# WIR 13 152

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MATH 152-Spring 2020
Week in Review XIII

TEXAS A&M UNIVERSITY
Math Learning Center

Rational Function 7.4
Numer $\neq \frac{d}{dx}$(Denom)
$\Rightarrow$ Partial Fractions

1. Compute each of the following integrals:

(a) $\displaystyle\int \frac{x-2}{x(x^2+1)}\,dx$   $x^3+x$

$$\left(\frac{x-2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}\right)x(x^2+1)$$

$$x-2 = A(x^2+1) + (Bx+C)x$$

① Nice values of x: $x=0$

$$-2 = A(0^2+1) + \cancel{(B(0)+C)(0)} \rightarrow A=-2$$

② Expand and match powers

$$x-2 = Ax^2 + A + Bx^2 + Cx$$

$x^2$: $0 = A+B$
$x$: $1 = C$
const: $-2 = A$

$A=-2$, $B=2$, $C=1$

so $\displaystyle\frac{x-2}{x(x^2+1)} = \frac{-2}{x} + \frac{2x+1}{x^2+1}$

$u = x^2+1$

$$\int \frac{x-2}{x(x^2+1)}\,dx = \int\left(\frac{-2}{x} + \frac{2x}{x^2+1} + \frac{1}{x^2+1}\right)dx$$

$$= \boxed{-2\ln|x| + \ln(x^2+1) + \tan^{-1}x + C}$$

Square Root
Difference of Squares 7.3
$\Rightarrow$ Trig Subst

(b) $\displaystyle\int_{\sqrt{2}}^{2} \frac{1}{\sqrt{x^2-1}}\,dx$

$x^2 - a^2 \Rightarrow x = a\sec\theta \Rightarrow x = \sec\theta$
$dx = \sec\theta\tan\theta\,d\theta$

$x=2=\sec\theta$
$x=\sqrt{2}=\sec\theta$

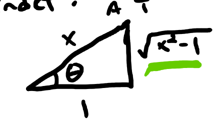
$\frac{1}{2} = \cos\theta \rightarrow \theta = \frac{\pi}{3}$
$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \cos\theta \rightarrow \theta = \frac{\pi}{4}$

$$= \int_{\pi/4}^{\pi/3} \frac{1}{\sqrt{\sec^2\theta - 1}}\sec\theta\tan\theta\,d\theta$$

$\tan^2\theta$

$$= \int_{\pi/4}^{\pi/3} \frac{1}{\cancel{\tan\theta}}\cdot\sec\theta\,\cancel{\tan\theta}\,d\theta$$

If indef: $\frac{H}{A}$ $\frac{x}{1} = \sec\theta$




$$= \ln|\sec\theta + \tan\theta|\Big|_{\pi/4}^{\pi/3}$$

$\dfrac{\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}} = \dfrac{\sqrt{3}/2}{1/2}$

$$= \ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right| - \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right|$$

$$= \boxed{\ln(2+\sqrt{3}) - \ln(\sqrt{2}+1)} = \ln\left(\frac{2+\sqrt{3}}{\sqrt{2}+1}\right)$$

Powers of Trig Function

$\Rightarrow$ Identities

7.2

"save" a trig du, use identities to convert the rest to u

(c) $\int \cos^3(2x)\,dx$

$$= \int \cos^2(2x) \cdot \underbrace{\cos(2x)\,dx}_{du \text{ when } u=\sin(2x)}$$

$\sin^2(2x)+\cos^2(2x)=1$

(Subst: 5.5)

$$= \int \left(1-\sin^2(2x)\right)\cos(2x)\,dx$$

Let $u=\sin(2x)$

$du = 2\cos(2x)\,dx$

$dx = \dfrac{du}{2\cos(2x)}$

$$= \int (1-u^2)\,\cancel{\cos(2x)}\cdot\frac{du}{2\cancel{\cos(2x)}}$$

$$= \frac{1}{2}\int (1-u^2)\,du$$

$$= \frac{1}{2}\left(u-\frac{1}{3}u^3\right)+C$$

$$= \frac{1}{2}\left(\sin(2x)-\frac{1}{3}\sin^3(2x)\right)+C$$

Multiplying unrelated functions

$\Rightarrow$ Integration by Parts

7.1

(d) $\int x\sin(2x)\,dx$

Let $u=x$ $\quad$ $dv=\sin(2x)\,dx$

$du = 1\,dx$ $\quad$ $v=\frac{1}{2}\left(-\cos(2x)\right)$

$$\int u\,dv = uv - \int v\,du$$

$$\int x\sin(2x) = x\left(\frac{-1}{2}\cos(2x)\right) + \int \frac{1}{2}\cos(2x)\cdot dx$$

