THE INSTITUTE OF APPLIED MATHEMATICS AND STATISTICS

INFINITE SERIES OF RANDOM VARIABLES

WITH REGULARLY VARYING TAILS

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Technical Report No. 83-24

November, 1983

The University of British Columbia
Vancouver, Canada

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Abstract We give conditions for the convergence of an infinite linear series of independent and identically distributed random variables, whose distribution has regularly varying tails. More importantly, we show that the distribution of such a series is tail equivalent to the distribution of its components. This enables us to define a quantity, which we call dispersion, measuring the relative thickness of the tails and thereby to compare different infinite series. The dispersion may be related to the ℓ_α metric for sequence spaces and this leads to a notion of linear projection which is useful for prediction of time series.

1. Introduction

We are concerned with random variables of the form $Y = \sum_{j=1}^{\infty} \rho_j W_j$ where $\{W_j\}$ is a sequence of independent random variables, all from a distribution F* with regularly varying tails. Let F be the distribution of $|W_j|$ and define $\bar{F}(t) = 1-F(t) = P[|W_j|>t]$. We say that $\bar{F}(t)$ is regularly varying with exponent $-\alpha$ ($\bar{F}\epsilon RV_{-\alpha}$) if for every $s>0$

$$\lim_{t\to\infty} \frac{\bar{F}(st)}{\bar{F}(t)} = s^{-\alpha}.$$

In fact, if the limit exists at all, then it will be of the form $s^{-\alpha}$ for some $\alpha>0$ and the convergence will be uniform on $[s_0,\infty)$ for any $s_0>0$. Furthermore, for any $\epsilon>0$ there exists $c>0$ such that $\bar{F}(t)\leq ct^{-\alpha+\epsilon}$ for all $t\geq s_0$. (See Feller II for a discussion of more general regularly varying functions.) The parameter α we call the tail index of F. Distributions with index α have moments up to (and perhaps including) order α, but higher moments do not exist. In particular, if $\alpha<2$ then the variance does not exist. More precisely, Feller (1971), p. 283, proves the following relationship between F and its truncated moments.

Lemma 1.1 Suppose $|W|$ has distribution F* with $\bar{F}(t) = P[|W|>t] \epsilon RV_{-\alpha}$. Define $m_\gamma(t) = E[|W|^\gamma 1_{|W|\leq t}]$ and, when it exists, $u_\gamma(t) = E[|W|^\gamma 1_{|W|>t}]$. Then for $\gamma\geq\alpha$, $t^{-\gamma}m_\gamma(t) \epsilon RV_{-\alpha}$ and $\lim_{t\to\infty} \frac{m_\gamma(t)}{t^\gamma\bar{F}(t)} = \frac{\alpha}{\gamma-\alpha}$, and when $u_\gamma(t)<\infty$, $t^{-\gamma}u_\gamma(t)\epsilon RV_{-\alpha}$ and $\lim_{t\to\infty} \frac{u_\gamma(t)}{t^\gamma\bar{F}(t)} = \frac{\alpha}{\alpha-\gamma}$.

Random variables with regularly varying tails exhibit a striking relationship between the distributions of sums and of maxima. The following is a modification of a theorem in Feller (1971), p. 278.

Lemma 1.2 Suppose $W_1, W_2, \ldots, W_n$ are independent. Let F_j be the distribution of $|W_j|$ and suppose $\lim_{t \to \infty} \dfrac{\bar{F}_j(t)}{\bar{F}(t)} = 1$, where $\bar{F} \in RV_{-\alpha}$. For real numbers $\rho_1, \ldots \rho_n$, define $\bar{G}(t) = P\left[\,\left|\,\sum_{j=1}^{n} \rho_j W_j\,\right| > t\,\right]$ and $\bar{H}(t) = P\left[\sup_{1 \leq j \leq n} |\rho_j W_j| > t\right]$. Then $\bar{G} \in RV_{-\alpha}$, $\bar{H} \in RV_{-\alpha}$ and

$$\lim_{t \to \infty} \frac{\bar{G}(t)}{\bar{F}(t)} = \lim_{t \to \infty} \frac{\bar{H}(t)}{\bar{F}(t)} = \sum_{j=1}^{n} |\rho_j|^{\alpha}.$$

Proof: The result, if true for n=2, extends by induction. We therefore consider only the case n=2. First,

$$\lim_{t \to \infty} \frac{\bar{H}(t)}{\bar{F}(t)} = \lim_{t \to \infty} \left(\frac{\bar{F}_1(t/|\rho_1|) + \bar{F}_2(t/|\rho_2|) - \bar{F}_1(t/|\rho_1|)\bar{F}_2(t/|\rho_2|)}{\bar{F}(t)} \right)$$

$$= |\rho_1|^{\alpha} + |\rho_2|^{\alpha} + 0$$

by the regular variation principle.

Second, by an application of the theorem in Feller (1971), p. 278,

$$\overline{\lim_{t \to \infty}} \frac{\bar{G}(t)}{\bar{F}(t)} \leq \lim_{t \to \infty} \frac{P[\,|\rho_1 W_1| + |\rho_2 W_2| > t\,]}{\bar{F}(t)}$$

$$= \lim_{t \to \infty} \frac{\bar{F}_1(t/|\rho_1|) + \bar{F}_2(t/|\rho_2|)}{\bar{F}(t)}$$

$$= |\rho_1|^{\alpha} + |\rho_2|^{\alpha}. \tag{1.1}$$

However, for any $\delta > 0$

$$\bar{G}(t) = P[|\rho_1 W_1 + \rho_2 W_2| > t]$$

$$\geq P[|\rho_1 W_1| > (1+\delta)t, |\rho_2 W_2| \leq \delta t] + P[|\rho_2 W_2| > (1+\delta)t, |\rho_1 W_1| \leq \delta t]$$

$$= \bar{F}_1\left(\frac{(1+\delta)t}{|\rho_1|}\right) F_2\left(\frac{\delta t}{|\rho_2|}\right) + \bar{F}_1\left(\frac{(1+\delta)t}{|\rho_2|}\right) F_2\left(\frac{\delta t}{|\rho_1|}\right)$$

