WIR 12 152

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MATH 152-Spring 2020
Week in Review XII

Texas A&M University
Math Learning Center

1 Section 11.10 Taylor Series

Definition of Taylor Series

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad c_n = \frac{f^{(n)}(a)}{n!}$$

← Pattern (NOT Simplify)

Maclawin Series to Know (a=0)

1. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
2. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \longrightarrow \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$
3. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ Geometric (11.9)

1. Use the definition to find the Taylor series of the following functions:

Cannot find Taylor Series at nonzero centers from the Maclaurin Series

$\cancel{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$ useless

(a) $f(x) = e^{-x}$ centered at $a = -1$

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad c_n = \frac{f^{(n)}(a)}{n!} \text{ first}$$

$$f^{0}(-1) = e^{-(-1)} = 1e^{1} = (-1)^0 e^1$$

$$f'(x) = (-1)e^{-x} \Rightarrow f'(-1) = (-1)e^{-(-1)} = (-1)^1 e^1$$

$$f''(x) = (-1)(-1)e^{-x} = (-1)^2 e^{-x} \Rightarrow f''(-1) = (-1)^2 e^1$$

$$f'''(x) = (-1)^2(-1)e^{-x} = (-1)^3 e^{-x} \Rightarrow f'''(-1) = (-1)^3 e^1$$

$$f^{(n)}(-1) = (-1)^n e^1 \quad \text{so} \quad c_n = \frac{f^n(-1)}{n!} = \frac{(-1)^n e^1}{n!}$$

$$\therefore \boxed{e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n e^1}{n!}(x+1)^n}$$

$\sum c_n (x-a)^n$

(b) $f(x) = \sin(x)$ centered at $a = \frac{\pi}{2}$

$n=0$

$f^{(0)}(x) = \sin x \Rightarrow f^{(0)}(\frac{\pi}{2}) = \sin\frac{\pi}{2} = 1$

$f^{(1)}(x) = \cos x \Rightarrow f'(\frac{\pi}{2}) = \cos\frac{\pi}{2} = 0$

$f''(x) = -\sin x \Rightarrow f''(\frac{\pi}{2}) = -\sin\frac{\pi}{2} = -1$

$f'''(x) = -\cos x \Rightarrow f'''(\frac{\pi}{2}) = -\cos\frac{\pi}{2} = 0$

$f^{(4)}(x) = \sin x \Rightarrow f^{(4)}(\frac{\pi}{2}) = \sin\frac{\pi}{2} = 1$

0, -1, 0, ⋮

only even derivatives

$$\boxed{\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(x-\frac{\pi}{2}\right)^{2n}}$$

Alt 1, -1, 1, -1, ...

$f^{(2n)}(\frac{\pi}{2}) = (-1)^n$ or $(-1)^{n+1}$?

$n=0 \Rightarrow 1 = (-1)^0$ or $(-1)^{0+1}$?

Alt $\frac{1}{0!}(x-\frac{\pi}{2})^0 + \frac{0}{1!}(x-\frac{\pi}{2})^1 + \frac{-1}{2!}(x-\frac{\pi}{2})^2 + \ldots$

$c_n = \frac{(-1)^n}{(2n)!}$

(c) $f(x) = \ln(x)$ centered at $a = 2$ $\quad (x-2)^n$

$f^{(0)}(x) = \ln(x) \Rightarrow f^{(0)}(2) = \ln 2 \rightarrow$ Set aside

Starts $n=1$

$f'(x) = \frac{1}{x} = x^{-1} \Rightarrow f'(2) = 2^{-1} = (-1)^0 \cdot 0! \cdot 2^{-1}$

$f''(x) = -1x^{-2} \Rightarrow f''(2) = (-1)(2)^{-2}$ $1!$

$f'''(x) = (-1)(-2)x^{-3} = (-1)^2(1\cdot 2)x^{-3} \Rightarrow f'''(2) = (-1)^2(1\cdot 2)(2)^{-3}$ $2!$