$$= \boxed{-\frac{1}{2}x\cos(2x)+\frac{1}{2}\cdot\frac{1}{2}\sin(2x)+C}$$

$$= \frac{-1}{2}x\cos(2x)+\frac{1}{4}\sin(2x)+C$$

$\frac{d}{dx}$(stuff) in numerator

$\Rightarrow$ Substitution

5.5

(e) $\int_0^{\ln(3)} \frac{e^x}{\sqrt{(e^x+1)}}\,dx$

Let $u = e^x + 1$

$du = e^x\,dx$

$dx = \frac{du}{e^x}$

$x = \ln(3) \rightarrow u = e^{\ln(3)} + 1 = 4$

if $x = 0 \rightarrow u = e^0 + 1 = 2$

$$= \int_2^4 \frac{\cancel{e^x}}{\sqrt{u}} \cdot \frac{du}{\cancel{e^x}}$$

$$= \int_2^4 u^{-1/2}\,du$$

$$= 2u^{1/2} \Big|_2^4$$

$$= \boxed{2 \cdot \underbrace{4^{1/2}}_{\sqrt{4}} - 2 \cdot 2^{1/2}} = 4 - 2\sqrt{2}$$

7.8

2. Compute $\int_0^\infty \left(\frac{2}{2x+1} - \frac{1}{x+3}\right) dx$

$$= \lim_{N\to\infty} \int_0^N \left(\frac{2}{2x+1} - \frac{1}{x+3}\right) dx$$

$u = 2x+1$, $du = 2\,dx$ $\quad$ $u = x+3$, $du = 1\,dx$

$$= \lim_{N\to\infty} \ln|2x+1| - \ln|x+3| \Big|_0^N$$

$$= \lim_{N\to\infty} \left(\ln(2N+1) - \ln(N+3)\right) - \left(\underbrace{\ln(1)}_{0} - \ln(3)\right)$$

Properties of logs

$$= \lim_{N\to\infty} \ln\left(\frac{2N+1}{N+3}\right) + \ln 3$$

2

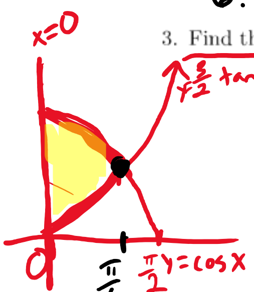
$\Rightarrow$ integral converges to $\boxed{\ln(2) + \ln(3)} = \ln(6)$

6.1

3. Find the area of the region bounded by the graphs of $x = 0$, $y = \frac{3}{2}\tan x$, and $y = \cos x$.

1) Plot
2) Find intersections
3) Integrate
$\int T-B\,dx$
or $\int R-L\,dy$

(1)




(2) Intersection:

$$\frac{3}{2}\tan x = \cos x$$

$$\frac{3\sin x}{2\cos x} = \frac{\cos x}{1}$$

$$3\sin x = 2\cos^2 x$$

$(1-\sin^2 x)$

$$3\sin x = 2 - 2\sin^2 x$$

$$2\sin^2 x + 3\sin x - 2 = 0$$

$$(2\sin x - 1)(\sin x + 2) = 0$$

$$\sin x = \tfrac{1}{2} \quad \text{or} \quad \cancel{\sin x = -2}$$

$$x = \frac{\pi}{6}$$

Polynomial with "sin x" as variable

$$2u^2 + 3u - 2 = 0$$

$$(2u-1)(u+2) = 0$$

(3) $$\int_0^{\pi/6} \left(\cos x - \frac{3}{2}\tan x\right) dx$$

$\frac{\sin x}{\cos x}$ $u = \cos x$ $du = -\sin x$

$$= \sin x + \frac{3}{2}\ln|\cos x| \Big|_0^{\pi/6}$$

$$= \left(\sin\frac{\pi}{6} + \frac{3}{2}\ln\left|\cos\frac{\pi}{6}\right|\right) - \left(\sin 0 + \frac{3}{2}\ln|\cos 0|\right)$$

