A note on a simple Markov bilinear stochastic process

Daren B.H. Cline *, Huay-min H. Pu

Department of Statistics, Texas A&M University, College Station TX 77843, USA

Received September 2001

Abstract

In this paper we note that while the results of a 1981 paper of H. Tong's are generally valid and can be strengthened, there is a special case that behaves differently. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Bilinear process; ARCH-type errors; Markov process; Drift criteria; Ergodicity; Geometric ergodicity; Limiting distribution

1. Introduction

Following Tong (1981), we adopt the name "simple Markov bilinear stochastic process" for a Markov process $\{X_n\}$ defined by

$$X_n = aX_{n-1} + be_nX_{n-1} + e_n, \tag{1.1}$$

where $\{e_n\}$ is a sequence of i.i.d. random variables with a density function that is positive and lower semicontinuous on the real line $\mathbb{R}$, a and b are constants. It is an AR(1) process with ARCH(1)-type errors. This model is useful for modeling financial time series in which the current volatility depends on the past value, including on its sign. This asymmetry has been pointed out to be a characteristic feature of financial time series. (See Rabemanajara and Zakoian, 1993 and the references therein.)

Tong (1981) studies this process and concludes that a sufficient condition for ergodicity is $E(|a + be_n|) < 1$. This conclusion was proved by assuming that $\{e_n\}$ is a Gaussian white noise process and using the ergodic drift criterion with the test function $V(x) = |x|$ (see Section 3). Tong remarks at the end of his paper that the normality assumption of e_n can be replaced by an assumption of absolutely continuous distribution with finite mean. At first glance, the proof can go through with the weaker

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* Corresponding author. Tel.: +1-797-845-3181; fax: +1-797-845-3144.

E-mail address: dcline@stat.tamu.edu (D.B.H. Cline).

assumption with no problem. The conclusion only holds, however, when $a \neq 1$. The story is different if $a = 1$. Of course, if $E(e_n) = 0$ then $E(|a + be_n|) > |E(a + be_n)| = |a|$, and hence $E(|a + be_n|) < 1$ implies $|a| < 1$. But if $a = 1$ and $E(e_n) \neq 0$ then there can exist $b \in \mathbb{R}$ such that $E(|1 + be_n|) < 1$ and yet $\{X_n\}$ is not ergodic. Why does the drift criterion fail in case $a = 1$?

We will investigate the case $a = 1$ in Section 2, answer the above question and show a stronger result, namely geometric ergodicity, for the case $a \neq 1$ in Section 3. Finally, in Section 4, we mention some other AR(1) processes with ARCH(1)-type errors that have been studied recently.

2. Ergodicity, stationary distribution and limiting distribution

It is well known that there is a unique stationary distribution π for an ergodic Markov process and, for every initial state x,

$$P^n(x, A) \to \pi(A) \quad \text{as } n \to \infty \text{ for all measurable sets } A, \tag{2.1}$$

where $P^n(\cdot, \cdot)$ is the nth transition probability for the Markov process. On the other hand, a Markov process may have a stationary distribution π without being ergodic and (2.1) may fail for some x. We shall see that $\{X_n\}$ defined by (1.1) is not ergodic when $a = 1$ even if $E(|1 + be_n|^r) < 1$ for some $r \in (0, 1]$, but in this case, $\{X_n\}$ has a unique stationary distribution which is the weak limit of $P^n(x, \cdot)$ as $n \to \infty$ for every $x \in \mathbb{R}$.

Theorem 2.1. *If $a = 1$, the process $\{X_n\}$ on $\mathbb{R}$ is not ψ-irreducible for any measure ψ on $\mathcal{B}(\mathbb{R})$ and hence cannot be ergodic.*

Proof. It is clear that if $X_j = -1/b$ for some j, then $X_n = -1/b$ for all $n \geq j$. Also, if $x \neq -1/b$ then

$$P\left(x, \left\{-\frac{1}{b}\right\}\right) = P\left(x + (bx + 1)e_n = -\frac{1}{b}\right) = 0.$$

Thus, both $\{-1/b\}$ and $\mathbb{R} \setminus \{-1/b\}$ are absorbing for $\{X_n\}$ and the conclusion follows. $\square$

For $x \in \mathbb{R}$, let δ_x denote the degenerate distribution concentrated at x.

