WIR10 152

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MATH 152-Spring 2020
Week in Review X

TEXAS A&M UNIVERSITY
Math Learning Center

Section 11.4

Comparison/Limit Comparison Test

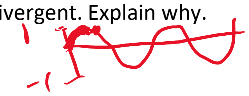
Comparison Test: Suppose $0 \leq a_n < b_n$ for $n > N$

1) If $\sum b_n$ converges, then $\sum a_n$ converges
2) If $\sum a_n$ diverges, then $\sum b_n$ diverges

(NO knowledge otherwise)

Limit Comparison Test: If $a_n > 0$ and $b_n > 0$, and $0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$ then $\sum a_n$ and $\sum b_n$ both converge or both diverge

Determine whether the following series are convergent. or divergent. Explain why.



(a) $\sum_{n=1}^{\infty} \frac{\sin\left(\frac{1}{n}\right)}{n^2}$

for $n \geq 1$, $0 < \sin\left(\frac{1}{n}\right) \leq 1$ $\div n^2$

$0 < \frac{\sin\left(\frac{1}{n}\right)}{n^2} \leq \frac{1}{n^2}$

$\left(\sum \frac{1}{n^p} \text{ converges if } p > 1\right)$

We know $\sum \frac{1}{n^2}$ converges by P-Test

$\therefore \sum_{n=1}^{\infty} \frac{\sin\left(\frac{1}{n}\right)}{n^2}$ converges by Comparison to $\sum \frac{1}{n^2}$

$b_n > a_n$ $\quad$ $a_n > b_n$

$$0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$$

dominating terms

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+2)(n-3)}$

$$\sum_{n=4}^{\infty} \frac{n}{(n+2)(n-3)} \qquad \frac{n}{n^2-n-6}$$

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$a_n = \frac{n}{n^2} = \frac{1}{n}$ $\quad \sum \frac{1}{n}$ diverges by P-Test
want it smaller

$b_n = \frac{n}{n^2-n-6}$

Comparison: $a_n < b_n$?

$$\frac{1}{n} \cancel{<} \frac{n}{n^2-n-6} ?$$

$$n^2 - n - 6 < n^2 ?$$
$$-n^2 \qquad -n^2$$

$$-n - 6 < 0$$
$$+6 \qquad +6$$

$$\frac{-n}{-1} < \frac{6}{-1}$$

$n > -6$ True because n starts at 4

since smaller series diverges, then $\sum^{\infty} \frac{n}{(n+2)(n-3)}$ diverges by Comparison

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+2)(n+3)}$

$$\sum_{n=0}^{\infty} \frac{n}{(n+2)(n+3)} \qquad \frac{n}{n^2+5n+6}$$

compare with $\frac{n}{n^2} = \frac{1}{n}$ $\quad \sum \frac{1}{n}$ diverges

$$\frac{1}{n} \cancel{<} \frac{n}{n^2+5n+6}$$

$$n^2 + 5n + 6 < n^2$$
$$5n < -6$$
$$n < -\frac{6}{5}$$

"wrong" inequality $\Rightarrow$ Use Limit Comp

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{n}{n^2+5n+6}} = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{n^2+5n+6}{n} = \lim_{n\to\infty} \frac{n^2+5n+6}{n^2} = 1$$

$0 < 1 < \infty$

$\therefore$ both behave the same. $\therefore \sum_{n=0}^{\infty} \frac{n}{(n+2)(n+3)}$ diverges by LCT with $\sum \frac{1}{n}$

(d) $\sum_{n=1}^{\infty} \frac{\sqrt{n+n^2}}{n+n^3}$ Compare with $\frac{\sqrt{n^2}}{n^3} = \frac{n}{n^3} = \frac{1}{n^2}$ $\sum \frac{1}{n^2}$ converges by P-Test

LCT : $\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{\sqrt{n+n^2}}{n+n^3}} = \lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{n+n^3}{\sqrt{n+n^2}}$ n^4

$= \lim_{n\to\infty} \frac{n+n^3}{\sqrt{n^5+n^4}}$ $\frac{n^3}{\sqrt{n^6}} = \frac{n^3}{n^3} = 1$

$0 < 1 < \infty$

Since $\sum \frac{1}{n^2}$ converges, $\sum_{n=1}^{\infty} \frac{\sqrt{n+n^2}}{n+n^3}$ Converges by LCT .