And from this we calculate

$$\lim_{t \to \infty} \frac{\bar{G}(t)}{\bar{F}(t)} \geq (1+\delta)^{-\alpha}|\rho_1|^\alpha + (1+\delta)^{-\alpha}|\rho_2|^\alpha. \tag{1.2}$$

Since δ is arbitrary, then (1.2) combined with (1.1) gives us the result. #

Lemma 1.2 tells us in particular, how to compare the tail of the distribution of a sum with the tail of the distribution of each component. We will extend this result to infinite series in Section 2. Whenever two distributions with regularly varying tails (say F_1 and F_2) satisfy

$$\lim_{t \to \infty} \frac{\bar{F}_1(t)}{\bar{F}_2(t)}$$ exists and is nonzero, we say that F_1 and F_2 are tail equiva-

lent. The limiting ratio gives us a convenient means to compare the probability of large values of random variables from the two distributions. In particular, we may be interested in the probability that

$$|Y_1| = \left|\sum_{j=1}^{\infty} \rho_{1j} W_j\right| \text{ is large relative to the probability that}$$

$$|Y_2| = \left|\sum_{j=1}^{\infty} \rho_{2j} W_j\right| \text{ is large.} \text{ For example, } Y_1 \text{ and } Y_2 \text{ might be the pre-}$$

diction errors from alternate methods of predicting a time series and we may prefer to choose the predictor which has the least chance of large errors (see Cline and Brockwell (1983)).

The limiting ratio of probabilities for $Y = \sum_{j=1}^{\infty} \rho_j W_j$ and W_1 will turn out to be $\sum_{j=1}^{\infty} |\rho_j|^{\alpha}$, a quantity we will call the _dispersion of Y._ When comparing variables on the linear space generated by a given sequence $\{W_j\}$, the dispersion is a useful measure of distance. This leads to the concept of minimum dispersion projection for variables in this linear space. Section 3 investigates this notion.

2. Existence and Tail Behavior of Infinite Series

We start with an application of Kolmogorov's **three series theorem** to series of regularly varying tail variables.

__Theorem 2.1__ Suppose $\{W_j\}$ are iid $F*$ and $\bar{F}(t) = P[|W_j|>t]\epsilon RV_{-\alpha}$.

Then $Y = \lim_{n\to\infty} \sum_{j=1}^{n} \rho_j W_j$ exists almost surely if either

1) $\sum_{j=1}^{\infty} |\rho_j|^{\delta} < \infty$ for some $\delta < \alpha$, $\delta \leq 1$

or

ii) EW_j exists and equals 0, and $\sum_{j=1}^{\infty} |\rho_j|^{\delta} < \infty$

for some $\delta < \alpha, \delta \leq 2$ (or $\delta = 1$ if $\alpha = 1$).

__Proof:__

1) The series $Y = \sum_{j=1}^{\infty} \rho_j W_j$ is absolutely convergent if and only if for all $v>0$

$$\sum_{j=1}^{\infty} P[|\rho_j W_j|>v] = \sum_{j=1}^{\infty} \bar{F}(v/|\rho_j|) < \infty$$

and

$$\sum_{j=1}^{\infty} E\left[|\rho_j W_j| 1_{|\rho_j W_j| \leq v}\right] = \sum_{j=1}^{\infty} |\rho_j| m_1(v/|\rho_j|) < \infty.$$

(The third series is not necessary to prove absolute convergence.)

Since by Lemma 1.1,

$$\lim_{t \to \infty} \frac{\bar{F}(t)}{t^{-1}m_1(t)} = \begin{cases} \frac{1-\alpha}{\alpha} & \text{if } \alpha \leq 1 \\ 0 & \text{otherwise} \end{cases},$$

then the terms in the first series are dominated by the terms in the second and it suffices to show the second series converges.

If $\alpha \leq 1$ then $t^{-1}m_1(t) \epsilon RV_{-\alpha}$ and so there exists a $c>0$ such that for any $s > \frac{v}{\sup_j |\rho_j|}$, $s^{-1}m_1(s) \leq cs^{-\delta}$, if $\delta < \alpha$. If $\alpha > 1$ then $m_1(t) \to E|W_j|$ and so we can use the bound $s^{-1}m_1(s) \leq cs^{-\delta}$ if $\delta \leq 1$. In either case,

$$\sum_{j=1}^{\infty} |\rho_j| m_1(v/|\rho_j|) \leq cv^{-\delta} \sum_{j=1}^{\infty} |\rho_j|^{\delta} < \infty.$$

Thus, condition i) is sufficient for absolute convergence of Y.

ii) If $EW_j = 0$ (in which case $\alpha \geq 1$), then

$$\left| E\left[W_j 1_{|W_j| \leq t} \right] \right| = \left| E\left[-W_j 1_{|W_1| > t} \right] \right|$$

$$\leq E\left[|W_j| 1_{|W_j| > t} \right]$$

$$= u_1(t)$$

For Y to exist, it suffices to prove that

$$\sum_{j=1}^{\infty} P[|\rho_j W_j| > v] = \sum_{j=1}^{\infty} \bar{F}(v/|\rho_j|) < \infty$$

$$\sum_{j=1}^{\infty} \left| \rho_j E\left[W_j 1_{|\rho_j W_j| \leq v} \right] \right| \leq \sum_{j=1}^{\infty} |\rho_j| u_1(v/|\rho_j|) < \infty$$

and

$$\sum_{j=1}^{\infty} E\left[(\rho_j W_j)^2 1_{|\rho_j W_j| \leq v} \right] = \sum_{j=1}^{\infty} \rho_j^2 m_2(v/|\rho_j|) < \infty.$$

From Lemma 2.1

$$\lim_{t\to\infty} \frac{\bar{F}(t)}{t^{-1}u_1(t)} = \frac{\alpha-1}{\alpha}$$

$$\lim_{t\to\infty} \frac{\bar{F}(t)}{t^{-2}m_2(t)} = \begin{cases} \dfrac{2-\alpha}{\alpha} & \text{if } \alpha\leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Thus if $\alpha=1$, convergence of the second series is sufficient and if $\alpha>1$, convergence of the third is sufficient. For $\alpha=1$, since $u_1(t)\to 0$ there exists c such that for all $s\geq\dfrac{v}{\sup_j|\rho_j|}$, $u_1(s)\leq c$ so that

$$\sum_{j=1}^{\infty} |\rho_j|u_1(v/|\rho_j|) \leq c\sum_{j=1}^{\infty}|\rho_j|$$

and hence condition ii) guarantees that Y exists.

