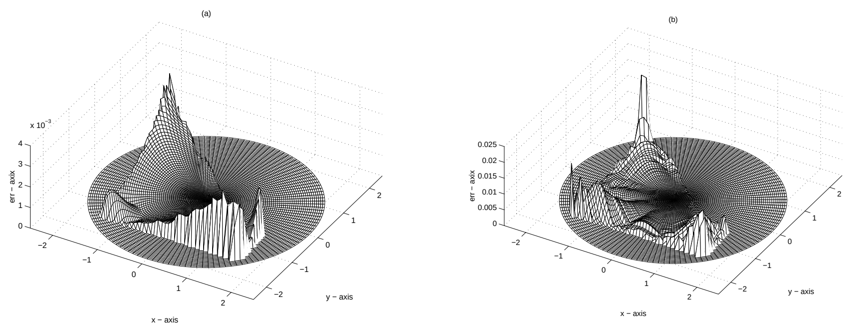

FIG. 6.3. *Errors in the domain when f is extended by* (a) *the formula in the domain* ω, (b) *zero.*

6.2. Exterior problems. Below, we show the performance of the method on two exterior problems.

Example 6.3. We solve the same problem as defined by (6.3) in Example 6.1 except that the domain ω is now the exterior of a square centered at the origin with sides parallel to the axes and of length of 2 units. For this problem, we consider exterior Dirichlet problem (6.4) with the embedding domain Ω as the exterior of a disc with its center at the origin and radius 0.99 unit.

Similar to Example 6.1, the solutions of the homogeneous Dirichlet problems in Ω are found by the Poisson formula

$$y(v)(z) = \frac{-1}{2\pi r}\int_{|\zeta|=r} v(\zeta)\frac{r^2 - |z|^2}{|z-\zeta|^2} dS_\zeta. \tag{6.14}$$

The spaces $U, \mathcal{U}, H$, and $\mathcal{H}$ are the same as in Example 6.1, and the integrals on the boundary γ use the same formula (6.7).

The problem in ω we have numerically solved has had $g_\gamma(x_1, x_2) = x_1 x_2$ and $f = 0$. Evidently, we take $y_f = 0$. In this case, we do not know the exact solution of the problem, but we recall from previous examples that the error on the boundary γ was very close to that in domain ω. Hence Table 6.6 shows the maximum relative errors between the exact prescribed data and the computed solutions on boundary γ against various values of n (the number of the nodes in tangential direction, i.e., number of nodes on Γ) while keeping the number of mesh points on γ fixed at $m = 120$ (corresponding to a mesh size of $1/15$ on γ) for all these computations.

We found that the smaller diagonal element during the Gauss elimination method is of the order 10^{-15} for $n = 120$ and of the order 10^{-14} for $n = 118$, and it is greater

TABLE 6.6
Errors obtained for the exterior Dirichlet problem.

n	$\mathrm{err_b}$
120	0.10995E-03
118	0.93472E-04
116	0.24253E-05
115	0.38082E-03
110	0.33003E-02
100	0.55797E-01
90	0.18828E+00
60	0.21087E+00
30	0.77558E+00

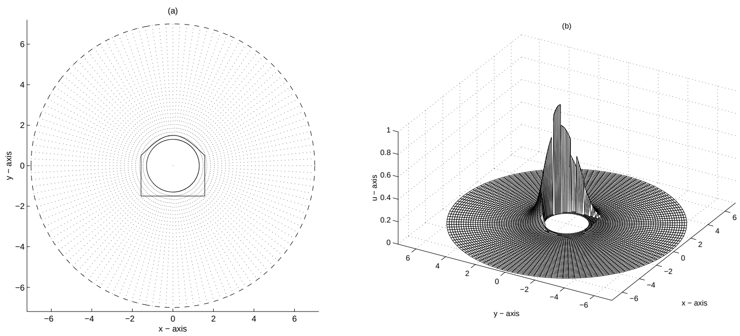

FIG. 6.4. (a) *Domains,* (b) *exact solution.*

than 10^{-12} for $n = 30, \ldots, 116$. We see in Table 6.6 that for $n > 116$, the error increases when the pivots in the Gauss elimination method become very small.