$$\boxed{\frac{1}{2} + \frac{3}{2}\ln\left(\frac{\sqrt{3}}{2}\right)}$$

note $\int \tan x\,dx = \ln|\sec x| + C$
$= -\ln|\cos x| + C$

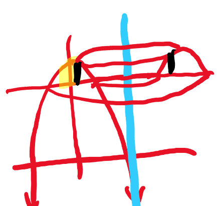
6.2 – 6.3
slices shells

4. The region bounded by $y = 4 - x^2$ and $y = 3$ is revolved (rotated) around the line $x = 2$. Write an integral to find the volume.

"y" axis

| | x-axis | y-axis |
|---|---|---|
| f(x) | slices | shells |
| g(y) | shells | slices |

Shells $V = \int 2\pi\, r \cdot h \cdot dr$ (R-L, T-B)



$$\boxed{= \int_{-1}^{1} 2\pi(2-x)\left((4-x^2)-3\right)dx}$$

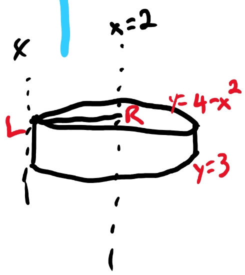
Boundaries come from intersections

$$4 - x^2 = 3$$

$$0 = x^2 - 1$$

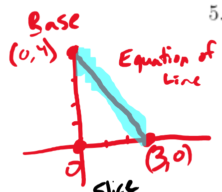
$$0 = (x+1)(x-1)$$

$$x = -1, 1$$



## 6.2

5. Find the volume of the solid whose base is the triangular region with vertices (0, 0), (3, 0), and (0, 4) and whose cross-sections perpendicular to the $x$-axis are semicircles.

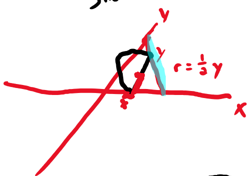



Equation of Line

$m = \frac{-4}{3}$

$b = 4$

$y = \frac{-4}{3}x + 4$





$V_{slice} = \frac{1}{2}\pi r^2 \, dx$

$= \frac{1}{2}\pi\left(\frac{1}{2}y\right)^2 dx$

slices

$V = \int_0^3 \frac{1}{8}\pi y^2 \, dx$

$= \int_0^3 \frac{1}{8}\pi\left(\frac{-4}{3}x + 4\right)^2 dx$ OR Let $u = -\frac{4}{3}x + 4$

$= \frac{1}{8}\pi \int_0^3 \left(\frac{16}{9}x^2 - \frac{32}{3}x + 16\right) dx$

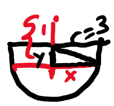
$= \frac{1}{8}\pi\left(\frac{16}{27}x^3 - \frac{16}{3}x^2 + 16x \right|_0^3$

$= \frac{1}{8}\pi\left(\left(\frac{16}{27}\cdot 3^3 - \frac{16}{3}\cdot 3^2 + 16(3)\right) - 0\right)$ All

$= \frac{1}{8}\pi(16) = \boxed{2\pi}$

## 6.4

6. Consider a trough in the shape of a half-cylinder of radius 3 feet and length 8 feet (diameter at the top). It is full of water to a depth of 3 feet. Find an integral that gives the work necessary to pump all of the water to a point 1 foot above the top of the trough.