For $\alpha>1$, we can find c such that $s^{-2}m_2(s)\leq cs^{-\delta}$ where $\delta<\alpha$, $\delta\leq 2$, so that

$$\sum_{j=1}^{\infty} |\rho_{j_1}|^2 m_2(v/|\rho_j|) \leq cv^{2-\delta}\sum_{j=1}^{\infty}|\rho_j|^\delta$$

and again condition ii) is sufficient. #

We remark that when $\sum_{j=1}^{\infty}\rho_j W_j$ is absolutely convergent then $\sup_j|\rho_j W_j|$ exists almost surely, also.

Sometimes the condition $\sum_{j=1}^{\infty}|\rho_j|^\alpha<\infty$ is sufficient for the existence of $\sum_{j=1}^{\infty}\rho_j W_j$. For an example, assume the W_j are symmetric about 0 and $\bar{F}\epsilon RV_{-\alpha}$, $0<\alpha<2$. In this case it is sufficient to show

$$\sum_{j=1}^{\infty} \bar{F}(v/|\rho_j|) < \infty \text{ for all } u>0.$$

If $\bar{F}$ satisfies $\overline{\lim_{t\to\infty}} t^{\alpha}\bar{F}(t) < \infty$, then there exists c such that for

$$t > \frac{u}{\sup_j |\rho_j|}, \quad \bar{F}(t) \le ct^{-\alpha}. \quad \text{Thus}$$

$$\sum_{j=1}^{\infty} \bar{F}(v/|\rho_j|) \le cv^{-\alpha} \sum_{j=1}^{\infty} |\rho_j|^{\alpha} < \infty$$

and hence $\sum_{j=1}^{\infty} \rho_j W_j$ exists almost surely.

On the other hand, a counterexample is the following. Suppose $\{W_j\}$ are distributed so that for $t \ge e^{1/\alpha}$, $P[|W_j|>t] = \bar{F}(t) = \alpha e\, t^{-\alpha}\ln t$.

Suppose also that $\rho_j = (j(\ln j)^2)^{-1/\alpha}$; $j \ge 3$. Then $\sum_{j=3}^{\infty} \rho_j^{\alpha} < \infty$, but with j_0 chosen large enough,

$$\sum_{j=3}^{\infty} \bar{F}(1/\rho_j) = e \sum_{j=3}^{\infty} \frac{\ln(j(\ln j)^2)}{j(\ln j)^2}$$

$$\ge e \sum_{j=3}^{\infty} \frac{1}{j\ln j} = \infty.$$

Therefore $\sum_{j=3}^{\infty} \rho_j W_j$ almost surely does not exist.

Lemma 2.2 Suppose F* is a probability measure for W with F the distribution of $|W|$ and $\bar{F}(t) = P[|W|>t]\epsilon RV_{-\alpha}, \alpha > 0$. Suppose also that $\{\rho_j\}$ satisfies $\sum_{j=1}^{\infty} |\rho_j|^{\delta} < \infty$ for some $\delta<\alpha$. Then

$$\text{i)} \quad \lim_{t\to\infty} \frac{\sum_{j=1}^{\infty} \bar{F}(t/|\rho_j|)}{\bar{F}(t)} = \sum_{j=1}^{\infty} |\rho_j|^{\alpha}$$

and

$$\text{ii)} \quad \lim_{t\to\infty} \frac{1- \prod_{j=1}^{\infty} F(t/|\rho_j|)}{\bar{F}(t)} = \sum_{j=1}^{\infty} |\rho_j|^{\alpha}.$$

Proof:

i) Let $m = \sup_j |\rho_j|$. There exists $c > 0$, $t_0 > 0$ such that for all $t > t_0$, $y > 1/m$,

$$\frac{\bar{F}(ty)}{\bar{F}(t)} \leq cy^{-\delta}.$$

Therefore $\dfrac{1}{\bar{F}(t)} \displaystyle\sum_{j=1}^{\infty} \bar{F}(t/|\rho_j|) \leq \sum_{j=1}^{\infty} |\rho_j|^{\delta} < \infty$, $t \geq t_0$.

Since $\displaystyle\lim_{t\to\infty} \frac{\bar{F}(t/|\rho_j|)}{\bar{F}(t)} = |\rho_j|^{\alpha}$

then by dominated convergence the result i) holds.

ii) By i) $\displaystyle\sum_{j=1}^{\infty} \bar{F}(t/|\rho_j|) < \infty$ for all $t > 0$ and $\sup_j \bar{F}(t/|\rho_j|) \to 0$ as $t \to \infty$.

We can therefore exchange $\bar{F}$ with $\ln F$ and $1 - \displaystyle\prod_{j=1}^{\infty} F(t/|\rho_j|)$ with $\displaystyle\sum_{j=1}^{\infty} \ln F(t/|\rho_j|)$ to get

$$\lim_{t\to\infty} \frac{\displaystyle\sum_{j=1}^{\infty} \ln F(t/|\rho_j|)}{1 - \displaystyle\prod_{j=1}^{\infty} F(t/|\rho_j|)} = \lim_{t\to\infty} \frac{\displaystyle\sum_{j=1}^{\infty} \bar{F}(t/|\rho_j|)}{\displaystyle\sum_{j=1}^{\infty} \ln F(t/|\rho_j|)} = 1$$

With i), this implies ii). #

The main result of this section is next.