Example 6.4. Here we solve the same problem as defined by (6.8) in Example 6.2 except that the domain ω now is the open complement of the domain bounded by the straight lines $x_1 = -\pi/2$, $x_1 = \pi/2$, and $x_2 = -1.5$ and the curve $y = 0.5 + \cos(x)$. For this problem, the embedding domain Ω is taken to be the exterior of a disc with its center at the origin and radius 1.3 unit (see Figure 6.4, (a)).

We approximate the solution of this problem by a solution of the exterior Neumann problem

$$\begin{aligned} &\Delta y(v) - \sigma^2 y(v) = f \text{ in } \Omega, \\ &\tfrac{\partial y(v)}{\partial n_A(\Omega)} = v \text{ on } \Gamma, \end{aligned} \tag{6.15}$$

where Γ is the inner boundary of the embedding domain Ω. Similar to Example 6.2, we have taken $\sigma^2 = 0.75$ in numerical computations.

As before, functions f and v are approximated by the discrete Fourier transforms (6.10). Then the solution of problem (6.15) admits representation given by (6.11) and (6.12) except that the Fourier coefficients $y_k(r)$ and $y_k'(r)$ are now given by

$$\begin{aligned} y_k(r) &= -\int_R^r \rho K_k(\sigma r) I_k(\sigma\rho) f_k(\rho) d\rho - \int_r^\infty \rho I_k(\sigma r) K_k(\sigma\rho) f_k(\rho) d\rho \\ &\quad - \frac{K_k(\sigma r)}{K_{k-1}(\sigma R) + K_{k+1}(\sigma R)} \int_R^\infty \rho [I_{k-1}(\sigma R) + I_{k+1}(\sigma R)] K_k(\sigma\rho) f_k(\rho) d\rho, \\ y_k'(r) &= \frac{K_k(\sigma r)}{K_{k-1}(\sigma R) + K_{k+1}(\sigma R)} \frac{2}{\sigma} v_k. \end{aligned} \tag{6.16}$$

Above, R is the radius of the disc whose complement is the domain Ω, and I_k and K_k are the modified Bessel functions of first and second kinds, respectively. In order to compute the solution of problem (6.15) at mesh points of the domain Ω with n equidistant nodes in the tangential direction and l equidistant nodes in the radial direction, we use the algorithm proposed in [4]. This algorithm uses (6.16) and the fast Fourier transforms to compute y_f and $y'(v)$ in (6.12). For numerical computations, the domain Ω is considered to be the annulus with the radii R and R_∞, where R_∞ is chosen very large so that its effect is minimal on the accuracy of the solutions.

The spaces U and H are the same as in Example 6.2. Also, the values of y_f and $y'(\varphi_)$ at the mesh points of the boundary γ were obtained by interpolating the values of the function at mesh points on Ω.

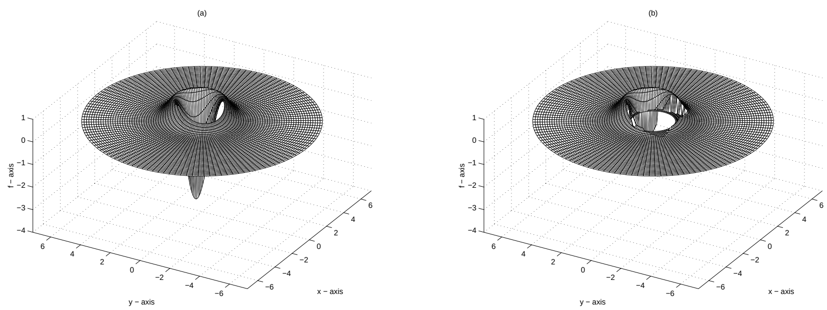

FIG. 6.5. *Extension of f by* (a) *the formula in the domain ω,* (b) *zero.*

For numerical purposes in this example, we have considered (6.8) with $f(x_1, x_2) = [4(x_1^2 + x_2^2 - x_1 - x_2) - 2 + \sigma^2]e^{-x_1^2 - x_2^2 + x_1 + x_2}$ and $g_\gamma(x_1, x_2) = e^{-x_1^2 - x_2^2 + x_1 + x_2}$. This problem has the exact solution $y(x_1, x_2) = e^{-x_1^2 - x_2^2 + x_1 + x_2}$ (it lies in $H^1(\omega)$ and satisfies the equation and the boundary conditions of problem (6.8)), which is plotted in Figure 6.4 (b).