$f^{(4)}(x) = (-1)^2(1\cdot 2)(-3)x^{-4} = (-1)^3(1\cdot 2\cdot 3)x^{-4}$

$\Rightarrow f^{(4)}(2) = (-1)^3(1\cdot 2\cdot 3)(2)^{-4}$ $3!$

$$\boxed{\ln(x) = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 2^n}(x-2)^n}$$

Why ln 2 off to side

Integrated a Geometric Series

$C + \sum$

$f^{(n)}(2) = (-1)^{n-1}(n-1)!\, 2^{-n}$

$c_n = \frac{f^{n}(2)}{n!} = \frac{(-1)^{n-1}\cancel{(n-1)!}\,2^{-n}}{n!} = \frac{(-1)^{n-1}}{n 2^n}$

$n(n-1)!$

Simplify

2. Use known Maclaurin Series to find the Maclaurin series of the following: $(a=0)$

(a) $f(x) = x\sin(2x)$

Series we know $= \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$\sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}$$

$$x\cdot\sin(2x) = x\cdot\sum_{n=0}^{\infty} \frac{(-1)^n\cdot 2^{2n+1} x^{2n+1}}{(2n+1)!}$$

$$x\sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n\cdot 2^{2n+1}\cdot x^1\cdot x^{2n+1}}{(2n+1)!}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+2}}{(2n+1)!}}$$

(b) $f(x) = \int_0^x e^{-t^2}\,dt$

Start with $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} \quad = (-1)^n (t^2)^n = (-1)^n t^{2n}$$

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$$

$$\int_0^x e^{-t^2}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)\,n!}\Bigg|_{t=0}^{t=x}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\,n!} - \sum_{n=0}^{\infty} \frac{(-1)^n 0^{2n+1}}{(2n+1)\,n!}$$

$$\boxed{\int_0^x e^{-t^2}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\,n!}}$$

3. Find the coefficient of x^2 in the Maclaurin series for $f(x) = \frac{1}{1-x} \cdot e^x$. Could use definition

We know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\left(\frac{1}{1-x}\right)(e^x) = (1 + x + x^2 + x^3 + \ldots)(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots)$$

$$= \# + \# x + \frac{1}{2}x^2 + 1x^2 + 1x^2$$

$$= \# + \# x + \boxed{\frac{5}{2}} x^2$$

Side Note $= 1 + 2x + \frac{5}{2}x^2 + \ldots$

2 Section 11.11 Taylor Polynomials

$$T_N(x) = \sum_{n=0}^{N} \frac{f^n(a)}{n!}(x-a)^n$$

(compute all N derivatives no pattern needed)

N^{th} degree Taylor Polynomial

1. Approximate $f(x) = \sec x$ by a Taylor polynomial of degree 2 at $x = 0$.

$\sec x = \frac{1}{\cos x}$

$\frac{1}{\sum c_n x^n} \neq \sum \frac{1}{c_n x^n}$

$\frac{1}{2+3}$ ~~$=$~~ $\frac{1}{2}+\frac{1}{3}$

$f^{(0)}(x) = \sec x \Rightarrow f^{(0)}(0) = \sec(0) = 1 \quad \left(\frac{1}{\cos 0}\right)$

$f'(x) = \sec x \tan x \Rightarrow f'(0) = \underset{1}{\sec(0)}\cdot\underset{0}{\tan(0)} = 0$

Product Rule!

$f''(x) = (\sec x \tan x)(\tan x) + (\sec x)(\sec^2 x)$

$= \sec x \tan^2 x + \sec^3 x$

$\Rightarrow f''(0) = \underset{1}{\sec 0} \cdot \underset{0}{(\tan 0)^2} + \underset{1}{(\sec 0)^3} = 1$

$\sec x \approx T_2(x) = \frac{1}{0!}(x-0)^0 + \frac{0}{1!}(x-0)^1 + \frac{1}{2!}(x-0)^2$