Theorem 2.3 Suppose $\{W_j\} \sim$ iid F^* where $\bar{F}(t) = P[|W_j| > t] \epsilon RV_{-\alpha}$ and suppose $\{\rho_j\}$ satisfy $\displaystyle\sum_{j=1}^{\infty} |\rho_j|^{\delta} < \infty$ for some $\delta < \min(1,\alpha)$. Let $\bar{G}(t) = P[|\displaystyle\sum_{j=1}^{\infty} \rho_j W_j| > t]$ and $\bar{H}(t) = P[\sup_j |\rho_j W_j| > t]$. Then $\displaystyle\lim_{t\to\infty} \frac{\bar{G}(t)}{\bar{F}(t)} = \lim_{t\to\infty} \frac{\bar{H}(t)}{\bar{F}(t)} = \sum_{j=1}^{\infty} |\rho_j|^{\alpha}$.

Proof: Set $Y = \displaystyle\sum_{j=1}^{\infty} \rho_j W_j$, $Y_n = \displaystyle\sum_{j=1}^{n} \rho_j W_j$, $Z_1 = \displaystyle\sum_{j=1}^{\infty} |\rho_j W_j|$ and $Z_\infty = \sup_j |\rho_j W_j|$.

By Theorem 2.1, Z_1 exists almost surely and hence Z_∞ and Y do also.

Since $H(t) = \prod\limits_{j=1}^{\infty} (1-\bar{F}(t/|\rho_j|))$, Lemma 2.2 immediately gives the second

conclusion.

To prove the first, we let G_n be the distribution of Y_n. Then

for any $n \geq 1$, $\varepsilon > 0$,

$$P[|Y| > t] \geq P[|Y_n| > (1+\varepsilon)t, \ |Y-Y_n| < \varepsilon t]$$

$$= \bar{G}_n((1+\varepsilon)t)P[|Y-Y_n| < \varepsilon t].$$

Thus,

$$\lim_{t \to \infty} \frac{\bar{G}(t)}{\bar{F}(t)} \geq \lim_{t \to \infty} \frac{\bar{G}_n((1+\varepsilon)t)}{\bar{F}(t)} P[|Y-Y_n| < \varepsilon t]$$

$$= (1+\varepsilon)^{-\alpha} \sum_{j=1}^{n} |\rho_j|^{\alpha},$$

where the limit is obtained by using Lemma 1.2 and the fact that $\bar{F} \varepsilon RV_{-\alpha}$.
Since both n and ε are arbitrary,

$$\lim_{t \to \infty} \frac{\bar{G}(t)}{\bar{F}(t)} \geq \sum_{j=1}^{\infty} |\rho_j|^{\alpha} \qquad (2.1)$$

The alternate inequality is first proven for $\alpha < 1$. Define

$\phi(\lambda) = E e^{-\lambda |W_j|}$. When $\alpha < 1$, then by Feller (1971), p. 447,

$$\lim_{t \to \infty} \frac{1 - \phi(1/t)}{\bar{F}(t)} = \Gamma(1-\alpha) \qquad (2.2)$$

This indicates that $1 - \phi(1/t)$ is a regularly varying distribution tail,
so that Lemma 2.2 applies.

$$\lim_{t \to \infty} \frac{1 - \prod\limits_{j=1}^{\infty} \phi(|\rho_j|/t)}{1 - \phi(1/t)} = \sum_{j=1}^{\infty} |\rho_j|^{\alpha} \qquad (2.3)$$

Of course, $Ee^{-\lambda Z_1} = \prod\limits_{j=1}^{\infty} \phi(\lambda|\rho_j|)$. Let H_1 be the distribution of Z_1.

Applying the theorem in Feller again, $\bar{H}_1 \in RV_{-\alpha}$ and

$$\lim_{t\to\infty} \frac{1 - \prod\limits_{j=1}^{\infty} \phi(|\rho_j|/t)}{\bar{H}_1(t)} = \Gamma(1-\alpha). \tag{2.4}$$

Combining (2.2), (2.3) and (2.4) we have

$$\lim_{t\to\infty} \frac{\bar{H}_1(t)}{\bar{F}(t)} = \sum_{j=1}^{\infty} |\rho_j|^{\alpha}.$$

Since $|Y| \le Z_1$, then

$$\overline{\lim_{t\to\infty}} \frac{\bar{G}(t)}{\bar{F}(t)} \le \lim_{t\to\infty} \frac{\bar{H}_1(t)}{\bar{F}(t)} = \sum_{j=1}^{\infty} |\rho_j|^{\alpha}$$

which, together with (2.1), proves the result for $\alpha < 1$.

When $\alpha \ge 1$, then $a = \sum\limits_{j=1}^{\infty} |\rho_j| < \infty$. Let $\gamma \in (\alpha, \alpha/\delta)$ and $p_j = \frac{1}{a}|\rho_j|$. By

Holder's inequality, with $\{p_j\}$ as the probability measure,

$$Z_1 = a \sum_{j=1}^{\infty} |W_j| p_j$$

$$\le a \left(\sum_{j-1}^{\infty} |W_j|^{\gamma} p_j \right)^{1/\gamma}$$

$$= a^{1-1/\gamma} \left(\sum_{j=1}^{\infty} |W_j|^{\gamma} |\rho_j| \right)^{1/\gamma} \tag{2.5}$$

The distribution of $|W_j|^{\gamma}$ is $F(t^{1/\gamma})$ and has index $\alpha/\gamma < 1$. Letting

$V = \sum\limits_{j=1}^{\infty} |W_j|^{\gamma}|\rho_j|$ and relying on the result for index less than 1,

$$\lim_{t \to \infty} \frac{P[V > t]}{\bar{F}(t^{1/\gamma})} = \sum_{j=1}^{\infty} |\rho_j|^{\alpha/\gamma} < \infty. \tag{2.6}$$