Numerical computations show that $|y(r)| \leq 0.104E - 16$ for $r > 7$, where r is the distance of the point from the origin. Hence we have taken $R_\infty = 7$ in these computations. As in Example 6.2, we have extended f outside of ω in two different ways: (i) by the above formula, and (ii) by zero. These extensions are plotted in Figure 6.5. We have taken $m = 360$, the number of the mesh points on the boundary γ.

The error tables are similar to those in Example 6.2. Tables 6.7 and 6.8 correspond to the case when the extension of f in Ω is made with the formula in ω, and Tables 6.9 and 6.10 correspond to the extension made by zero. In Tables 6.7 and 6.9, we show the arithmetic mean of the absolute errors computed on the boundary γ by taking the average over $m = 360$ boundary points. On the other hand, we show in Tables 6.8 and 6.10 the errors computed in the domain ω by taking the average over all mesh points in ω. It is worth noting in these tables that, this time, the errors on the boundary γ are less than those in the domain ω, and the two extensions of f give solutions with

TABLE 6.7
Errors on $\gamma - f$ extended with the formula in ω.

n/δ_r	0.1	0.05	0.02	0.01
8	0.13247E-01	0.13231E-01	0.13233E-01	0.13233E-01
16	0.25712E-02	0.25628E-02	0.25500E-02	0.25496E-02
32	0.59286E-03	0.58076E-03	0.57869E-03	0.57859E-03
64	0.18186E-03	0.15977E-03	0.15536E-03	0.15462E-03
128	0.63343E-04	0.51301E-04	0.45571E-04	0.45775E-04

TABLE 6.8
Errors in $\omega - f$ extended with the formula in ω.

n/δ_r	0.1	0.05	0.02	0.01
8	0.29115E-02	0.28034E-02	0.26385E-02	0.26264E-02
16	0.11901E-02	0.10997E-02	0.10582E-02	0.10493E-02
32	0.66451E-03	0.62745E-03	0.61610E-03	0.61432E-03
64	0.56566E-03	0.54864E-03	0.54777E-03	0.54815E-03
128	0.60927E-03	0.53842E-03	0.53886E-03	0.53988E-03

TABLE 6.9
Errors on $\gamma - f$ extended by zero.

n/δ_r	0.1	0.05	0.02	0.01
8	0.13045E-01	0.13030E-01	0.13016E-01	0.13016E-01
16	0.26982E-02	0.26937E-02	0.27057E-02	0.26834E-02
32	0.61089E-03	0.61377E-03	0.63730E-03	0.64057E-03
64	0.17974E-03	0.15425E-03	0.16077E-03	0.16632E-03
128	0.63717E-04	0.52191E-04	0.53991E-04	0.57498E-04

TABLE 6.10
Errors in $\omega - f$ extended by zero.

n/δ_r	0.1	0.05	0.02	0.01
8	0.28435E-02	0.27346E-02	0.25635E-02	0.25483E-02
16	0.11929E-02	0.11024E-02	0.10533E-02	0.10487E-02
32	0.67268E-03	0.64000E-03	0.63015E-03	0.63105E-03
64	0.58007E-03	0.56682E-03	0.56861E-03	0.57090E-03
128	0.60658E-03	0.55784E-03	0.56152E-03	0.56343E-03

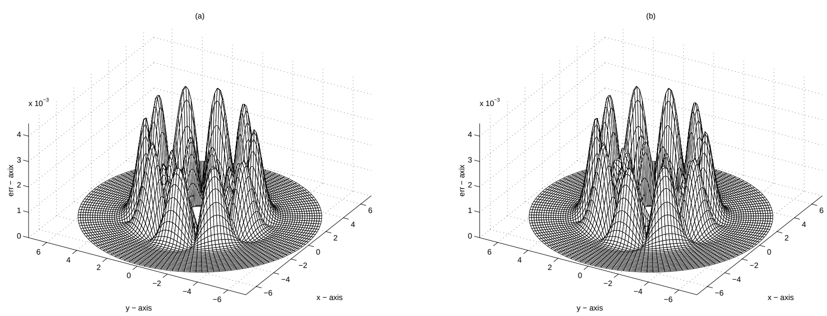

FIG. 6.6. *Errors in the domain when f is extended by* (a) *the formula in the domain ω,* (b) *zero.*

errors of the same order. In Figure 6.6, we have plotted the absolute error at the mesh nodes in the domain ω when $n = 128$ and $\delta_r = 0.01$ for these two extensions of f.

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