2 Section 11.5 Alternating Series Test

$a_0 - a_1 + a_2 - a_3 + a_4 - \ldots$

If $a_n > 0$, a_n decreasing, and $a_n \to 0$, $(-1)^n$ or $(-1)^{n+1}$

then the Alternating Series $\sum (-1)^{n} a_n$ converges

If $a_n \to L \neq 0$, series diverges by Test for Divergence

Error Estimates : if $s_N = \sum_{n=0}^{N} (-1)^n a_n$, then

$|s - s_N| \leq a_{N+1}$

1. Determine whether the following series are convergent or divergent. Explain why.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \to$ AST

$a_n = \left| \frac{(-1)^{n+1}}{n} \right| = \frac{1}{n} > 0$

a_n decreasing $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2} < 0$

$\lim_{n\to\infty} \frac{1}{n} = 0$

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by AST

(b) $\sum_{n=1}^{\infty} (-1)^{n+1} 3n^2 e^{-n^3}$

$a_n = \left|(-1)^{n+1} 3n^2 e^{-n^3}\right| = 3n^2 e^{-n^3} > 0$

a_n dec $\frac{d}{dx}\left(3x^2 e^{-x^3}\right) < 0$

$\lim_{n\to\infty} 3n^2 e^{-n^3} = \lim_{n\to\infty} \frac{3n^2}{e^{n^3}} = 0$

$\therefore \sum_{n=1}^{\infty} (-1)^{n+1} 3n^2 e^{-n^3}$ converges by AST

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{n}$

$a_n = \left|\frac{(-1)^n(n+1)}{n}\right| = \frac{n+1}{n} > 0$

a_n dec $\left(\frac{d}{dx}\left(\frac{x+1}{x}\right) = -\frac{1}{x^2} < 0\right)$

$\lim_{n\to\infty} \frac{n+1}{n} = 1$

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{n}$ diverges by Test for Divergence

(d) $\sum_{n=1}^{\infty} \frac{(-1)^n(2+\cos n)}{n^2}$

$a_n = \frac{2+\cos(n)}{n^2} > 0$

$\frac{d}{dx}\left(\frac{2+\cos x}{x^2}\right) = \frac{x^2(-\sin x) - (2+\cos x)(2x)}{x^4}$

$= \frac{-x\sin x - 4 - 2\cos x}{x^3} < 0$

$\lim_{n\to\infty} \frac{2+\cos(n)}{n^2} = 0$

$-1 \le \cos(n) \le 1$ add 2 everywhere

$1 \le 2+\cos n \le 3$ $\div n^2$

$\frac{1}{n^2} \le \frac{2+\cos n}{n^2} < \frac{3}{n^2}$

$\to 0 \quad \to 0 \quad \to 0$

by Squeeze theorem

Alt Series

2. How many terms of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$ do you need to add to estimate the sum to within an error of 10^{-6}?

$a_n = \frac{1}{n^3}$

$|S - S_N| < a_{N+1} < 10^{-6}$

$\frac{1}{(N+1)^3} \cancel{\leq} \frac{1}{10^6}$

$(10^6)^{1/3} \leq ((N+1)^3)^{1/3}$

$10^2 \leq N+1$

$N+1 \geq 100$

$N \geq 99$ at least 99 terms

If ask error after 50 terms?

$|S - S_{50}| < a_{51} = \frac{1}{51^3}$

3 Section 11.6 Absolute Convergence and Ratio Test

Absolute Conv : $\sum |a_n|$ converges

if $\sum |a_n|$ conv, then $\sum a_n$ conv

Ratio Test – if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ then $\sum a_n$ absolutely converges

if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ then $\sum a_n$ diverges

(if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ then ?)

1. For the convergent series in #1 of the previous section, determine which are absolutely convergent.

1a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by AST

$a_n = \left|\frac{(-1)^{n+1}}{n}\right| = \frac{1}{n}$ does $\sum \frac{1}{n}$ converge? NO by P-Test

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges, but NOT absolutely

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots$ diverges

BUT $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$ converges

1b) $\sum_{n=1}^{\infty} (-1)^{n+1} 3n^2 e^{-n^3}$ converges by AST

$a_n = 3n^2 e^{-n^3}$ $\quad \sum_{n=1}^{\infty} 3n^2 e^{-n^3}$ → Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\cancel{3}(n+1)^2 e^{-(n+1)^3}}{\cancel{3}n^2 e^{-n^3}}$$

$$= \lim_{n\to\infty} \frac{n^2+2n+1}{n^2} \cdot \frac{e^{n^3}}{e^{(n^3+3n^2+3n+1)}}$$

$$= \lim_{n\to\infty} \underbrace{\frac{n^2+2n+1}{n^2}}_{1} \cdot \frac{\cancel{e^{n^3}}}{\cancel{e^{n^3}} \cdot \underbrace{e^{3n^2} \cdot e^{3n} \cdot e^{1}}_{0}} = 0 < 1$$