From (2.5) and (2.6) we can calculate,

$$\varlimsup_{t \to \infty} \frac{\bar{H}_1(t)}{\bar{F}(t)} \leq \lim_{t \to \infty} \frac{P[a^{1-1/\gamma} V^{1/\gamma} > t]}{\bar{F}(t)}$$

$$= \lim_{s \to \infty} \frac{P[V > a^{1-\gamma} s]}{F(s^{1/\gamma})}$$

$$= (a^{1-\gamma})^{-\alpha/\gamma} \sum_{j=1}^{\infty} |\rho_j|^{\alpha/\gamma}$$

$$= \left(\sum_{j=1}^{\infty} |\rho_j| \right)^{\alpha - \alpha/\gamma} \sum_{j=1}^{\infty} |\rho_j|^{\alpha/\gamma}.$$

Since $\gamma > \alpha$ is arbitrary,

$$\varlimsup_{t \to \infty} \frac{\bar{H}_1(t)}{\bar{F}(t)} \leq \left(\sum_{j=1}^{\infty} |\rho_j| \right)^{\alpha}. \tag{2.7}$$

This is still not strong enough to prove our result. However, with $Y_n = \sum_{j=1}^{n} \rho_j W_j$ and $\epsilon < \frac{1}{2}$,

$$P[|Y| > t] \leq P[|Y_n| > (1-\epsilon)t] + P[|Y-Y_n| > (1-\epsilon)t]$$

$$+ P[|Y_n| > \epsilon t, \ |Y-Y_n| > \epsilon t].$$

$$= \bar{G}_n((1-\epsilon)t) + \bar{G}_{-n}((1-\epsilon)t) + \bar{G}_n(\epsilon t)\bar{G}_{-n}(\epsilon t) \tag{2.8}$$

where G_{-n} is the distribution of $Y-Y_n$, which is independent of Y_n. By Lemma 1.2 and an inequality similar to (2.7), respectively,

$$\lim_{t \to \infty} \frac{\overline{G}_n(t)}{\overline{F}(t)} = \sum_{j=1}^{n} |\rho_j|^\alpha$$

and

$$\overline{\lim_{t \to \infty}} \frac{\overline{G}_{-n}(t)}{\overline{F}(t)} \leq \overline{\lim_{t \to \infty}} \frac{P\left[\sum_{j=n+1}^{\infty} |\rho_j W_j| > t\right]}{\overline{F}(t)}$$

$$\leq \left(\sum_{j=n+1}^{\infty} |\rho_j|\right)^\alpha.$$

Using these in (2.8),

$$\overline{\lim_{t \to \infty}} \frac{\overline{G}(t)}{\overline{F}(t)} \leq (1-\varepsilon)^{-\alpha} \sum_{j=1}^{n} |\rho_j|^\alpha + (1-\varepsilon)^{-\alpha}\left(\sum_{j=n+1}^{\infty} |\rho_j|\right)^\alpha.$$

Since n and ε are arbitrary, then

$$\overline{\lim_{t \to \infty}} \frac{\overline{G}(t)}{\overline{F}(t)} \leq \sum_{j=1}^{\infty} |\rho_j|^\alpha$$

and with (2.1) we have our result for $\alpha \geq 1$. #

The quantity $\sum_{j=1}^{\infty} |\rho_j|^\alpha$ we call the dispersion of Y (disp(Y)).
after a similar usage by Stuck (1978). This theorem indicates that disp(Y) is a measure of the probability of large values of Y. If $\{W_j\}$ are symmetric stable (α) in distribution, then Y will also be symmetric stable(α) and $(\text{disp}(Y))^{1/\alpha}$ will be the ratio of Y's scale parameter to W_j's scale. Section 3 demonstrates how dispersion may be used as a measure of distance between random variables which are infinite series in $\{W_j\}$.

Corollary 2.4 Let $Z_\gamma = \left[\sum\limits_{j=1}^{\infty} |\rho_j W_j|^\gamma\right]^{1/\gamma}$ for $\gamma \geq \delta$, then

$$\lim_{t\to\infty} \frac{P[Z_\gamma > t]}{P[|W| > y]} = \sum_{j=1}^{\infty} |\rho_j|^\alpha.$$

Proof: $|W_j|^\gamma$ has distribution tail $\bar{F}_\gamma(t) = \bar{F}(t^{1/\gamma}) \epsilon RV_{-\alpha/\gamma}$. For

$\delta_1 = \delta/\gamma$, $\delta_1 \leq 1$ and $\delta_1 < \alpha/\gamma$ and $\sum\limits_{j=1}^{\infty} \left(|\rho_j|^\gamma\right)^{\delta_1} < \infty$. Thus Z_γ exists

almost surely. Let H_γ be the distribution of $(Z_\gamma)^\gamma = \sum\limits_{j=1}^{\infty} |\rho_j W_j|^\gamma$. By

the theorem,

$$\lim_{t\to\infty} \frac{\bar{H}_\gamma(t)}{\bar{F}_\gamma(t)} = \sum_{j-1}^{\infty} \left(|\rho_j|^\gamma\right)^{\alpha/\gamma} = \sum_{j=1}^{\infty} |\rho_j|^\alpha.$$

Thus

$$\lim_{t\to\infty} \frac{P[Z_\gamma > t]}{P[|W| > t]} = \lim_{t\to\infty} \frac{\bar{H}_\gamma(t^{1/\gamma})}{\bar{F}(t^{1/\gamma})} = \sum_{j=1}^{\infty} |\rho_j|^\alpha. \quad \#$$

3. Dispersion as a Metric

In this section we define a metric for infinite series of regularly varying variables and a corresponding projection operator. We also elaborate on the nature of the projection operator. As before, the sequence $\{W_j\}$ will be independent and identically distributed, $\bar{F}(t) = P[|W_j|>t]$ is the tail of $|W_j|$ and regularly varying with exponent $-\alpha$. Recall that we have defined the dispersion of $Y = \sum_{j=1}^{\infty} \rho_j W_j$ by

$$\text{disp}(Y) = \sum_{j=1}^{\infty} |\rho_j|^{\alpha} \ .$$

Let $\delta>0$ satisfy $\delta < \min(1,\alpha)$. Define now the (random) linear space for given sequence $\{W_j\}$,

$$S = \{Y = \sum_{j=1}^{\infty} \rho_j W_j \text{ such that } \sum_{j=1}^{\infty} |\rho_j|^{\delta} < \infty\}.$$