$= \boxed{1 + \frac{1}{2}x^2}$

2. Approximate $f(x) = \sqrt{x}$ by a Taylor polynomial of degree 3 at $x = 4$. $= a$

$f^{(0)}(x) = x^{1/2} \Rightarrow f^{(0)}(4) = 4^{1/2} = \sqrt{4} = 2$

$f^{(1)}(x) = \frac{1}{2}x^{-1/2} \Rightarrow f'(4) = \frac{1}{2}\cdot 4^{-1/2} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

$f^{(2)}(x) = -\frac{1}{4}x^{-3/2} \Rightarrow f^{(2)}(4) = -\frac{1}{4}(4)^{-3/2} = -\frac{1}{4}\cdot\frac{1}{(\sqrt{4})^3} = \frac{-1}{32}$

$f^{(3)}(x) = \frac{3}{8}x^{-5/2} \Rightarrow f^{(3)}(4) = \frac{3}{8}(4)^{-5/2} = \frac{3}{8}\cdot\frac{1}{\underset{32}{(\sqrt{4})^5}} = \frac{3}{256}$

$-\frac{1}{32}\cdot\frac{1}{2}$ $\quad$ $\frac{3}{256}\cdot\frac{1}{6_2}$

$\sqrt{x} \approx T_3(x) = \frac{2}{0!}(x-4)^0 + \frac{1/4}{1!}(x-4)^1 + \frac{-1/32}{2!}(x-4)^2 + \frac{\frac{3}{256}}{3!}(x-4)^3$

$= \boxed{2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3}$

Convergence Tests $\sum a_n$

1) Does $\lim_{n\to\infty} a_n \neq 0$? YES series diverges by Test for Divergence

NO 2) Factorials and/or Exponentials? (other than $(-1)^n$) YES Ratio Test

NO 3) is there $(-1)^n$? YES AST (may have to continue for abs convergence)

NO 4) Ratio of Powers? YES LCT

NO 5) is sine/cosine? YES Comparison Test ($\frac{\sin}{\cos} \le 1$)

NO 6) Hope to Integrate it

3 Exam III Review

1. Determine if the following series are absolutely convergent, convergent but not absolutely, or divergent. Explain which test you use and show clearly that all conditions are met.

(a) $\sum_{n=1}^{\infty} \frac{n^3}{n^5+5n^4+7}$

Ratio of Powers $\to$ Comparison or Limit Comparison Test

Compare with $\sum \frac{n^3}{n^5} = \sum \frac{1}{n^2}$ KNOW this series converges by P-Test ($\sum \frac{1}{n^p}$ conv if $p>1$)

Let $a_n = \frac{n^3}{n^5+5n^4+7} > 0 \quad b_n = \frac{1}{n^2} > 0$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3}{n^5+5n^4+7} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{n^5}{n^5+5n^4+7} = 1$$

So $\sum a_n$ and $\sum b_n$ do the same thing

$\sum_{n=1}^{\infty} \frac{n^3}{n^5+5n^4+7}$ converges by LCT with $\sum \frac{1}{n^2}$

Since terms are all positive, series converges absolutely ($\sum |a_n| = \sum a_n$)

(Could also use Comparison Test)

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+3}$

AST Let $b_n = |a_n| = \frac{1}{n+3}$

$b_n > 0$? YES

b_n decreasing? YES $f(x) = \frac{1}{x+3} \to f'(x) = \frac{-1}{(x+3)^2} < 0$ so f dec

$b_n \to 0$? YES $\lim_{n\to\infty} \frac{1}{n+3} = 0$

$\therefore$ $\boxed{\text{Series converges by AST}}$

Absolute convergence?