Theorem 2.2. *Suppose $a = 1$ and $E(\log(|1 + be_n|)) < 0$. Then $\delta_{-1/b}$ is the unique stationary distribution for $\{X_n\}$ and (2.1) holds only when $x = -1/b$. Nevertheless, $P^n(x, \cdot)$ converges weakly to $\delta_{-1/b}$ (in notation, $P^n(x, \cdot) \Rightarrow \delta_{-1/b}$) for every $x \in \mathbb{R}$.*

Proof. It is trivial to see that $\delta_{-1/b}$ is a stationary distribution for $\{X_n\}$ and (2.1) holds only when $x = -1/b$. To show the weak convergence, let $\eta = E(\log(|1 + be_n|))$ and $Y_n = X_n + (1/b)$. Now assume $y \neq 0$ and $Y_0 = y$. Then

$$Y_n = (1 + be_n)Y_{n-1} = (1 + be_n)(1 + be_{n-1})Y_{n-2} = \cdots = \prod_{j=1}^{n}(1 + be_j)y$$

and $Y_n^{1/n} \to e^\eta < 1$ by the strong law of large numbers. Hence,

$$P(|Y_n| > \varepsilon | Y_0 = y) \to 0 \quad \text{as } n \to \infty, \tag{2.2}$$

Trivially, (2.2) also holds for $y = 0$. It is easy to see from (2.2) that, for every $x \in \mathbb{R}$,

$$P^n(x, \cdot) \Rightarrow \delta_{-1/b} \quad \text{as } n \to \infty. \tag{2.3}$$

Now, suppose that π is a stationary distribution for $\{X_n\}$, then for every $A \in \mathscr{B}(\mathbb{R})$ and every $n \geqslant 1$,

$$\pi(A) = \int P^n(x, A)\pi(dx).$$

This and (2.3) imply $\pi = \delta_{-1/b}$. Hence $\delta_{-1/b}$ is the only stationary distribution. $\square$

3. The drift criteria

Suppose $\{X_n\}$ is a Markov process on a normed space $\mathbb{X}$ and $\mathscr{B}(\mathbb{X})$ is the Borel σ-field on $\mathbb{X}$ with topology generated by the norm. Here, we mention two drift criteria, namely, the ergodicity drift criterion and the geometric ergodicity drift criterion.

The (geometric) ergodicity drift criterion. There exist a nonnegative $\mathscr{B}(\mathbb{X})$-measurable function V on $\mathbb{X}$ (with $V(\cdot) \geqslant 1$), a small set $K \subset \mathbb{X}$, finite positive constants $c(c < 1)$ and B such that

$$E(V(X_1) - V(x)|X_0 = x) \leqslant -c \text{ for every } x \notin K, \tag{3.1}$$

$$(E(V(X_1) - V(x)|X_0 = x) \leqslant -cV(x) \text{ for every } x \notin K) \tag{3.1g}$$

and

$$E(V(X_1) - V(x)|X_0 = x) \leqslant B \text{ for every } x \in K, \tag{3.2}$$

where the extra conditions needed for the geometric ergodicity are put in the parentheses.

If the process $\{X_n\}$ is aperiodic and ψ-irreducible for some nontrivial σ-finite measure ψ on $\mathscr{B}(\mathbb{X})$, then the drift criteria imply ergodicity and geometric ergodicity, respectively. In practice, K is often chosen to be a compact set and $\{X_n\}$ is in a "certain class of chains" which guarantees that every compact set is small. The "certain class" originally meant strong Feller chains (Tweedie, 1975) but now can be taken to mean T-chains (Meyn and Tweedie, 1992 or 1993).

In Tong (1981), it is asserted that $\{X_n\}$ defined by (1.1) is strong Feller and the ergodicity drift criterion is applied with $V(x) = |x|$, $c > 0$, $B = E(|e_n|)$ and $K = \{x \in \odot : |x| \leqslant (c + B)(1 - \eta)^{-1}\}$, where $\eta = E(|a + be_n|) < 1$, but the notation $\odot$ is not explained. Since $\{X_n\}$ is strong Feller on $\mathbb{R}\backslash\{-1/b\}$, not on $\mathbb{R}$, it is reasonable to assume that $\odot = \mathbb{R}\backslash\{-1/b\}$. Clearly (3.1) and (3.2) are satisfied and ergodicity is claimed. Note that in this proof, $E(e_n) = 0$ is not used. Also, (3.1) and (3.2) are satisfied no matter $a = 1$ or not. However, as we pointed out in the introduction, if $E(e_n) \neq 0$ then there exists $b \in \mathbb{R}$ such that $E(|1 + be_n|) < 1$. In case $a = 1$, $\{X_n\}$ restricted to $\odot = \mathbb{R}\backslash\{-1/b\}$ is aperiodic and ψ-irreducible (see Cline and Pu, 1998) and strong Feller, but Theorem 2.2 implies that this restricted process has no stationary distribution and hence cannot be ergodic even if $E(|1 + be_n|) < 1$. What could be going wrong here? The answer is: K is not compact (Theorem 3.1 below) and may not be small and hence we cannot say that the ergodicity drift criterion is satisfied even though (3.1) and (3.2) hold. (In fact, this together with Theorem 2.2 shows that K is not small for $\{X_n\}$.)