We remark that in fact we need only work with a space of equivalence classes which are well defined by the distribution structure, but S is a convenient means to express this. For $Y_1 = \sum_{j=1}^{\infty} \rho_{1j} W_j$, $Y_2 = \sum_{j=1}^{\infty} \rho_{2j} W_j$ we define

$$d(Y_1,Y_2) = \begin{cases} \sum_{j=1}^{\infty} |\rho_{1j}-\rho_{2j}|^{\alpha} = \text{disp}(Y_1-Y_2) & \text{if } \alpha \leq 1 \\[2em] \left(\sum_{j=1}^{\infty} |\rho_{1j}-\rho_{2j}|^{\alpha}\right)^{1/\alpha} = (\text{disp}(Y_1-Y_2))^{1/\alpha} & \text{if } \alpha > 1. \end{cases}$$

<u>Lemma 3.1</u> d is a metric on S.

<u>Proof</u>: The only condition not obvious is that $d(Y_1, Y_2) = 0$ if and only if $Y_1 = Y_2$ almost surely. Clearly, if $d(Y_1, Y_2) = 0$ then $\rho_{1j} = \rho_{2j}$ for all j and hence

$$Y_1 - Y_2 = \lim_{n \to \infty} \left(\sum_{j=1}^{n} \rho_{1j} W_j - \sum_{j=1}^{n} \rho_{2j} W_j \right)$$

$$= \sum_{j=1}^{\infty} (\rho_{1j} - \rho_{2j}) W_j$$

$$= 0$$

On the other hand, if $Y_1 - Y_2 = 0$ almost surely, then by Theorem 2.3,

$$0 = \lim_{t \to \infty} \frac{P[|Y_1 - Y_2| > t]}{P[|W_1| > t]} = \sum_{j=1}^{\infty} |\rho_{1j} - \rho_{2j}|^{\alpha}. \quad \#$$

Therefore, we see that dispersion is not only a measure of tail thickness, but can also be used to define distance between two random variables in S. We remark that if, as in the example given in Section 2, $\sum_{j=1}^{\infty} |\rho_j|^{\alpha} < \infty$ is sufficient for existence, then the use of $\delta < \alpha$ is not required in the definition of S.

We recognize that when $\alpha=2$, S is a subspace of a Hilbert space. In this case and when $\alpha>2$ so that variances are finite, it is usually most convenient to consider a Hilbert space setting. However, we are primarily interested in cases where $\alpha<2$. To consider projection operators in S, let $X_1, \ldots, X_n \in S$, then for any $Y \in S$ define the projection operator $P_{\underline{X}}$ by

$$P_{\underline{X}} Y = \{\hat{Y} = \underline{a}'\underline{X} \text{ such that } disp(Y-\hat{Y}) \text{ is minimum}\}.$$

<u>Theorem 3.2</u>   Assume $\alpha>1$ and suppose $X_i = \sum_{j=1}^{\infty} \pi_{ij} W_j \in S$, $i \leq n$. Assume for each $m \geq n$, $\Pi^{(m)} = [\pi_{ij}]_{i=1 \ j=1}^{n \ m}$ is of full rank n. Suppose also that $Y = \sum_{j=1}^{\infty} \rho_j W_j \in S$. Then $P_{\underline{X}} Y$ has a unique element. Furthermore, if $X_i^{(m)} = \sum_{j=1}^{m} \rho_{ij} W_j$, then

$$\hat{Y} = P_{\underline{X}} Y = \lim_{m \to \infty} P_{\underline{X}^{(m)}}(Y), \text{ almost surely.}$$

<u>Proof</u>: We start by assuming $X_i = \sum_{j=1}^{m} \pi_{ij} W_j$, $Y = \sum_{j=1}^{m} \rho_j W_j$ where $n \leq m$ and m is finite. We wish to minimize

$$h_m(\underline{a}) = disp(Y-\underline{a}'\underline{X})$$

$$= \sum_{j=1}^{m} |\rho_j - \underline{a}'\underline{\pi}_j|^{\alpha} \tag{3.1}$$

where $\underline{\pi}_j$ is the jth column of $\Pi^{(m)} = [\pi_{ij}]_{i=1 \ j=1}^{n \ m}$. We have assumed $\Pi^{(m)}$ has full row rank n. Define

$$D_j = \{\underline{a} \epsilon R^n \text{ such that } \underline{a}'\underline{\pi}_j = \rho_j\}$$

and $g_j(\underline{a}) = |\rho_j - \underline{a}'\underline{\pi}_j|^\alpha$.

For $\underline{a} \notin D_j$ (using $[x]^\gamma = \text{sgn}(x)|x|^\gamma$),

$$\frac{\partial g_j(\underline{a})}{\partial \underline{a}} = \alpha \underline{\pi}_j [\underline{a}'\underline{\pi}_j - \rho_j]^{\alpha-1}$$

and $\quad \dfrac{\partial^2 g_j(\underline{a})}{\partial \underline{a}\underline{a}'} = \alpha(\alpha-1)\underline{\pi}_j\underline{\pi}_j' \, |\underline{a}\underline{\pi}_j - \rho_j|^{\alpha-2}.$ (3.2)

Since $\alpha > 1$ and $\underline{\pi}_j\underline{\pi}_j'$ is nonnegative definite then g_j is convex off of D_j. In fact, g_j is minimized on D_j so that g_j is everywhere convex. We can actually go a step further and say that for $\underline{a}_1, \underline{a}_2 \epsilon R^n$, $\lambda \epsilon (0,1)$,

$$\lambda g_j(\underline{a}_1) + (1-\lambda)g_j(\underline{a}_2) \geq g_j(\lambda\underline{a}_1 + (1-\lambda)\underline{a}_2)$$

with equality iff $\underline{a}_1'\underline{\pi}_j = \underline{a}_2'\underline{\pi}_j$. That is, g_j is strictly convex except along lines orthogonal to $\underline{\pi}_j$. Equality cannot hold for every j, since $\Pi^{(m)}$ is full rank, so that $h = \sum\limits_{j=1}^{m} g_j$ must be strictly convex. Furthermore, as $\max\limits_{1 \leq j \leq n} |a_j| \to \infty$, $h(\underline{a}) \to \infty$. Thus h must have a unique minimum.

