WIR 11 152

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TEXAS A&M UNIVERSITY
Math Learning Center

1 Section 11.8 Power Series $\sum_{n=0}^{\infty} c_n (x-a)^n$

a = center
c_n = coefficients

Interval of Convergence = values of x which make series converge

1) radius of convergence → ratio test
$|x-a| < R \leftarrow$ radius of conv

2) test endpoints by substituting into series for x

1. Find the radius and interval of convergence for the following series: $x^n = (x-0)^n \quad a=0$

(a) $\sum_{n=1}^{\infty} x^n$

$a_n = x^n$ $\quad x^{(n+1)-n}$

Ratio Test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\cancel{x^{n+1}}}{\cancel{x^n}} \right|$

$= \lim_{n\to\infty} |x|$ $\quad |x-a| < R = 1$ so ROC = 1

$= \boxed{|x| < 1}$ series converges ($|x| > 1$ series diverges)

$|x| < a$ means $-a < x < a$

$-1 < x < 1 \leftarrow$ interval length = 2

ROC = $\frac{1}{2}(2) = 1$

Test Endpoints:

$x = -1$: $\sum_{n=1}^{\infty} (-1)^n$ series diverges by Test for Divergence

$x = 1$: $\sum_{n=1}^{\infty} 1^n = \sum_{n=1}^{\infty} 1$ series diverges by Test for Divergence

interval of convergence is $\boxed{x \in (-1, 1)}$

NOTE: $\sum_{n=1}^{\infty} x^n = x + x^2 + x^3 + \ldots$ Geometric $|r| = |x| < 1$

$\frac{a}{1-r} = \frac{x}{1-x}$

(b) $\sum_{n=1}^{\infty} n! x^n$ $a_n = n! \, x^n$

Ratio Test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)! \, x^{n+1}}{n! \, x^n} \right|$

$(n+1)! = (n+1) n!$

$= \lim_{n\to\infty} \frac{(n+1)\cancel{n!}}{\cancel{n!}} \left| \frac{x \cancel{x^{n}}}{\cancel{x^n}} \right|$

$= \lim_{n\to\infty} (n+1) |x| \overset{?}{<} 1$

$(n+1) \to \infty$ unless $x = 0$ ← center

ROC $= 0$

"interval" of convergence $\boxed{x=0}$ or $[0,0]$

(c) $\sum_{n=1}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ $a_n = \frac{x^{2n+1}}{(2n+1)!}$

Ratio Test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{2(n+1)+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| \overset{?}{<} 1$

$x^{2(n+1)+1} = x^{2n+2+1}$

$(2n+3)! =$
$(2n+3)(2n+2)(2n+1)!$

$= \lim_{n\to\infty} \left| \frac{x^{2n+3}}{x^{2n+1}} \right| \cdot \frac{(2n+1)!}{(2n+3)!} \overset{?}{<} 1$

$= \lim_{n\to\infty} x^2 \cdot \frac{\cancel{(2n+1)!}}{(2n+3)(2n+2)\cancel{(2n+1)!}} < 1$

$\to 0$

$0 < 1$

true for ALL values of x

$\boxed{\text{ROC} = \infty \quad \text{interval: } (-\infty, \infty)}$

(d) $\sum_{n=1}^{\infty} \frac{(x-2)^n}{\sqrt{n+1}\,6^n} = a_n$

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{\sqrt{n+1+1}\cdot 6^{n+1}}\cdot\frac{\sqrt{n+1}\cdot 6^n}{(x-2)^n}\right| \overset{?}{<} 1$$

$$= \lim_{n\to\infty}\left|\frac{(x-2)^{\cancel{n+1}}}{(x-\cancel{2})^{\cancel{n}}}\right|\cdot\frac{\sqrt{n+1}}{\sqrt{n+2}}\cdot\frac{6^{\cancel{n}}1}{6^{\cancel{n+1}}6} \overset{?}{<} 1$$

$$\lim_{n\to\infty} |x-2|\cdot\frac{\sqrt{n+1}}{\sqrt{n+2}}\cdot\frac{1}{6} \overset{?}{<} 1$$

$$6\cdot\frac{|x-2|}{6} < 1\cdot 6 \to |x-2| < 6 = ROC$$

$$-6 < x-2 < 6$$
$$+2 \quad +2 \quad +2$$
$$-4 < x < 8$$

(length = 12 ÷ 2 = 6 = ROC)
center is $x=2$

Test endpoints

$x=-4$ $\quad \sum_{n=1}^{\infty}\frac{(-4-2)^n}{\sqrt{n+1}\,6^n} = \sum_{n=1}^{\infty}\frac{(-6)^n}{\sqrt{n+1}\cdot 6^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$ converges by AST (include $x=-4$)