Theorem 3.1. *Let* $\odot = \mathbb{R} \setminus \{-1/b\}$. *If* $\eta = E(|1 + be_n|) < 1$ *and* $B = E(|e_n|)$, *then* $K = \{x \in \odot : |x| \leqslant (c + B)(1 - \eta)^{-1}\}$ *is not compact for any* $c > 0$.

Proof. First, we note that $E(|1 + be_n|) < 1$ implies $b \neq 0$. Clearly, $-1/b \notin K$ and K cannot be compact unless $(c + B)(1 - \eta)^{-1} < |b|^{-1}$. Now, $\eta = E(|1 + be_n|) \geqslant 1 - |b|B$ and hence $(1 - \eta) \leqslant |b|B$. Thus, for any $c > 0$, $(c + B)(1 - \eta)^{-1} \geqslant B(1 - \eta)^{-1} \geqslant |b|^{-1}$ and K is not compact. $\square$

Remark. If the underlying space is $\mathbb{R}$, then K is relatively compact since it is bounded. In the case $a = 1$, however, we cannot apply the drift criteria to $\{X_n\}$ on $\mathbb{R}$, since it is not ψ-irreducible for any ψ (Theorem 2.1).

Now assume $a \neq 1$. Then $\{X_n\}$ defined by (1.1) is weak Feller on $\mathbb{R}$, aperiodic and μ-irreducible, where μ is the Lebesgue measure on $\mathscr{B}(\mathbb{R})$ since the density function of e_n is positive everywhere on $\mathbb{R}$. Clearly, the support of μ is $\mathbb{R}$ which has nonempty interior. This implies that $\{X_n\}$ is a T-chain (Meyn and Tweedie, 1993) and hence the use of the drift criteria with compact set K is secured. Since the set $K = \{x \in \mathbb{R} : |x| \leqslant C\}$ is compact for every $C > 0$, we can easily get (3.1g) and (3.2) by taking $V(x) = 1 + |x|^r$. Hence, we have the following theorem which may be considered known by now, but we provide it here for completeness.

Theorem 3.2. *Suppose* $\{X_n\}$ *is defined by* (1.1) *with* $a \neq 1$. *If there exists* $r > 0$ *such that* $E(|e_n|^r) < \infty$ *and* $E(|a + be_n|^r) < 1$ *then* $\{X_n\}$ *is geometrically ergodic.*

Corollary 3.3. *Suppose* $\{X_n\}$ *is defined by* (1.1) *with* $a \neq 1$ *and there exists* $r > 0$ *such that* $E(|e_n|^r) < \infty$. *Let* f *denote the density function of* e_n.

(i) *If* $\sup_{u \in \mathbb{R}} f(u) = M < \infty$, *then* $E(|\log(|a + be_n|)|) < \infty$.
(ii) *If* $E(|\log(|a + be_n|)|) < \infty$ *and* $E(\log(|a + be_n|)) < 0$, *then* $\{X_n\}$ *is geometrically ergodic.*

Proof. Both conclusions are obvious if $b = 0$. Assume $b \neq 0$. Note that if $\sup_{u \in \mathbb{R}} f(u) = M < \infty$, then

$$E((\log|a + be_n|)1_{|a+be_n| \leqslant 1}) = \int (\log(|a + bu|)1_{|a+bu| \leqslant 1} f(u)\, \mathrm{d}u$$

$$= \int_{-1}^{1} \log|v| f\left(\frac{v - a}{b}\right) b^{-1}\, \mathrm{d}v \geqslant 2Mb^{-1} \int_{0}^{1} \log(v)\, \mathrm{d}v > -\infty.$$

It follows from the above and the assumption $E(|e_n|^r) < \infty$ for some $r > 0$ that (i) is proved.

Since

$$\log(|a + be_n|) \leqslant \frac{|a + be_n|^s - 1}{s} \leqslant \frac{|a + be_n|^r - 1}{r} \quad \text{for all } s \in (0, r],$$

we can use the dominated convergence theorem to conclude that

$$\lim_{s \downarrow 0} E\left(\frac{|a + be_n|^s - 1}{s}\right) = E(\log(|a + be_n|)) < 0.$$

Thus, there exists $s > 0$ such that $E(|a + be_n|^s) < 1$ for some $s > 0$ and (ii) follows from Theorem 3.2 immediately. $\square$

Remarks. (i) As Quinn (1982) points out, it is interesting to note that there are combinations of a, b and the distribution of e_n that allow ergodicity even if $|a| > 1$.