$\therefore$ $\sum (-1)^{n+1} 3n^2 e^{-n^3}$ converges absolutely by Ratio Test

1c) diverged into N/A

1d) $\sum_{n=1}^{\infty} \frac{(-1)^n(2+\cos n)}{n^2}$ $a_n = \frac{2+\cos n}{n^2}$ → Comparison Test

$\cos(n) < 1$ add 2

$2+\cos(n) < 3$ $\div n^2$

$\frac{2+\cos(n)}{n^2} < \frac{3}{n^2}$ $\sum \frac{3}{n^2}$ converges by P-Test

(larger series converges)

$\therefore \sum \frac{2+\cos(n)}{n^2}$ converges by Comparison with $\sum \frac{3}{n^2}$

$\therefore \sum \frac{(-1)^n(2+\cos(n))}{n^2}$ converges absolutely

2. Determine if the following series are absolutely convergent, convergent (but not absolutely), or divergent:

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{n! 3^n}$ factorial or exponential → Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{4^{n+1}}{(n+1)!\,3^{n+1}} \cdot \frac{n!\,3^n}{4^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{n!}\,1}{\cancel{(n+1)!}\,(n+1)\cancel{n!}} \cdot \frac{\cancel{3^n}\,1}{3^{n+1}\ \cancel{3^n}\cdot 3^1} \cdot \frac{4^{n+1}\ \cancel{4^n}\cdot 4^1}{\cancel{4^n}\,1}$$

OR $3^{n-(n+1)} = 3^{n-n-1}$ $= 3^{-1}$

$n! = n(n-1)(n-2)\ldots(2)(1)$

$(n+1)! = (n+1)n(n-1)(n-2)\ldots(2)(1)$

$(n+1)! = (n+1)\,n!$

$$= \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{1}{3} \cdot \frac{4}{1} = 0 < 1$$

$\frac{1}{n+1} \to 0$

$\therefore$ series converges absolutely by Ratio Test

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}4^n}{n3^n}$ Exponentials → Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{4^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n\,3^n}{4^n}$$

$$= \lim_{n\to\infty} \underbrace{\frac{n}{n+1}}_{\text{NO cancel}} \cdot \frac{3^n}{3^{n+1}} \cdot \frac{4^{n+1}}{4^n}$$

$$= \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{1}{3} \cdot \frac{4}{1} = \frac{4}{3} > 1$$

∴ series diverges by Ratio Test

3. Find the values of x for which the series $\sum_{n=2}^{\infty} \frac{(-3)^n(x-1)^n}{\sqrt{n}}$ converges. Power Series → Exponential → Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3^{n+1}(x-1)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{3^n (x-1)^n}\right|$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} \cdot \frac{3^{n+1}}{3^n} \cdot \frac{|x-1|^{n+1}}{|x-1|^n} < 1 \quad \text{(we want)}$$

General Rule

if $|x| < a$ $(a > 0)$

then $-a < x < a$

$|x| < a$: $-a \quad 0 \quad a$

$$3|x-1| < 1$$

$$|x-1| < \frac{1}{3}$$

$$-\frac{1}{3} < x-1 < \frac{1}{3}$$

$+1 \quad +1 \quad +1$

$$\frac{2}{3} < x < \frac{4}{3}$$

series converges absolutely when $x \in \left(\frac{2}{3}, \frac{4}{3}\right)$

Series diverges when $x < \frac{2}{3}$ and $x > \frac{4}{3}$

$x = \frac{2}{3}$: $\sum_{n=1}^{\infty} \frac{(-3)^n\left(\frac{2}{3}-1\right)^n}{\sqrt{n}} = \sum \frac{1}{\sqrt{n}}$ diverges

$x = \frac{4}{3}$: $\sum_{n=1}^{\infty} \frac{(-3)^n\left(\frac{4}{3}-1\right)^n}{\sqrt{n}} = \sum \frac{(-1)^n}{\sqrt{n}}$ conv by AST

$$x \in \left(\frac{2}{3}, \frac{4}{3}\right]$$

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1 Section 11.4

Convergence Tests $\sum a_n$

1) Does $\lim_{n\to\infty} a_n \neq 0$? YES series diverges by Test for Divergence

NO 2) Factorials and/or Exponentials? YES Ratio Test

NO 3) is there $(-1)^n$? YES AST (may have to continue for abs convergence)

NO 4) Ratio of Powers? YES LCT

NO 5) is sine/cosine? YES Comparison Test ($\begin{matrix}\sin\\ \cos\end{matrix} \leq 1$)

NO 6) Hope to Integrate it

1. Determine whether the following series are convergent or divergent. Explain why.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin\left(\frac{1}{n}\right)}{n^2}$