$x=8$ $\quad \sum_{n=1}^{\infty}\frac{(8-2)^n}{\sqrt{n+1}\,6^n} = \sum_{n=1}^{\infty}\frac{\cancel{6^n}}{\sqrt{n+1}\cdot\cancel{6^n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}}$ diverges by LCT with $\sum\frac{1}{n^{1/2}}$ (exclude $x=8$)

$\boxed{\text{interval} = [-4, 8)}$

(e) $\sum_{n=0}^{\infty}\frac{(3x-2)^n}{n^2+4}$ a_n

Ratio Test

$$\lim_{n\to\infty}\left|\frac{(3x-2)^{n+1}}{(n+1)^2+4}\cdot\frac{n^2+4}{(3x-2)^n}\right| \overset{?}{<} 1$$

$$\lim_{n\to\infty}\left|\frac{(3x-2)^{\cancel{n+1}}}{(3\cancel{x-2})^{\cancel{n}}}\right|\cdot\frac{n^2+4}{n^2+2n+5} \to 1 \overset{?}{<} 1$$

$$|3x-2|\cdot 1 < 1$$

$$-1 < 3x-2 < 1 \longrightarrow \frac{1}{3} < \frac{3x}{3} < \frac{3}{3} \to \frac{1}{3} < x < 1$$
$$+2 \quad +2 \quad +2$$

length $= \frac{2}{3}\cdot\frac{1}{2}$

$\boxed{ROC = \frac{1}{3}}$

Test Endpoints

$x=\frac{1}{3}$ $\quad \sum_{n=0}^{\infty}\frac{(3\cdot\frac{1}{3}-2)^n}{n^2+4} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+4}$ conv by AST (so include $x=\frac{1}{3}$)

$x=1$ $\quad \sum_{n=0}^{\infty}\frac{(3\cdot 1-2)^n}{n^2+4} = \sum_{n=0}^{\infty}\frac{1}{n^2+4}$ conv by LCT with $\sum\frac{1}{n^2}$ (so include $x=1$)

interval = $\boxed{x\in\left[\frac{1}{3}, 1\right]}$

2. If the series $\sum_{n=1}^{\infty} c_n(x-\underset{=a}{6})^n$ has radius of convergence $r = \underset{=R}{3}$, find the values of x for which we know the series is convergent.

$|x-a| < R$ converges

$$|x-6| < 3$$

$$-3 < x-6 < 3$$

$$+6 \quad +6 \quad +6$$

$\boxed{\text{we know for } 3 < x < 9 \text{ series converges}}$

we know series diverges when $x<3$ and $x>9$

at $x=3, 9$ $\to$ we know <u>NOTHING</u> (since c_n's unknown)

$-\frac{1}{3} < x < \frac{1}{3}$

3. The ~~radius~~ of convergence of the series $\sum_{n=1}^{\infty} \frac{(-3)^n(x-1)^n}{\sqrt{n}}$ is $\frac{1}{3}$. Find the interval of convergence.

$$|x-a| < \frac{1}{3}$$

$$-\frac{1}{3} < x-1 < \frac{1}{3} \quad \text{so} \quad \frac{2}{3} < x < \frac{4}{3}$$

$$+1 \quad +1 \quad +1$$

Test Endpoints

$x = \frac{2}{3}$: $\sum_{n=1}^{\infty} \frac{(-3)^n(\frac{2}{3}-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-3)^n(-\frac{1}{3})^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1^n = 1}{\sqrt{n}}$ diverges by P Test

$x = \frac{4}{3}$: $\sum_{n=1}^{\infty} \frac{(-3)^n(\frac{4}{3}-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-3)^n(\frac{1}{3})^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by AST

interval: $\boxed{x \in \left(\frac{2}{3}, \frac{4}{3}\right]}$

2 Section 11.9 Functions as Power Series

Key: Geometric series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ ← will have $x/x^2/x^3$/etc in it

How? Suppose $r = x$: $\frac{a}{1-x}$

$$\frac{d}{dx}\left(\frac{a}{1-x}\right) = -a(1-x)^{-2}(-1) = \frac{a}{(1-x)^2} = \frac{d}{dx}\sum_{n=0}^{\infty} ax^n = \sum_{n=1}^{\infty} nax^{n-1}$$

$a(1-x)^{-1}$

$$\int\left(\frac{a}{1-x}\right)dx = -a\ln|1-x| + C = \int\left(\sum_{n=0}^{\infty} ax^n\right)dx$$

1. Given $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, find a power series for $f'(x)$ and $\int f(x)\,dx$. What is $f(x)$?