Does $\sum b_n$ converge? $\sum_{n=0}^{\infty} \frac{1}{n+3}$

Compare with $\sum \frac{1}{n}$ KNOW this diverges by p-test $1 \not> 1$

LCT $\lim_{n\to\infty} \frac{b_n}{c_n} = \lim_{n\to\infty} \frac{1}{n+3} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n}{n+3} = 1$

$\sum b_n$ and $\sum c_n$ both do the same thing

$\sum_{n=0}^{\infty} \frac{1}{n+3}$ diverges by LCT with $\sum \frac{1}{n}$

So $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+3}$ converges, but NOT absolutely

2. At least how many terms are needed to sum the series in #1(b) to within $\frac{1}{10000}$? (Alt Series Error Estimation)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+3}$$

$|s - s_N| = R_N \le b_{N+1} < \frac{1}{10000}$ where $b_n = |a_n|$

$b_n = \frac{1}{n+3}$ so $b_{N+1} = \frac{1}{N+1+3} \le \frac{1}{10000}$

$\frac{1}{N+4} \le \frac{1}{10000}$

$10000 \le N + 4$
-4

$N \ge 9996$

$$S_{9996} = \sum_{n=0}^{9996} a_n$$

has $\boxed{9997 \text{ terms}}$

Power Series

n in exponent $\rightarrow$ Ratio Test

3. Find the radius and interval of convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n 3^n (x-2)^n}{(n+1)2^{2n}}$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{3^{n+1}(x-2)^{n+1}}{((n+1)+1)2^{2(n+1)}} \cdot \frac{(n+1)2^{2n}}{3^n(x-2)^n} \right| < 1 \quad \text{want}$$

$$= \lim_{n\to\infty} \left| \frac{3^{n+1}}{3^n} \cdot \frac{(x-2)^{n+1}}{(x-2)^n} \cdot \frac{n+1}{n+2} \cdot \frac{2^{2n}}{2^{2n+2}} \right| < 1$$

$$= \lim_{n\to\infty} \left| 3 \cdot (x-2) \cdot \frac{n+1}{n+2} \cdot \frac{1}{4} \right| < 1$$

$$= \frac{4}{3} \cdot \frac{3}{4} |x-2| \cdot 1 < 1 \cdot \frac{4}{3} \rightarrow |x-2| < \boxed{\frac{4}{3} = ROC}$$

$|x-a| < R$ radius of conv

$-\frac{4}{3} < x - 2 < \frac{4}{3}$
$+2 = \frac{6}{3}$ $\quad +2 \quad$ $+2 = \frac{6}{3}$

$\frac{2}{3} < x < \frac{10}{3}$

Test endpoints

$x = \frac{2}{3}$: $\sum_{n=0}^{\infty} \frac{(-1)^n 3^n (\frac{2}{3} - 2)^n}{(n+1) \cdot 2^{2n}} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{3 \cdot -\frac{4}{3}}{4}\right)^n}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^n}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges by LCT with $\sum \frac{1}{n}$

$-\frac{4}{3}$, -1, $(2^2)^n = 4^n$, $(-1)^{2n} = 1$

$x = \frac{10}{3}$: $\sum_{n=0}^{\infty} \frac{(-1)^n 3^n (\frac{10}{3} - 2)^n}{(n+1) \, 4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n 1^n}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ Converges by AST

$\frac{4}{3}$

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So our $\boxed{\text{interval of conv is } x \in \left(\frac{2}{3}, \frac{10}{3}\right]}$

Maclaurian BUT Don't use definition!

4. Write $f(x) = \ln(1 + x^3)$ as a power series centered at $a = 0$ and find its radius of convergence. (HINT: start with a power series for $f'(x)$)

$$\ln(1+x^3) = \sum_{n=0}^{\infty} \frac{3(-1)^n x^{3n+3}}{3n+3} \cancel{+C}$$

$f(x) = \ln(1+x^3)$

Deriv $\frac{d}{dx}$

$$f'(x) = \frac{3x^2}{1+x^3} = \frac{3x^2}{1-(-x^3)}$$

Geometric Sum (11.9)

$$\frac{a}{1-r} = \sum_{n=0}^{\infty} ar^n$$

Integrate dx

$$\sum_{n=0}^{\infty} (3x^2)(-x^3)^n = \sum_{n=0}^{\infty} 3x^2(-1)^n x^{3n} = \sum_{n=0}^{\infty} 3(-1)^n x^{3n+2}$$

$+C$: if $x=0$, $f(0) = \ln(1+0^3) = \ln(1) = 0$, so $C=0$