$x^2 + y^2 = 9$

$x = \sqrt{9 - y^2}$

$V = \boxed{\int_0^3 16\rho g\sqrt{9-y^2}\,(y+1)\,dy}$

(OR) $\int_{-3}^{0} 16\rho g\sqrt{9-y^2}\,(1-y)\,dy$

y is dist from top to slice


halfway distance across slice

$V_{slice} = 2x \cdot 8 \cdot dy$

weight $= \rho g\,\text{Vol} = 16\rho g\, x\, dy$

Work = weight · distance $(y+1)$

10.9/ ~~10.10~~

7. Write a power series for the function $f(x) = \ln(1+2x)$ centered at $x = 0$.

Taylor Series: $f^{0}(x) = \ln(1+2x)$ so $f^{0}(0) = \ln(1+2\cdot 0) = \ln 1 = 0$

$f'(x) = \frac{2}{1+2x}$ $\quad \frac{a}{1-r}$ Geometric Sum $\quad \int x^n\,dx$

$1-(-2x)$

U-Turn

$f(x) = \ln(1+2x)$

$\frac{d}{dx}$

$\frac{2}{1-(-2x)}$ $\quad \frac{a}{1-r} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} 2(-2x)^n$ $\quad -1\cdot 2\cdot x$

$\int dx$

$C + \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1} x^{n+1}}{n+1}$

Properties of Exponents

$\sum_{n=0} 2\cdot(-1)^n (2)^n x^n$

$(-1)^n 2^{n+1} x^n$

ROC

$|r| < 1$

$|-2x| < 1$

$2|x| \le 1$

$|x| \le \frac{1}{2}$ ROC

$$\boxed{\ln(1+2x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1} x^{n+1}}{n+1}}$$

~~10.9~~/10.10? $\quad$ exponential $\rightarrow$ $\cancel{\frac{a}{1-r}}$ $\quad$ CANNOT use known Maclaurin Series

8. Write a power series for the function $f(x) = e^{-x}$ centered at $x = 1$.

$f^{(0)}(x) = e^{-x} \rightarrow f^{(0)}(1) = e^{-1} = (-1)^0 e^{-1}$

$f'(x) = -1e^{-x} \rightarrow f'(1) = (-1)^1 e^{-1}$

$f''(x) = (-1)(-1)e^{-x} \rightarrow f''(1) = (-1)^2 e^{-1}$

$f'''(x) = (-1)^2(-1)e^{-x} \rightarrow f'''(1) = (-1)^3 e^{-1}$

$f^{(n)}(1) = (-1)^n e^{-1}$ $\quad$ so $c_n = \frac{f^{n}(1)}{n!} = \frac{(-1)^n e^{-1}}{n!}$

So $e^{-x} = \sum_{n=0}^{\infty} c_n (x-1)^n = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n e^{-1}}{n!} (x-1)^n}$

6.4

9. A spring stretches 1 foot beyond its natural length under a force of 100 pounds. How much work is done in stretching it 3 feet beyond its natural length?

Hooke's Law

$F(x) = kx$

$100 = k(1)$

$k = 100$

If given work, integrate $kx$ first, then solve for $k$

$W = \int_a^b kx \, dx$

$W = \int_0^3 100x \, dx$

$= 50x^2 \Big|_0^3$

$= 50 \cdot 3^2 - 50 \cdot 0^2$

$= \boxed{450 \text{ ft} \cdot \text{lbs}}$

$x$ = displacement from natural length

11.2-11.6

10. Determine whether the following series converge or diverge. Name and apply an appropriate test and state all the conditions that must be satisfied.

10.4

(a) $\sum_{n=0}^{\infty} \frac{n^2}{\sqrt{n^5}+10}$

Ratio of Powers → LCT $\quad 2-\frac{5}{2}=-\frac{1}{2}$

Compare with $\sum_{n=1}^{\infty} \frac{n^2}{n^{5/2}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ Diverges by P-Test ($p<1$)

$a_n = \frac{n^2}{\sqrt{n^5+10}} > 0 \quad b_n = \frac{1}{n^{1/2}} > 0$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{\sqrt{n^5+10}} \cdot \frac{n^{1/2}}{1}$$

$$= \lim_{n\to\infty} \frac{n^{5/2}}{\sqrt{n^5+10}} \qquad \frac{n^{5/2}}{n^{5/2}} = 1$$