The argument that h is strictly convex and has a unique minimum holds even when the series are infinite, that is, when $Y = \sum\limits_{j=1}^{\infty} \rho_j W_j$, $X_i = \sum\limits_{j=1}^{\infty} \pi_{ij} W_j$ and

$$h(\underline{a}) = \text{disp}(Y - \underline{a}'\underline{X})$$

$$= \sum\limits_{j=1}^{\infty} |\rho_j - \underline{a}'\underline{\pi}_j|^\alpha.$$

Let $\underline{a}_0$ be the unique minimum of $h(\underline{a})$ and set (for $m \geq n$)

$$h_m(\underline{a}) = \sum_{j=1}^{m} |\rho_j - \underline{a}'\pi_j|^\alpha$$ with unique minimum $\underline{a}_m$. Suppose for some subse-

quence $\{\underline{a}_{m_k}\}$, $\max_{1 \leq j \leq n} |a_{m_k j}| \to \infty$. Then

$$\lim_{k \to \infty} h_{m_k}(\underline{a}_{m_k}) \geq \lim_{k \to \infty} h_n(\underline{a}_{m_k}) = \infty.$$

But for any m,

$$h_m(\underline{a}_m) = \inf_{\underline{a}} h_m(\underline{a})$$

$$\leq \inf_{\underline{a}} h(\underline{a})$$

$$= h(\underline{a}_0). \tag{3.3}$$

Thus, the sequence $\{\underline{a}_m\}$ must be compact and every subsequence must have a convergent subsequence. Suppose $\underline{a}_{m_k} \to \underline{a}_1$. Then from (3.3)

$$h_{m_k}(\underline{a}_{m_k}) \leq h(\underline{a}_0)$$

$$\leq h(\underline{a}_1).$$

But since $h_m \uparrow h$ and the functions are all continuous, convergence is locally uniform, by an application of Dini's Theorem. This means

$$\lim_{k \to \infty} h_{m_k}(\underline{a}_{m_k}) = h(\underline{a}_1)$$

which implies $h(\underline{a}_1) = h(\underline{a}_0)$ and hence $\underline{a}_1 = \underline{a}_0$. Thus $\underline{a}_0 = \lim_{m \to \infty} \underline{a}_m$ and

$$P_{\underline{X}} Y = \underline{a}_0' \underline{X} = \lim_{m \to \infty} \underline{a}_m' \underline{X} = \lim_{m \to \infty} P_{\underline{X}^{(m)}}(Y). \#$$

To actually calculate $\underline{a}_m$ is not easy unless either $m=n$

$(\underline{a}_n = (\Pi^{(n)'})^{-1}\underline{\rho})$ or $\alpha=2$ $(\underline{a}_m = (\Pi^{(m)}\Pi^{(m)'})^{-1}\Pi^{(m)}\underline{\rho})$. An iterative

procedure would be

$$\underline{a}_{m,1} = (\Pi^{(m)}\Pi^{(m)'})^{-1}\Pi^{(m)}\underline{\rho}^{\bullet}$$

$$\underline{a}_{m,k+1} = \underline{a}_{m,k} - [(\Pi^{(m)}\Pi^{(m)'})^{-1}\Pi^{(m)}\underline{\ell}^{(m)}(\underline{a}_{m,k})]$$

where $\ell_j^{(m)}(\underline{a}) = [\underline{a}'\underline{\pi}_j^{(m)}-\rho_j]^{\alpha-1}$ (using $[x]^{\ell} = sgn(x)|x|^{\ell}$), $1 \leq j \leq m$.

Even though the mapping $Y \to \hat{Y} = P_{\underline{X}}Y$ is unique it will not be a

linear mapping (except when $\alpha=2$ or $m=n$). (See the example at the end

of the section.)

Theorem 3.3 Assume that $\underline{X}$ and Y are as in Theorem 3.2, except assume

$\alpha \leq 1$. To minimize $disp(Y-\underline{a}'\underline{X})$, it suffices to consider $\underline{a} \in \bar{E}$, the closure

of $E = \{\underline{a} \in R^n : \underline{a}'\underline{\pi}_j = \rho_j$ for at least n values of j$\}$.

Proof: As before, we seek to minimize

$$h^{(m)}(\underline{a}) = \sum_{j=1}^{m} |\rho_j - \underline{a}'\underline{\pi}_j|^{\alpha}, \quad m \geq n. \tag{3.5}$$

Define again $D_j = \{\underline{a} \in R^n : \underline{a}'\underline{\pi}_j = \rho_j\}$ and $g_j(\underline{a}) = |\underline{a}'\underline{\pi}_j-\rho_j|^{\alpha}$. The matrix

of second derivatives given in (3.2) indicate that at every $\underline{a} \notin D_j$, g_j

is concave since $\alpha \leq 1$. However, g_j is minimized on D_j. Since

$h^{(m)} = \sum_{j=1}^{m} g_j$ is continuous everywhere, concave at all $\underline{a} \notin \bigcup_{j=1}^{m} D_j$ and

infinite at infinity, then $h^{(m)}$ must therefore be minimized on $\bigcup_{j=1}^{m} D_j$.

(This is not to say that points of minimum are exclusively in this set.)

Now consider the set

$$E_m = \{\underline{a} \epsilon R^n : \underline{a}'\underline{\pi}_j = \rho_j \text{ for at least n values of } j\epsilon\{1,2,\ldots,m\}\}.$$

Since $\Pi^{(m)} = [\pi_{ij}]_{i=1}^{n}{}_{j=1}^{m}$ has rank n, then $D_j \cap E_m$ is non empty.