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$= 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \dots$$

$$f'(x) = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\int f(x)\,dx = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \dots + C - 1$$

$$= 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \dots + C$$

$$= f(x) + C$$

$\frac{d}{dx} f(x) = f(x)$

$\int f(x)\,dx = f(x) + C \longrightarrow f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

NOTE

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) = \sum_{n=0}^{\infty}\left(\frac{d}{dx}x^n\right)\cdot\frac{1}{n!} = \sum_{n=1}^{\infty} \frac{\cancel{n}}{\cancel{n!}\,(n-1)!} x^{n-1} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

2. Express the following functions as power series, and state the radius of convergence.

$f(x) = \arctan x$

$\frac{d}{dx}$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} \quad \frac{a}{1-r} = \sum_{n=0}^{\infty} 1(-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\sum_{n=0}^{\infty} ar^n$$

$$(-1x^2)^n = (-1)^n \cdot (x^2)^n = (-1)^n x^{2n}$$

integrate dx $\quad \sum_{n=0}^{\infty} \int (-1)^n x^{2n} \, dx$ No x's

$$\boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}$$

$\text{ROC} = |r| < 1$

$|-x^2| < 1$

$\sqrt{x^2} < \sqrt{1}$ $\quad |x| < 1$

$-1 < x < 1$ $\quad \boxed{\text{ROC} = 1}$

~~(a) $f(x) = \arctan x$~~

(b) $f(x) = \dfrac{x^2}{(1+x)^2}$ $= \cancel{\left(\dfrac{x}{1+x}\right)^2}$ ← Geom

~~$(a+b+c+\cdots)^2 = a^2+b^2+c^2+\cdots$~~

$$\frac{x^2}{(1+x)^2}$$

$\div x^2$

$$\frac{1}{(1+x)^2} = (1+x)^{-2}$$

integrate

$$-(1+x)^{-1} = \frac{-1}{1+x} = \frac{-1}{1-(-x)} \qquad \frac{a}{1-r} \qquad = \sum_{n=0}^{\infty} (-1)(-x)^n$$

$$\sum_{n=0}^{\infty} ar^n$$

$$(-1)(-1\cdot x)^n = (-1)^1(-1)^n x^n$$

$$\frac{d}{dx}\sum a_n x^n = \sum \frac{d}{dx}(a_n x^n)$$

(n=0 is constant)

$$\sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1}$$

$\cdot x^2$

$$\sum_{n=1}^{\infty} (-1)^{n+1} n \underbrace{x^{n-1}\cdot x^2}_{x^{n+1}} = \boxed{\sum_{n=1}^{\infty} (-1)^{n+1} n x^{n+1}}$$

(c) $f(x) = \dfrac{x}{3-x}$ if this was a "1" we'd have it

$$\frac{x}{3-x} \begin{matrix}\div 3\\ \div 3\end{matrix}$$

$$= \frac{x/3}{1-x/3} \qquad \frac{a}{1-r}$$

$$= \sum_{n=0}^{\infty} \frac{x}{3}\left(\frac{x}{3}\right)^n \qquad \sum_{n=0}^{\infty} ar^n \quad \text{Use Properties of Exponents}$$

$$= \sum_{n=0}^{\infty} \frac{x^1}{3^1}\cdot\frac{x^n}{3^n}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{x^{n+1}}{3^{n+1}}}$$

3. Find the sum of $\sum_{n=1}^{\infty} nx^n$.

$x \sum_{n=1}^{\infty} nx^{n-1}$

series $\div x \downarrow$ integrate $\downarrow$ $\uparrow \frac{d}{dx}$ $\uparrow \cdot x$

$\sum x^n = \frac{1}{1-x}$

$= 1x^1 + 2x^2 + 3x^3 + 4x^4 + \dots$

$= x(1 + 2x + 3x^2 + 4x^3 + \dots)$

$= x \cdot \sum_{n=1}^{\infty} \frac{d}{dx}(x^n)$

$= x \cdot \frac{d}{dx} \sum_{n=0}^{\infty} x^n$

$1 + x + x^2 + \dots = \frac{1}{1-x}$

$= x \cdot \frac{d}{dx}\left(\frac{1}{1-x}\right)$ $\quad (1-x)^{-1}$

$= x \cdot (-1)(1-x)^{-2}(-1)$

$= \boxed{\frac{x}{(1-x)^2}}$

4. If $g(x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots + \frac{1}{n}x^n + \cdots$, find $g\left(\frac{1}{2}\right)$

$g(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$

$\frac{d}{dx}$

$a = 1$, $1 + x + x^2 + x^3 + \dots$, $r = x$

$\sum_{n=0}^{\infty} x^n$ Geometric $= \frac{1}{1-x}$

integrate dx

$\int \frac{1}{1-x}\, dx$

$u = 1-x$

$du = -1\, dx$

$= -\int \frac{1}{u}\, du = -\ln|u|$

$g(x) = -\ln|1-x| + C$

$g(0) =$

if $x = 0$, $0 = -\ln(1) + C$

$C = 0$

$g(x) = -\ln|1-x|$

so $g\left(\frac{1}{2}\right) = -\ln\left|1 - \frac{1}{2}\right| = \boxed{-\ln\left(\frac{1}{2}\right)} = \ln(2)$