Converges to 1

∴ both series do same thing

Diverges by LCT with $\sum \frac{1}{n^{1/2}}$

10.6

(b) $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$

Factorials → Ratio Test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{((n+1)!)^2}{(2(n+1))!} \cdot \frac{(2n)!}{(n!)^2} \overset{?}{<} 1$$

$(n+1)! = (n+1)n!$ $\qquad (2n+2)! = (2n+2)(2n+1)\cdot(2n)!$

$$= \lim_{n\to\infty} \underbrace{\frac{(n+1)!}{n!}}_{(n+1)} \cdot \underbrace{\frac{(n+1)!}{n!}}_{(n+1)} \cdot \underbrace{\frac{(2n)!}{(2n+2)!}}_{\frac{1}{(2n+2)(2n+1)}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2} = \frac{1}{4} < 1 \checkmark$$

∴ series converges absolutely by ratio test

10.3 Nothing else works → try integral test

(c) $\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$

$f(x) = \frac{\ln x}{x}$ f positive, decreasing, cts

$\lim_{N\to\infty} \int_2^N \frac{\ln x}{\cancel{x}}\, dx$ $\quad \cancel{x}\cdot du$

$u = \ln x$
$du = \frac{1}{x}\, dx$
$dx = x \cdot du$

$= \lim_{N\to\infty} \int_{x=2}^{N} u\, du$

$= \lim_{N\to\infty} \frac{1}{2}u^2 \Big|_{x=2}^{N}$

$= \lim_{N\to\infty} \frac{1}{2}(\ln x)^2 \Big|_2^N = \lim_{N\to\infty} \frac{1}{2}(\ln N)^2 - \frac{1}{2}(\ln 2)^2$ $\quad \infty$

∴ series diverges by Integral Test

10.4

NOTE: $\ln(n) > 1$ for $n >$ some #

$\frac{\ln(n)}{n} > \frac{1}{n}$ $\quad \Sigma\frac{1}{n}$ diverges so series diverges by Comparison with $\Sigma\frac{1}{n}$

10.7

11. Find the radius and interval of convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\sqrt{n+7}}$. Power Series → Ratio Test

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{x^{(n+1)}}{\sqrt{(n+1)+7}} \cdot \frac{\sqrt{n+7}}{x^n} \right| \overset{\text{WANT}}{<} 1$

$= \lim_{n\to\infty} \left| \frac{\cancel{x}^{n+1}}{\cancel{x}^n} \cdot \frac{\sqrt{n+7}}{\sqrt{n+8}} \right| < 1$

$= \lim_{n\to\infty} |x| \cdot \frac{\sqrt{n+7}}{\sqrt{n+8}} < 1$ $\quad \to 1$

$|x| < 1$ $\quad \boxed{ROC = 1}$ $\quad -1 < x < 1$

Test

$x = -1$ $\quad \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^n}{\sqrt{n+7}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+7}}$ diverges by LCT with $\Sigma\frac{1}{n^{1/2}}$

$x = 1$ $\quad \sum_{n=0}^{\infty} \frac{(-1)^n(1)^n}{\sqrt{n+7}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+7}}$ converges by AST

so interval is $\boxed{x \in (-1, 1]}$

9

11.9

12. Find the second degree Taylor polynomial for $f(x) = \sqrt{x}$ at $x = 1$.

$x^{1/2}$

$c_n = \frac{f^{n}(a)}{n!}$

$f^{(0)}(x) = x^{1/2} \rightarrow f^{(0)}(1) = 1^{1/2} = 1 \quad c_0 = \frac{f^{(0)}}{0!} \cdot (x-1)^0$

$f'(x) = \frac{1}{2}x^{-1/2} \rightarrow f'(1) = \frac{1}{2}(1)^{-1/2} = \frac{1}{2} \quad c_1 = \frac{f^{(1)}}{1!} \quad (x-1)^1$

$f''(x) = -\frac{1}{4}x^{-3/2} \rightarrow f''(1) = -\frac{1}{4}(1)^{-3/2} = -\frac{1}{4} \quad c_2 = \frac{f^{(2)}}{2!} \quad (x-1)^2$

$T_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1/4}{2!}(x-1)^2$

$\boxed{T_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2} \approx \sqrt{x}$ near $x=1$