(ii) Suppose $E(|\log(|x+e_n|)|) < \infty$ for all $x \in \mathbb{R}$. Quinn (1982) shows that if $E(\log(|a+be_n|)) < 0$, then (1.1) admits a strictly stationary solution:

$$X_n = e_n + \sum_{j=1}^{\infty} \left\{ \prod_{i=0}^{j-1} (a + be_{n-i}) \right\} e_{n-j}.$$

In fact, the series $\sum_{j=1}^{\infty} \{ \prod_{i=0}^{j-1} (a + be_{n-i}) \}$ is almost surely absolutely summable. Hence,

$$X_n = e_n + \frac{1}{b} \sum_{j=1}^{\infty} \left\{ \prod_{i=0}^{j} (a + be_{n-i}) - a \prod_{i=0}^{j-1} (a + be_{n-i}) \right\}$$

$$= \frac{-a}{b} + \frac{1-a}{b} \sum_{j=1}^{\infty} \prod_{i=0}^{j-1} (a + be_{n-i}).$$

In particular, if $a = 1$ then $X_n = -1/b$ almost surely, as predicted by Theorem 2.2.

4. Other AR(1) processes with ARCH(1)-type errors

The process $\{X_n\}$ defined by

$$X_n = aX_{n-1} + \sqrt{1 + b^2 X_{n-1}^2}\, e_n, \tag{4.1}$$

with $b > 0$ is an AR(1) process with ARCH(1) errors, the simplest AR-ARCH model in which the current volatility depends on the past value only through its magnitude. Borkovec and Klüppelberg (2001) mention that $E(\log(|a + be_n|)) < 0$ is a sufficient condition for geometric ergodicity, under general assumptions on e_n. This may be proved by an argument similar to our Corollary 3.3.

An AR(1) process with general ARCH(1)-type errors defined by

$$X_n = aX_{n-1} + (\beta(X_{n-1})X_{n-1} + \delta)e_n \tag{4.2}$$

is studied by Ferrant et al. (2000), where $\{e_n\}$ is the same as stated at the beginning of this paper, a and δ are real numbers, and the function $\beta(\cdot)$ satisfies some regularity condition so that $\{X_n\}$ is an aperiodic, ψ-irreducible T-chain. A sufficient condition for geometric ergodicity is found to be

$$|a| + \sup_{x \in \mathbb{R}} |\beta(x)| E(|e_n|) < 1 \tag{4.3}$$

by using the geometric ergodicity drift criterion with $V(x) = 1 + |x|$. This model can handle the asymmetry in volatility and includes model (1.1), but $|a|$ has to be less than 1 if (4.3) is satisfied.

Pu and Cline (2001) study nonlinear AR(p) processes with ARCH(p)-type errors with affine threshold. Included there is the example (case $p = 1$)

$$X_n = a^*(X_{n-1}) + b^*(X_{n-1})e_n + a_0(X_{n-1}) + b_0(X_{n-1})e_n, \tag{4.4}$$

where $a^*(x) = a_1 1_{x<0} + a_2 1_{x>0}$, $b^*(x) = b_1 1_{x<0} + b_2 1_{x>0}$ and $a_0(x) = o(|x|)$, $b_0(x) = o(|x|)$ as $|x| \to \infty$ and $b^*(x) + b_0(x) \neq 0$ for $x \in \mathbb{R}$. Also, $\{e_n\}$ is the same as in the above with $E(|e_n|^r) < \infty$ for some $r > 0$ and $\sup_{u \in \mathbb{R}} f(u) < \infty$ as in Corollary 3.3(i). A sufficient condition for geometric ergodicity in case $|a_i| + |b_i| \neq 0$, $i = 1, 2$, is

$$P(a_2 + b_2 e_1 \leqslant 0)E(\log|a_1 + b_1 e_1|) + P(a_1 + b_1 e_1 \leqslant 0)E(\log|a_2 + b_2 e_1|) < 0.$$

If $a_1 = b_1 = 0$ and $b_2 \neq 0$ (or $a_2 = b_2 = 0$ and $b_1 \neq 0$) then $\{X_n\}$ is geometrically ergodic. This result is obtained by using a new approach called the *piggyback method* (Cline and Pu, 2001). The condition is sharp, but the model does not include (1.1) since it requires that $b^*(x) + b_0(x) \neq 0$ for all $x \in \mathbb{R}$.

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