Suppose $\underline{a}_1 \epsilon D_j$ and $\underline{a}_2 \epsilon D_j \cap E_m$. From (3.4) we clearly have

$h^{(m)}(\underline{a}_1) \geq h^{(m)}(\underline{a}_2)$. Thus $h^{(m)}$ will be minimized on the set

$$\bigcup_{j=1}^{m} (D_j \cap E_m) = E_m.$$

Suppose $\underline{a}_m \epsilon E_m$ minimizes $h^{(m)} = \sum_{j=1}^{m} g_j$. To minimize $h = \sum_{j=1}^{\infty} g_j$ we

consider the sequence $\{\underline{a}_m\}$. As in Theorem 3.2, this sequence must be

compact, and hence there exists a subsequence $\underline{a}_{m_k} \to \underline{a}_0 \epsilon \bar{E}$ where

$E = \lim_{m \to \infty} E_m$. That $\underline{a}_0$ will minimize h is also true, and this is argued

as in the previous theorem. #

The point of minimum $\underline{a}_m$ for $h^{(m)}$ will not necessarily be unique,

except when m=n. In that case, $\underline{a}_m = (\Pi^{(m)'})^{-1}\underline{\rho}$ and the mapping $Y \to P_{\underline{X}}Y$

is linear. When m=n+1, however, such a linear mapping can still be

defined, even when there is not a unique minimum.

<u>Theorem 3.4</u> Suppose $Y = \sum_{j=1}^{\infty} \rho_j W_j \epsilon S$, $\alpha \leq 1$ and suppose, for $1 \leq i \leq n$,

$X_i = \sum_{j=1}^{n+1} \pi_{ij} W_j$ and $\Pi = [\pi_{ij}]$ has rank n. Then there exists a linear

mapping $Y \to \hat{Y}$ into span $\{X_1, \ldots, X_n\}$ which minimizes $disp(Y-\hat{Y})$.

Proof: Let $Z = \sum\limits_{j=n+2}^{\infty} \rho_j W_j$ so that $Y = Z + \underline{\rho}'\underline{W}$, where $\underline{\rho}' = (\rho_1, \ldots, \rho_{n+1})$ and $\underline{W}' = (W_1, \ldots, W_{n+1})$.

We wish to minimize

$$h(\underline{a}) = disp(Y - \underline{a}'\underline{X})$$

$$= disp(Z) + \sum_{j=1}^{n+1} |\rho_j - \underline{a}'\underline{\pi}_j|^{\alpha}.$$

Since $\alpha \leq 1$, then according to Theorem 3.4 a solution is given by $\underline{a}$ satisfying $\underline{a}'\underline{\pi}_j = \rho_j$ for at least n values of $j \in \{1, 2, \ldots, n+1\}$. If $\underline{\rho} = \Pi'\underline{a}_0$ for some $\underline{a}_0 \epsilon R^n$, then $Y = Z + \underline{a}_0'\underline{X}$. Clearly, in this case, $\hat{Y} = \underline{a}_0'\underline{X}$ is the unique solution and the mapping is linear.

On the other hand, if $\underline{\rho}$ is not in the row space of Π, then it suffices to consider $\underline{a}$ such that $\underline{a}'\underline{\pi}_j = \rho_j$ for exactly n values of j. Suppose k is the one value for which $\underline{a}'\underline{\pi}_k \neq \rho_k$. Define Π_{-k} and $\underline{\rho}_{-k}$ to be Π and $\underline{\rho}$, respectively, with k^{th} column $(\underline{\pi}_k)$ and k^{th} element (ρ_k) removed. Then $\underline{a} = (\Pi'_{-k})^{-1}\underline{\rho}_{-k}$ and

$$\min_{\underline{a}} h(\underline{a}) = \min_{\underline{a}} \sum_{j=1}^{n+1} |\rho_j - \underline{a}'\underline{\pi}_j|^{\alpha} + disp(Z)$$

$$= \min_{1 \leq k \leq n+1} |\rho_k - \underline{\rho}'_{-k}(\Pi_{-k})^{-1}\underline{\pi}_k|^{\alpha} + disp(Z) \qquad (3.5)$$

By inverting $Q = [\Pi'\underline{\rho}]$ we can find the (n+1,k) element of Q^{-1}, namely

$$[\rho_k - \underline{\rho}'_{-k}(\Pi_{-k})^{-1}\underline{\pi}_k]^{-1} = (-1)^{k+n+1}det(\Pi_{-k})[det(Q)]^{-1}. \qquad (3.6)$$

Note that this factors into a part depending only on $\underline{\rho}$ and a part depending only on k.

Define $j_0 = \max\{k \leq n+1 : |\det(\Pi_{-k})^{-1}| \text{ is minimum}\}$. Then from (3.5) and (3.7) we have

$$\min_{\underline{a}} h(\underline{a}) = \left|\frac{\det(Q)}{\det(\Pi_{-j_0})}\right|^{\alpha} + \text{disp}(Z)$$

$$= |\rho_{j_0} - \underline{\rho}'_{-j_0} (\Pi_{-j_0})^{-1} \underline{\pi}_{j_0}|^{\alpha} + \text{disp}(Z).$$

And a point of minimum for h is $\underline{a}_0 = (\Pi'_{-j_0})^{-1} \underline{\rho}_{-j_0}$. If we define P to

be the matrix $(\Pi'_{-j_0})^{-1}$ where a column of zeroes is squeezed in to make

a new $j_0{}^{\text{th}}$ column, then $\underline{a}_0 = P\underline{\rho}$.

By this definition, $\hat{Y} = (P\underline{\rho})'\underline{X}$ defines a linear mapping on S and $\hat{Y}\epsilon P_{\underline{X}}Y$, so that it is a minimum dispersion projection of Y. #

The following example illustrates that $P_{\underline{X}}$ is not necessarily linear. With $X = W_1 + 2W_2 + W_3$, then $P_X(W_1 + W_2) \neq P_X(W_1) + P_X(W_2)$ for $\alpha = 1/2$, 1, 3/2.

α	1/2	1	3/2
$P_X(W_1)$	0	0	$\frac{2-\sqrt{2}}{4}$ X
$P_X(W_2)$	0	0	1/3 X
$P_X(W_1 + W_2)$	1/2 X	1/2 X	1/2 X

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